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Positive Solutions for a Nonlocal Fractional Boundary Value Problem with r-Laplacian Operator

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Abstract

We study the existence and nonexistence of positive solutions for a Riemann–Liouville fractional differential equation with r-Laplacian operator and a nonnegative nonlinearity, subject to some nonlocal boundary conditions which contain Riemann–Stieltjes integrals, various fractional derivatives and a positive constant. In the proof of our main existence result we use the Schauder fixed point theorem.

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1 Introduction

We consider the fractional differential equation

$$D_{0+}^{\alpha}(\varphi_r(D_{0+}^{\beta}u(t))) + a(t)f(u(t)) = 0, \;\; t \in (0,1), \tag{1.1}$$

supplemented with the nonlocal boundary conditions

$$\begin{cases} u^{(j)}(0) = 0, \; j = 0, \ldots, n-2; \;\; D_{0+}^{\beta}u(0) = 0, \\ D_{0+}^{\gamma_0}u(1) = \displaystyle\sum_{i=1}^{p} \int_0^1 D_{0+}^{\gamma_i}u(t)\, dH_i(t) + a_0, \end{cases} \tag{1.2}$$

where $\alpha \in (0,1]$, $\beta \in (n-1,n]$, $n \in \mathbb{N}$, $n \geq 3$, $p \in \mathbb{N}$, $\gamma_i \in \mathbb{R}$ for all $i = 0, 1, \ldots, p$, $0 \leq \gamma_1 < \gamma_2 < \cdots < \gamma_p \leq \gamma_0 < \beta - 1$, $\gamma_0 \geq 1$, $r > 1$, $\varphi_r(s) = |s|^{r-2}s$, $\varphi_r^{-1} = \varphi_\varrho$,

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$\varrho = r/(r-1)$, a and f are nonnegative nonlinear functions, a_0 is a positive constant, the integrals from (1.2) are Riemann–Stieltjes integrals with H_i, $i = 1, \dots, p$ functions of bounded variation, and D_{0+}^k denotes the Riemann–Liouville fractional derivative of order k (for $k = \alpha$, β, γ_i for $i = 0, \dots, p$).

In this paper we present sufficient conditions for the nonlinearities a and f such that problem (1.1)–(1.2) has at least one positive solution, or it has no positive solution. We apply the Schauder fixed point theorem to prove our main existence result. A positive solution of problem (1.1)–(1.2) is a function $u \in C([0,1], \mathbb{R}_+)$ satisfying (1.1) and (1.2), with $u(t) > 0$ for all $t \in (0,1]$, ($\mathbb{R}_+ = [0,\infty)$). The existence of multiple positive solutions for the fractional differential equation

$$D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, \;\; t \in (0,1), \tag{1.3}$$

subject to the boundary conditions

$$u^{(j)}(0) = 0, \;\; j = 0, \dots, m-2; \;\; D_{0+}^{\gamma_0} u(1) = \sum_{i=1}^{p} \int_0^1 D_{0+}^{\gamma_i} u(t)\, dH_i(t),$$

where $\alpha \in \mathbb{R}$, $\alpha \in (m-1, m]$, $m \in \mathbb{N}$, $m \geq 3$, and the function f from (1.3) may change sign and may be singular at various points, was investigated in the paper [1]. The authors used in [1] various height functions of the nonlinearity defined on special bounded sets, and some theorems from the fixed point index theory. We also mention the recent paper [20], where the authors studied the system of Riemann–Liouville fractional differential equations with p-Laplacian operators

$$\begin{cases} D_{0+}^{\alpha_1}(\varphi_{\varrho_1}(D_{0+}^{\beta_1} u(t))) + \lambda f(t, u(t), v(t)) = 0, \;\; t \in (0,1), \\ D_{0+}^{\alpha_2}(\varphi_{\varrho_2}(D_{0+}^{\beta_2} v(t))) + \mu g(t, u(t), v(t)) = 0, \;\; t \in (0,1), \end{cases} \tag{1.4}$$

