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Existence of Solutions for Implicit Impulsive Differential Systems with Coupled Nonlocal Conditions

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Dedicated to Johnny Henderson on the occasion of his 70th birthday.

Abstract

The main purpose of this paper is to establish the existence of solution for implicit impulsive differential systems with coupled nonlocal conditions. We employ a vectorial version of Krasnoselskiĭ's fixed point theorem in generalized Banach space to overcome the lack of complete continuity of the associated integral operators. Moreover, the sufficient conditions for the existence will be reduce on the subinterval in which our nonlocal conditions act. An example is presented to illustrate the efficiency of the result obtained.

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1 Introduction

The nonlocal boundary value problems of ordinary differential equations play an important role in both theory and application, and as a consequence, have attracted a great deal of interest over the years. They are often used to model various phenomena in physics, biology, chemistry. Nonlocal problems for different classes of differential equations and systems are intensively studied in the literature by a variety of methods (see for example [6, 8–11, 17, 18, 22–29, 31, 32]). Impulsive differential equations with nonlocal conditions have been studied by many authors, see [2, 11, 12, 15, 19–21] and references therein. Recently, many authors have studied existence of solution for system of impulsive differential equations by using vector versions of fixed point theorems; see, for example, [1, 3, 5, 7, 13, 14, 16]. We consider the following system of implicit impulsive differential with nonlocal conditions:

$$\begin{cases} x'(t) = g_1(t, x(t), y(t)) + h_1(t, x'(t), y'(t)), & t \in J\backslash\{t_1, \dots, t_m\}; \\ y'(t) = g_2(t, x(t), y(t)) + h_2(t, x'(t), y'(t)), & t \in J\backslash\{t_1, \dots, t_m\}; \\ \Delta x(t_k) = I_k(x(t_k)), & k = 1, 2, \dots, m; \\ \Delta y(t_k) = J_k(y(t_k)), & k = 1, 2, \dots, m; \\ x(0) = \alpha[x]; & \\ y(0) = \beta[y], & \end{cases} \tag{1.1}$$

where $J = [0, 1]$, $h_i, g_i : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions for $i = 1, 2,\ 0 < t_1 < t_2 < \cdots < t_m < 1$, $J_k, I_k \in C(\mathbb{R}, \mathbb{R}),\ k \in \{1, 2, \cdots, m\},\ \Delta x(t_k) = x(t_k^+) - x(t_k^-)$ and $\Delta y(t_k) = y(t_k^+) - y(t_k^-)$ in which $x(t_k^+)$ and $x(t_k^+)$ $(x(t_k^-), y(t_k^-))$ denote the right (left) limit of $x(t)$ and $y(t)$ at $t = t_k$, respectively. Next $\alpha,\ \beta$ are linear functionals given by Stieltjes integrals

$$\alpha[v] = \int_0^{\tilde{t}} v(s)dA(s),$$

$$\beta[v] = \int_0^{\tilde{t}} v(s)dB(s),$$

where $\tilde{t} \in (t_m, 1]$ is fixed.

In [4], Bolojan and Precup studied the existence of solutions for the following first-order implicit differential systems with nonlocal conditions

$$\begin{cases} x'(t) & = g_1(t, x(t), y(t)) + h_1(t, x'(t), y'(t)); \\ y'(t) & = g_2(t, x(t), y(t)) + h_2(t, x'(t), y'(t)); \\ x(0) & = \alpha[x]; \\ y(0) & = \beta[y], \end{cases} \tag{1.2}$$

where $h_i, g_i : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions for $i = 1, 2$, $\alpha, \beta : C(J) \to \mathbb{R}$ are continuous linear functionals with $\alpha[1] \neq 1$ and $\beta[1] \neq 1$.

The goal of this work is to carry the work in [5] over the impulsive system of equations (1.1). The paper is organized as follows: Some auxiliary results needed in this paper are gathered together in Section 2. In Section 3 we give our main existence result for solutions of the problem (1.1). Finally, in Section 4, an example is given to demonstrate the applicability of our result.