with the coupled nonlocal boundary conditions

$$\begin{cases} u^{(j)}(0) = 0, \;\; j = 0, \dots, n-2; \;\; D_{0+}^{\beta_1} u(0) = 0, \\ D_{0+}^{\gamma_0} u(1) = \displaystyle\sum_{i=1}^{p} \int_0^1 D_{0+}^{\gamma_i} v(t)\, dH_i(t), \\ v^{(j)}(0) = 0, \;\; j = 0, \dots, m-2; \;\; D_{0+}^{\beta_2} v(0) = 0, \\ D_{0+}^{\delta_0} v(1) = \displaystyle\sum_{i=1}^{q} \int_0^1 D_{0+}^{\delta_i} u(t)\, dK_i(t), \end{cases} \tag{1.5}$$

where $\alpha_1, \alpha_2 \in (0,1]$, $\beta_1 \in (n-1, n]$, $\beta_2 \in (m-1, m]$, $n, m \in \mathbb{N}$, $n, m \geq 3$, $p, q \in \mathbb{N}$, $\gamma_i \in \mathbb{R}$ for all $i = 0, 1, \dots, p$, $0 \leq \gamma_1 < \gamma_2 < \cdots < \gamma_p \leq \delta_0 < \beta_2 - 1$, $\delta_0 \geq 1$, $\delta_i \in \mathbb{R}$ for all $i = 0, 1, \dots, q$, $0 \leq \delta_1 < \delta_2 < \cdots < \delta_q \leq \gamma_0 < \beta_1 - 1$, $\gamma_0 \geq 1$, $\varrho_1, \varrho_2 > 1$, $\varphi_{\varrho_i}(s) = |s|^{\varrho_i - 2} s$, λ and μ are positive parameters, f and g are nonnegative continuous functions, and H_i, $i = 1, \dots, p$ and K_i, $i = 1, \dots, q$ are functions of bounded variation. In [20], the authors gave sufficient conditions on the functions f and g, and intervals for the parameters λ and μ such that problem (1.4)–(1.5) has positive solutions. The existence results for (1.4)–(1.5) were based on the Guo–Krasnosel'skii fixed point theorem of cone expansion and compression of norm type. In this last paper, the authors also investigated the nonexistence of positive solutions.

Fractional calculus and fractional differential equations describe many phenomena in several fields of engineering and scientific disciplines such as physics, biophysics, chemistry, biology (for example, the primary infection with HIV), economics, control theory, signal and image processing, thermoelasticity, aerodynamics, viscoelasticity, electromagnetics, and rheology (see the books [4, 5, 12, 13, 16–18], and the papers [3, 6–9, 14, 15, 19]). Fractional differential equations are also better tools for the description of hereditary properties of various materials and processes than the corresponding integer order differential equations. For some new results for fractional differential equations and systems of fractional differential equations, subject to various nonlocal boundary conditions, we refer the reader to the monographs [2, 10, 21].

2 Auxiliary Results

We consider the nonlinear fractional differential equation

$$D_{0+}^{\alpha}(\varphi_r(D_{0+}^{\beta}u(t))) + x(t) = 0, \;\; t \in (0,1), \tag{2.1}$$

with the boundary conditions

$$\begin{cases} u^{(j)}(0) = 0, \;\; j = 0, \ldots, n-2; \;\; D_{0+}^{\beta}u(0) = 0, \\ D_{0+}^{\gamma_0}u(1) = \displaystyle\sum_{i=1}^{p} \int_0^1 D_{0+}^{\gamma_i}u(t)\, dH_i(t), \end{cases} \tag{2.2}$$

where $x \in C[0,1]$. We denote by

$$\Delta = \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma_0)} - \sum_{i=1}^{p} \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma_i)} \int_0^1 s^{\beta-\gamma_i-1}\, dH_i(s).$$

Lemma 2.1. *If $\Delta \neq 0$, then the unique solution $u \in C[0,1]$ of problem* (2.1)–(2.2) *is given by*

$$u(t) = \int_0^1 \mathcal{G}(t,s)\varphi_{\varrho}(I_{0+}^{\alpha}x(s))\, ds, \;\; t \in [0,1], \tag{2.3}$$

where the Green function $\mathcal{G}$ is given by

$$\mathcal{G}(t,s) = g_1(t,s) + \frac{t^{\beta-1}}{\Delta} \sum_{i=1}^{p} \left(\int_0^1 g_{2i}(\tau,s)\, dH_i(\tau) \right), \;\; t,s \in [0,1], \tag{2.4}$$

with

$$g_1(t,\zeta) = \frac{1}{\Gamma(\beta)} \begin{cases} t^{\beta-1}(1-\zeta)^{\beta-\gamma_0-1} - (t-\zeta)^{\beta-1}, \ 0 \le \zeta \le t \le 1, \\ t^{\beta-1}(1-\zeta)^{\beta-\gamma_0-1}, \ 0 \le t \le \zeta \le 1, \end{cases}$$
$$g_{2i}(\tau,\zeta) = \frac{1}{\Gamma(\beta-\gamma_i)} \begin{cases} \tau^{\beta-\gamma_i-1}(1-\zeta)^{\beta-\gamma_0-1} - (\tau-\zeta)^{\beta-\gamma_i-1}, \\ \qquad 0 \le \zeta \le \tau \le 1, \\ \tau^{\beta-\gamma_i-1}(1-\zeta)^{\beta-\gamma_0-1}, \ 0 \le \tau \le \zeta \le 1, \end{cases} \tag{2.5}$$
$$i = 1,\dots,p.$$

Proof. We denote by $\varphi_r(D_{0+}^{\beta}u(t)) = v(t)$. Then problem (2.1)–(2.2) is equivalent to the following two boundary value problems

$$D_{0+}^{\alpha}v(t) + x(t) = 0, \ \ 0 < t < 1; \ \ v(0) = 0, \tag{2.6}$$

and

$$\begin{cases} D_{0+}^{\beta}u(t) = \varphi_{\varrho}(v(t)), \ \ 0 < t < 1; \\ u^{(j)}(0) = 0, \ j = 0,\dots,n-2; \ D_{0+}^{\gamma_0}u(1) = \displaystyle\sum_{i=1}^{p} \int_0^1 D_{0+}^{\gamma_i}u(t)\,dH_i(t). \end{cases} \tag{2.7}$$

For the first problem (2.6), the function

$$v(t) = -I_{0+}^{\alpha}x(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}x(s)\,ds, \ \ t \in [0,1], \tag{2.8}$$

is the unique solution $v \in C[0,1]$ of (2.6). For the second problem (2.7), if $\Delta \ne 0$, then by [1, Lemma 2.2], we deduce that the function

$$u(t) = -\int_0^1 \mathcal{G}(t,s)\varphi_{\varrho}(v(s))\,ds, \ \ t \in [0,1], \tag{2.9}$$

where $\mathcal{G}$ is given by (2.4), is the unique solution $u \in C[0,1]$ of problem (2.7). Now, by using relations (2.8) and (2.9), we find formula (2.3) for the unique solution $u \in C[0,1]$ of problem (2.1)–(2.2). □

By using the properties of the functions g_1, g_{2i}, $i = 1,\dots,p$, given by (2.5) (see [1] and [11]), we obtain the following properties of the Green function $\mathcal{G}$ that will be used in the next section.

Lemma 2.2. *Assume that $H_i : [0,1] \to \mathbb{R}$, $i = 1,\dots,p$ are nondecreasing functions and $\Delta > 0$. Then the Green function $\mathcal{G}$ given by* (2.4) *has the properties:*

a) $\mathcal{G} : [0,1]\times[0,1] \to [0,\infty)$ is a continuous function;

b) $\mathcal{G}(t,s) \le \mathcal{J}(s)$ for all t, $s \in [0,1]$, where

$$\mathcal{J}(s) = h(s) + \frac{1}{\Delta}\sum_{i=1}^{p}\int_0^1 g_{2i}(\tau,s)\,dH_i(\tau), \ \ s \in [0,1], \ \ \textit{and}$$
$$h(s) = \frac{1}{\Gamma(\beta)}[(1-s)^{\beta-\gamma_0-1} - (1-s)^{\beta-1}], \ \ s \in [0,1];$$

c) $\mathcal{G}(t,s) \ge t^{\beta-1}\mathcal{J}(s)$ for all t, $s \in [0,1]$.