2 Preliminaries

We recall the following fundamental definitions, properties and results which will be used in the sequel.

Definition 2.1. Let X be a nonempty set. By a vector-valued metric on X we mean a map $d : X \times X \to \mathbb{R}^n$ with the following properties:

(i) $d(u,v) \geq 0$ for all $u,\ v \in X$ if $d(u,v) = 0$ then $u = v$;

(ii) $d(u,v) = d(v,u)$ for all $u,\ v \in X$;

(iii) $d(u,v) \leq d(u,w) + d(w,v)$ for all $u, v, w \in X$.

Here, if $x, y \in \mathbb{R}^n$, $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n)$, by $x \leq y$ we mean $x_i \leq y_i$ for $i = 1, 2, \cdots, n$. We call the pair (X, d) a generalized metric space with

$$d(x,y) := \begin{pmatrix} d_1(x,y) \\ \vdots \\ d_n(x,y) \end{pmatrix}.$$

Notice that d is a generalized metric space on X if and only if $d_i,\ i = 1, 2, \cdots, n$ are metrics on X. Similarly, we speak about a vector-valued norm on a linear space X, as being a mapping $\|\cdot\| : X \to \mathbb{R}^n_+$ with $\|x\| = 0$ only for $x = 0$; $\|\lambda x\| = |\lambda|\|x\|$ for $x \in X$, $\lambda \in \mathbb{R}$, and $\|x + y\| \leq \|x\| + \|y\|$ for every $x,\ y \in X$. To any vector-valued norm $\|\cdot\|$ we can associate the vector valued metric $d(x,y) := \|x - y\|$, and we say that $(X, \|\cdot\|)$ is a generalized Banach space if X is complete with respect to d.

Definition 2.2. A square matrix of nonnegative real numbers is said to be convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1. In other words, this means that all the eigenvalues of M are in the open unit disc i.e. $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $det(M - \lambda I) = 0$, where I denote the unit matrix of $\mathcal{M}_{n\times n}(\mathbb{R})$.

Theorem 2.3 (See [30])**.** *Let $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$, the following assertions are equivalent:*

(a) *M is convergent towards zero;*

(b) *$M^k \to 0$ as $k \to \infty$;*

(c) *The matrix* $(I-M)$ *is nonsingular and*

$$(I-M)^{-1} = I + M + M^2 + \cdots + M^k + \cdots ;$$

(d) *The matrix* $(I-M)$ *is nonsingular and* $(I-M)^{-1}$ *has nonnegative elements.*

Definition 2.4. Let (X,d) be a generalized metric space. An operator $N : X \to X$ is called contractive associated with the above d on X, if there exists a convergent to zero matrix M such that

$$d(T(x), T(y)) \leq Md(x,y), \text{ for all } x, y \in X.$$

Lemma 2.5 (See [4]). *If* $A \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ *is a matrix with* $\rho(A) < 1$*, then* $\rho(A+B) < 1$ *for every matrix* $B \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ *whose elements are small enough.*

Now we present the version of Krasnoselskiĭ's fixed point theorem for a sum of two operators in vector Banach space.

Theorem 2.6 (See [4,13]). *Let* $(X, \|\cdot\|)$ *be a generalized Banach space,* D *a nonempty closed bounded convex subset of* X *and* $N : D \to X$ *such that:*

(i) $N = G + H$ *with* $G : D \to X$ *completely continuous and* $H : D \to X$ *a generalized contraction, i.e. there exists a matrix* $M \in \mathcal{M}_{n\times n}(\mathbb{R}^+)$ *with* $\rho(M) < 1$*, such that*

$$\|H(x) - H(y)\| \leq M\|x - y\|, \textit{ for all } x, y \in D;$$

(ii) $G(x) + H(y) \in D$ *for all* $x, y \in D$.

Then N *has at least one fixed point in* D.