Lemma 2.3. *Assume that $H_i : [0,1] \to \mathbb{R}$, $i = 1,\dots,p$ are nondecreasing functions, $\Delta > 0$, and $x \in C([0,1];\mathbb{R}_+)$. Then the solution u of problem* (2.1)–(2.2) *satisfies the inequality $u(t) \geq 0$ for all $t \in [0,1]$. Moreover, we have the inequality $u(t) \geq t^{\beta-1}u(t')$ for all $t, t' \in [0,1]$.*

Proof. Under the assumptions of this lemma, by using relation (2.3) and Lemma 2.2, we deduce that $u(t) \geq 0$ for all $t \in [0,1]$. Besides, by using again Lemma 2.2, we obtain

$$\begin{aligned} u(t) &= \int_0^1 \mathcal{G}(t,\zeta)\varphi_\varrho(I_{0+}^\alpha x(\zeta))\,d\zeta \geq \int_0^1 t^{\beta-1}\mathcal{J}(\zeta)\varphi_\varrho(I_{0+}^\alpha x(\zeta))\,d\zeta \\ &\geq t^{\beta-1}\int_0^1 \mathcal{G}(t',\zeta)\varphi_\varrho(I_{0+}^\alpha x(\zeta))\,d\zeta = t^{\beta-1}u(t'), \quad \forall\, t,t' \in [0,1]. \end{aligned}$$

This completes the proof. □

3 Main Results

In this section we investigate the existence and nonexistence of positive solutions for problem (1.1)–(1.2) under various assumptions on the nonlinearities a and f. We present now the assumptions that will be used in our main results:

$(H1)$ $\alpha \in (0,1]$, $\beta \in (n-1,n]$, $n \in \mathbb{N}$, $n \geq 3$, $p \in \mathbb{N}$, $\gamma_i \in \mathbb{R}$ for all $i = 0,1,\dots,p$, $0 \leq \gamma_1 < \gamma_2 < \cdots < \gamma_p \leq \gamma_0 < \beta_1 - 1$, $\gamma_0 \geq 1$, $H_i,\ i = 1,\dots,p$ are nondecreasing functions, $\Delta > 0$, $r > 1$, $\varphi_r(s) = |s|^{r-2}s$, $\varphi_r^{-1} = \varphi_\varrho$, $\varrho = r/(r-1)$.

$(H2)$ The function $a : [0,1] \to [0,\infty)$ is continuous and there exists $t_0 \in (0,1)$ such that $a(t_0) > 0$.

$(H3)$ The function $f : [0,\infty) \to [0,\infty)$ is continuous, and there exists $c_0 > 0$ such that $f(x) < \dfrac{c_0^{r-1}}{L}$ for all $x \in [0,c_0]$, where $L = \dfrac{\Lambda_0}{\Gamma(\alpha+1)}\left(\displaystyle\int_0^1 s^{\alpha(\varrho-1)}\mathcal{J}(s)\,ds\right)^{r-1}$, with $\Lambda_0 = \sup\limits_{t\in[0,1]} a(t)$, and $\mathcal{J}$ is defined in Lemma 2.2.

$(H4)$ The function $f : [0,\infty) \to [0,\infty)$ is continuous and satisfies the condition $\lim\limits_{u\to\infty} \dfrac{f(u)}{u^{r-1}} = \infty$.

Our first theorem is the following existence result for problem (1.1)–(1.2).

Theorem 3.1. *Assume that assumptions $(H1)$–$(H3)$ hold. Then problem* (1.1)–(1.2) *has at least one positive solution for $a_0 > 0$ sufficiently small.*

Proof. We consider the problem

$$\begin{cases} D_{0+}^{\alpha}\left(\varphi_r(D_{0+}^{\beta}k(t))\right) = 0, \;\; t\in(0,1), \\ k^{(j)}(0)=0, \;\; j=0,\dots,n-2, \;\; D_{0+}^{\beta}k(0)=0, \\ D_{0+}^{\gamma_0}k(1) = \displaystyle\sum_{i=1}^{p}\int_0^1 D_{0+}^{\gamma_i}k(t)\,dH_i(t)+1. \end{cases} \tag{3.1}$$

By splitting problem (3.1) into two boundary value problems (as in the first part of the proof of Lemma 2.1), we easily deduce that the unique solution from $C[0,1]$ of problem (3.1) is the function

$$k(t) = \frac{t^{\beta-1}}{\Delta}, \;\; t\in[0,1]. \tag{3.2}$$

By assumption $(H1)$ we obtain $k(t)>0$ for all $t\in(0,1]$.

We define the function $z(t)=u(t)-a_0k(t)$ for $t\in[0,1]$, where u is a solution of problem (1.1)–(1.2). Then (1.1)–(1.2) can be equivalently written as

$$D_{0+}^{\alpha}(\varphi_r(D_{0+}^{\beta}z(t))) + a(t)f(z(t)+a_0k(t)) = 0, \;\; t\in(0,1), \tag{3.3}$$

with the boundary conditions

$$\begin{cases} z^{(j)}(0)=0, \;\; j=0,\dots,n-2; \;\; D_{0+}^{\beta}z(0)=0, \\ D_{0+}^{\gamma_0}z(1) = \displaystyle\sum_{i=1}^{p}\int_0^1 D_{0+}^{\gamma_i}z(t)\,dH_i(t). \end{cases} \tag{3.4}$$

Using the Green function $\mathcal{G}$ from Section 2 (Lemma 2.1), the function z is a solution of problem (3.3)–(3.4) if and only if z is a solution for the nonlinear integral equation

$$z(t) = \int_0^1 \mathcal{G}(t,s)\varphi_\varrho(I_{0+}^{\alpha}(a(s)f(z(s)+a_0k(s))))\,ds, \;\; t\in[0,1], \tag{3.5}$$

where $k(t),\; t\in[0,1]$ is given by (3.2).

We consider the Banach space $\mathcal{X}=C[0,1]$ with the supremum norm $\|u\| = \sup_{t\in[0,1]}|u(t)|$, and define the set $\mathcal{V}=\{z\in\mathcal{X},\; 0\le z(t)\le c_0,\; \forall t\in[0,1]\}\subset\mathcal{X}$. We also define the operator $\mathcal{A}:\mathcal{V}\to\mathcal{X}$ by

$$(\mathcal{A}z)(t) = \int_0^1 \mathcal{G}(t,s)\varphi_\varrho(I_{0+}^{\alpha}(a(s)f(z(s)+a_0k(s))))\,ds, \;\; \forall t\in[0,1],$$

for all $z\in\mathcal{V}$. We remark that z is a solution of equation (3.5) if and only if z is a fixed point of operator $\mathcal{A}$. In what follows, we will investigate the existence of fixed points z of operator $\mathcal{A}$.

For sufficiently small $a_0 > 0$, by $(H3)$, we deduce that $f(z(t) + a_0 k(t)) \le c_0^{r-1}/L$ for all $t \in [0,1]$ and $z \in \mathcal{V}$. By using Lemma 2.2, we obtain $(\mathcal{A}z)(t) \ge 0$ for all $t \in [0,1]$ and $z \in \mathcal{V}$. In addition, for all $z \in \mathcal{V}$ we have

$$\begin{aligned} I_{0+}^{\alpha}(a(s)f(z(s)+a_0k(s))) &= \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1}a(\tau)f(z(\tau)+a_0k(\tau))\,d\tau \\ &\le \frac{c_0^{r-1}}{\Gamma(\alpha)L}\int_0^s (s-\tau)^{\alpha-1}a(\tau)\,d\tau \le \frac{c_0^{r-1}\Lambda_0}{\Gamma(\alpha)L}\int_0^s (s-\tau)^{\alpha-1}\,d\tau = \frac{c_0^{r-1}\Lambda_0 s^{\alpha}}{\Gamma(\alpha+1)L}, \end{aligned}$$

for all $s \in [0,1]$, where $\Lambda_0 = \sup\limits_{t\in[0,1]} a(t)$. So we find

$$(\mathcal{A}z)(t) \le \int_0^1 \mathcal{J}(s)\left(\frac{c_0^{r-1}\Lambda_0 s^{\alpha}}{\Gamma(\alpha+1)L}\right)^{\varrho-1} ds = \left(\frac{c_0^{r-1}\Lambda_0}{\Gamma(\alpha+1)L}\right)^{\varrho-1}\int_0^1 s^{\alpha(\varrho-1)}\mathcal{J}(s)\,ds = c_0,$$