3 Main Results

Let $J_0 = [0, t_0],\ J_k = (t_k, t_{k+1}],\ k = 1, \dots, m$, and let y_k be the restriction of a function y to J_k. In order to define mild solutions for problem (1.1), consider the space

$$PC(J,\mathbb{R}) = \{y : J \to \mathbb{R} : y_k \in C(J_k,\mathbb{R}), k = 1, \cdots, m,$$
$$y(t_k^-) \text{ and } y(t_k^+) \text{ exist } k = 1, \dots, m, \text{ and } y(t_k^-) = y(t)\}.$$

We use in $PC(J,\mathbb{R}) \times PC(J,\mathbb{R})$ the norm

$$\|(x,y)\|_{PC\times PC} := (\|x\|_{PC}, \|y\|_{PC}),$$

where

$$\|v\|_{PC} := \max\{\|v\|_{[0,\tilde{t}]}, \|v\|_{[\tilde{t},1]}\}$$

and the notation $\|v\|_{[0,\tilde{t}]}$ stands for sup-norm on $[0, \tilde{t}]$:

$$\|v\|_{[0,\tilde{t}]} = \sup_{[0,\tilde{t}]} |v(t)|,$$

while $\|v\|_{[\tilde{t},1]}$ denotes Bielecki-type norm on $[\tilde{t}, 1]$:

$$\|v\|_{[\tilde{t},1]} = \sup_{[0,\tilde{t}]} |v(t)| e^{-\tau(t-\eta)},$$

Here $\eta < \tilde{t}$ and $\tau > 0$ are given numbers. As we shall see, the joint role of the parameters η (any fixed number with $\eta < \tilde{t}$) and τ(chosen large enough) is to weaken the assumptions on $g_1(t, x, y)$, $g_2(t, x, y)$ when $t \in [\tilde{t}, 1]$. Then the norms of the functional $\alpha, \beta : PC(J, \mathbb{R}) \to \mathbb{R}$ are given by

$$\|\alpha\| = \sup_{\|v\|=1} \left| \int_0^{\tilde{t}} v(s) dA(s) \right|, \quad \|\beta\| = \sup_{\|v\|=1} \left| \int_0^{\tilde{t}} v(s) dB(s) \right|.$$

In order to obtain the equivalent integral form of the problem (1.1), we denote

$$u(t) = x'(t), \quad v(t) = y'(t), \quad t \neq t_k,\ k = 1, 2, \cdots, m. \tag{3.1}$$

Integrating (3.1) from 0 to t, we have

$$x(t) = x(0) + \int_0^t u(s) ds + \sum_{0<t_k<t} I_k(x(t_k)),$$

and

$$y(t) = y(0) + \int_0^t v(s) ds + \sum_{0<t_k<t} J_k(y(t_k)).$$

The conditions $x(0) = \alpha[x]$ and $y(0) = \beta[x]$ give

$$x(0) = \alpha \left[\sum_{0<t_k<.} I_k(x(t_k)) + \int_0^{.} u(s) ds \right] + \alpha[x(0)]$$

and

$$y(0) = \beta \left[\sum_{0<t_k<.} J_k(y(t_k)) + \int_0^{.} v(s) ds \right] + \alpha[y(0)].$$

Hence

$$x(0) = \alpha \left[\sum_{0<t_k<.} I_k(x(t_k)) + \int_0^{.} u(s) ds \right] + \alpha[1] x(0)$$

and

$$y(0) = \alpha \left[\sum_{0<t_k<\cdot} J_k(y(t_k)) + \int_0^{\cdot} u(s)ds \right] + \alpha[1]y(0).$$