for all $t \in [0,1]$. Therefore $\mathcal{A}(\mathcal{V}) \subset \mathcal{V}$. By using standard arguments, we deduce that $\mathcal{A}$ is completely continuous. Now by using the Schauder fixed point theorem, we conclude that $\mathcal{A}$ has a fixed point $z \in \mathcal{V}$ which is a solution for problem (3.3)–(3.4). Then our problem (1.1)–(1.2) has a positive solution $u = z + a_0 k$ for sufficiently small a_0, which satisfies the inequalities $\frac{a_0}{\Delta}t^{\beta-1} \le u(t) \le \frac{a_0}{\Delta}t^{\beta-1} + c_0$, for all $t \in [0,1]$. □

In what follows we present sufficient conditions for the nonexistence of positive solutions of (1.1)–(1.2).

Theorem 3.2. *Assume that assumptions* $(H1)$, $(H2)$ *and* $(H4)$ *hold. Then problem* (1.1)–(1.2) *has no positive solution for* $a_0 > 0$ *sufficiently large.*

Proof. We suppose that u is a positive solution of (1.1)–(1.2), ($u(t) > 0$ for all $t \in (0,1]$). Then $z = u - a_0 k$ is a solution for (3.3)–(3.4) (or equivalently, for equation (3.5)), where k is the solution of problem (3.1) given by (3.2), that is $k(t) = t^{\beta-1}/\Delta,\ t \in [0,1]$. By $(H2)$, there exists $c \in (0,1/2)$ such that $t_0 \in (c, 1-c)$ and then

$$\Lambda_1 = \int_c^{1-c} \mathcal{J}(s)\left(\int_c^s (s-\tau)^{\alpha-1}a(\tau)\,d\tau\right)^{\varrho-1} ds > 0. \tag{3.6}$$

By using Lemma 2.3, we have $z(t) \ge t^{\beta-1}\|z\|$ for all $t \in [0,1]$, and then $\inf\limits_{t\in[c,1-c]} z(t) \ge c^{\beta-1}\|z\|$. Using (3.2), we deduce

$$\inf_{t\in[c,1-c]} k(t) = k(c) = \frac{c^{\beta-1}}{\Delta} = c^{\beta-1}\|k\|,\ \left(\|k\| = \frac{1}{\Delta}\right).$$

Then we obtain

$$\begin{aligned} &\inf_{t\in[c,1-c]}(z(t)+a_0k(t)) \ge \inf_{t\in[c,1-c]} z(t) + a_0 \inf_{t\in[c,1-c]} k(t) \\ &\ge c^{\beta-1}\|z\| + a_0c^{\beta-1}\|k\| \ge c^{\beta-1}\|z+a_0k\|. \end{aligned}$$

We consider now $R = 2^{r-1}\Gamma(\alpha)(c^{2(\beta-1)}\Lambda_1)^{1-r}$, where Λ_1 is given by (3.6). By using $(H4)$, for R above, we conclude that there exists $M > 0$ such that $f(u) > Ru^{r-1}$ for all $u \geq M$. We consider $a_0 > 0$ sufficiently large such that $\inf\limits_{t\in[c,1-c]}(z(t) + a_0k(t)) \geq M$. Now by using Lemma 2.2 and the above considerations we find

$$\begin{aligned}
&I_{0+}^{\alpha}(a(s)f(z(s)+a_0k(s))) \geq \frac{1}{\Gamma(\alpha)}\int_c^s (s-\tau)^{\alpha-1}a(\tau)f(z(\tau)+a_0k(\tau))\,d\tau \\
&\geq \frac{R}{\Gamma(\alpha)}\int_c^s (s-\tau)^{\alpha-1}a(\tau)(z(\tau)+a_0k(\tau))^{r-1}\,d\tau \\
&\geq \frac{R}{\Gamma(\alpha)}\int_c^s (s-\tau)^{\alpha-1}a(\tau)\left(\inf_{\zeta\in[c,1-c]}(z(\zeta)+a_0k(\zeta))\right)^{r-1}\,d\tau \\
&\geq \frac{RM^{r-1}}{\Gamma(\alpha)}\int_c^s (s-\tau)^{\alpha-1}a(\tau)\,d\tau, \ \ \forall\, s\in[c,1-c],
\end{aligned}$$

and then

$$\begin{aligned}
z(c) &\geq \int_0^1 c^{\beta-1}\mathcal{J}(s)\varphi_\varrho(I_{0+}^{\alpha}(a(s)f(z(s)+a_0k(s))))\,ds \\
&\geq \int_c^{1-c} c^{\beta-1}\mathcal{J}(s)\varphi_\varrho\left(\frac{RM^{r-1}}{\Gamma(\alpha)}\int_c^s (s-\tau)^{\alpha-1}a(\tau)\,d\tau\right)ds \\
&= \frac{R^{\varrho-1}Mc^{\beta-1}}{(\Gamma(\alpha))^{\varrho-1}}\int_c^{1-c}\mathcal{J}(s)\left(\int_c^s (s-\tau)^{\alpha-1}a(\tau)\,d\tau\right)^{\varrho-1}ds > 0.
\end{aligned}$$

We deduce that $\|z\| \geq z(c) > 0$. In addition, from the above inequalities we obtain

$$\begin{aligned}
&I_{0+}^{\alpha}(a(s)f(z(s)+a_0k(s))) \geq \frac{R}{\Gamma(\alpha)}\int_c^s (s-\tau)^{\alpha-1}a(\tau) \\
&\quad\times\left(\inf_{\zeta\in[c,1-c]}(z(\zeta)+a_0k(\zeta))\right)^{r-1}\,d\tau \\
&\geq \frac{Rc^{(\beta-1)(r-1)}}{\Gamma(\alpha)}\|z+a_0k\|^{r-1}\int_c^s (s-\tau)^{\alpha-1}a(\tau)\,d\tau, \ \ \forall\, s\in[c,1-c],
\end{aligned}$$

and so

$$\begin{aligned}
z(c) &\geq \int_c^{1-c} c^{\beta-1}\mathcal{J}(s)\left(\frac{Rc^{(\beta-1)(r-1)}}{\Gamma(\alpha)}\right)^{\varrho-1}\|z+a_0k\| \\
&\quad\times\left(\int_c^s (s-\tau)^{\alpha-1}a(\tau)\,d\tau\right)^{\varrho-1}ds \\
&= \frac{c^{2(\beta-1)}R^{\varrho-1}}{(\Gamma(\alpha))^{\varrho-1}}\|z+a_0k\|\int_c^{1-c}\mathcal{J}(s)\left(\int_c^s (s-\tau)^{\alpha-1}a(\tau)\,d\tau\right)^{\varrho-1}ds \\
&= 2\|z+a_0k\| \geq 2\|z\|.
\end{aligned}$$

Therefore we conclude that $\|z\| \geq z(c) \geq 2\|z\|$, which is a contradiction because $\|z\| > 0$. Then, for a_0 sufficiently large, our problem (1.1)–(1.2) has no positive solution. □

4 An Example

Let $\alpha = 1/2$, $\beta = 7/3$ ($n = 3$), $r = 5$, $\varrho = 5/4$, $p = 2$, $\gamma_1 = 1/3$, $\gamma_2 = 3/4$, $\gamma_0 = 6/5$, $H_1(t) = t/2,\ t \in [0,1]$, $H_2(t) = \{1,\ t \in [0,1/2);\ 4/3,\ t \in [1/2,1]\}$, $a(t) = 1$ for all $t \in [0,1]$, and $f(x) = \dfrac{\widetilde{a}x^{\alpha_0}}{x^{\beta_0} + \widetilde{b}}$ for $x \geq 0$, where $\widetilde{a}, \widetilde{b}, \alpha_0, \beta_0 > 0$ and $\alpha_0 > \beta_0 + 4$.

We consider the fractional differential equation

$$D_{0+}^{1/2}\left(\varphi_5\left(D_{0+}^{7/3}u(t)\right)\right) + \frac{\widetilde{a}(u(t))^{\alpha_0}}{(u(t))^{\beta_0} + \widetilde{b}} = 0, \quad t \in (0,1), \tag{4.1}$$

with the nonlocal boundary conditions

$$\begin{cases} u(0) = u'(0) = 0, \quad D_{0+}^{7/3}u(0) = 0, \\ D_{0+}^{6/5}u(1) = \dfrac{1}{2}\displaystyle\int_0^1 D_{0+}^{1/3}u(t)\,dt + \dfrac{1}{3}D_{0+}^{3/4}u\left(\dfrac{1}{2}\right) + a_0. \end{cases} \tag{4.2}$$

We obtain $\Delta \approx 0.67363158 > 0$. So the assumptions $(H1)$ and $(H2)$ are satisfied. Because $\lim\limits_{x\to\infty} \dfrac{f(x)}{x^4} = 0$, the assumption $(H4)$ is also satisfied. In addition we find

$$\begin{aligned}
g_1(t,s) &= \frac{1}{\Gamma(7/3)}\begin{cases} t^{4/3}(1-s)^{2/15} - (t-s)^{4/3}, & 0 \leq s \leq t \leq 1, \\ t^{4/3}(1-s)^{2/15}, & 0 \leq t \leq s \leq 1, \end{cases} \\
g_{21}(t,s) &= \begin{cases} t(1-s)^{2/15} - (t-s), & 0 \leq s \leq t \leq 1, \\ t(1-s)^{2/15}, & 0 \leq t \leq s \leq 1, \end{cases} \\
g_{22}(t,s) &= \frac{1}{\Gamma(19/12)}\begin{cases} t^{7/12}(1-s)^{2/15} - (t-s)^{7/12}, & 0 \leq s \leq t \leq 1, \\ t^{7/12}(1-s)^{2/15}, & 0 \leq t \leq s \leq 1, \end{cases} \\
\mathcal{G}(t,s) &= g_1(t,s) + \frac{t^{4/3}}{\Delta}\left[\frac{1}{2}\int_0^1 g_{21}(\tau,s)\,d\tau + \frac{1}{3}g_{22}\left(\frac{1}{2},s\right)\right], \ (t,s) \in [0,1]\times[0,1], \\
h(s) &= \frac{1}{\Gamma(7/3)}\left[(1-s)^{2/15} - (1-s)^{4/3}\right], \quad s \in [0,1], \\
\mathcal{J}(s) &= \begin{cases} \dfrac{1}{\Gamma(7/3)}[(1-s)^{2/15} - (1-s)^{4/3}] + \dfrac{1}{\Delta}\left\{\dfrac{1}{4}(1-s)^{2/15} - \dfrac{1}{4}(1-s)^2 \right. \\ \left. + \dfrac{1}{3\Gamma(19/12)}\left[\left(\dfrac{1}{2}\right)^{7/12}(1-s)^{2/15} - \left(\dfrac{1}{2}-s\right)^{7/12}\right]\right\}, \quad 0 \leq s < \dfrac{1}{2}, \\ \dfrac{1}{\Gamma(7/3)}[(1-s)^{2/15} - (1-s)^{4/3}] + \dfrac{1}{\Delta}\left[\dfrac{1}{4}(1-s)^{2/15} - \dfrac{1}{4}(1-s)^2 \right. \\ \left. + \dfrac{1}{3\Gamma(19/12)}\left(\dfrac{1}{2}\right)^{7/12}(1-s)^{2/15}\right], \quad \dfrac{1}{2} \leq s \leq 1. \end{cases}
\end{aligned}$$

Then we obtain $\Lambda_0 = 1$ and $L \approx 0.3389556$. We choose $c_0 = 1$, and if we select $\widetilde{a}$ and $\widetilde{b}$ satisfying the condition $\dfrac{\widetilde{a}}{\widetilde{b}+1} < \dfrac{1}{L}$, then we conclude that $f(x) \leq \dfrac{\widetilde{a}}{\widetilde{b}+1} < \dfrac{1}{L}$

for all $x \in [0,1]$. For example, if $\widetilde{b} = 1$, then for $\widetilde{a} \leq 5.9$, the above condition for f is satisfied. So, assumption $(H3)$ is also satisfied. By Theorems 3.1 and 3.2, we deduce that problem (4.1)–(4.2) has at least one positive solution for sufficiently small $a_0 > 0$, and no positive solution for sufficiently large a_0.

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