Therefore

$$x(t) = (1-\alpha[1])^{-1}\alpha \left[\sum_{0<t_k<\cdot} I_k(x(t_k)) + \int_0^{\cdot} u(s)ds \right] + \int_0^t u(s)ds + \sum_{0<t_k<t} I_k(x(t_k))$$

and

$$y(t) = (1-\beta[1])^{-1}\beta \left[\sum_{0<t_k<\cdot} J_k(y(t_k)) + \int_0^{\cdot} v(s)ds \right] + \int_0^t v(s)ds + \sum_{0<t_k<t} J_k(y(t_k)).$$

On the other hand, we have that

$$\begin{aligned}
x(t_1) &= (1-\alpha[1])^{-1}\alpha \left[\int_0^{t_1} u(s)ds \right] + \int_0^{t_1} u(s)ds = (1-\alpha[1])^{-1} \int_0^{t_1} u(s)ds \\
x(t_2) &= (1-\alpha[1])^{-1} \left(I_1(x(t_1)) + \int_0^{t_2} u(s)ds \right) \\
x(t_3) &= (1-\alpha[1])^{-1} \left(I_1(x(t_1)) + I_2(x(t_2)) + \int_0^{t_3} u(s)ds \right) \\
&. \\
&. \\
&. \\
x(t_k) &= (1-\alpha[1])^{-1} \left(\sum_{0<t_i<t_k} I_i(x(t_i)) + \int_0^{t_k} u(s)ds \right).
\end{aligned}$$

Similarly we have

$$y(t_k) = (1-\beta[1])^{-1} \left(\sum_{0<t_i<t_k} J_i(y(t_i)) + \int_0^{t_k} v(s)ds \right)$$

with

$$y(t_1) = (1-\beta[1])^{-1} \int_0^{t_1} v(s)ds.$$

Let

$$G_1(u,v)(t) = g_1(t, (1-\alpha[1])^{-1}\alpha[h_1] + h_1(t), (1-\beta[1])^{-1}\beta[h_2] + h_2(t))$$

and

$$G_2(u,v)(t) = g_2(t, (1-\alpha[1])^{-1}\alpha[h_1] + h_1(t), (1-\beta[1])^{-1}\beta[h_2] + h_2(t))$$

where

$$h_1(t) = \sum_{0<t_k<t} I_k\left((1-\alpha[1])^{-1}\left(\sum_{0<t_i<t_k} I_i(x(t_i)) + \int_0^{t_k} u(s)ds\right)\right) + \int_0^t u(s)ds$$

and

$$h_2(t) = \sum_{0<t_k<t} J_k\left((1-\beta[1])^{-1}\left(\sum_{0<t_i<t_k} J_i(y(t_i)) + \int_0^{t_k} v(s)ds\right)\right) + \int_0^t v(s)ds.$$

Also, we define

$$H_1(u,v)(t) = h_1(t, u(t), v(t))$$

and

$$H_2(u,v)(t) = h_2(t, u(t), v(t)).$$

Then the problem (1.1) is equivalent to the system

$$\begin{cases} u = G_1(u,v) + H_1(u,v), \\ v = G_2(u,v) + H_2(u,v). \end{cases} \tag{3.2}$$

Consider the operator

$$T : PC(J,\mathbb{R}) \times PC(J,\mathbb{R}) \quad \to \quad PC(J,\mathbb{R}) \times PC(J,\mathbb{R})$$

defined by

$$T(x,y) = (T_1(x,y), T_2(x,y)), \;\; (x,y) \in PC \times PC \tag{3.3}$$

where

$$\begin{cases} T_1(u,v) = G_1(u,v) + H_1(u,v), \\ T_2(u,v) = G_2(u,v) + H_2(u,v). \end{cases}$$

Therefore, the system (3.2) can be regarded as a fixed point problem for the operator T.

We will consider the following assumptions:

(H_1) g_1 and g_2 are jointly continuous functions, there exists nonnegative coefficients a_i b_i c_i A_i B_i C_i such that:

$$|g_1(t,u,v)| \leq \begin{cases} a_1|u| + b_1|v| + c_1, & \text{if } t \in [0,\tilde{t}]; \\ A_1|u| + B_1|v| + C_1, & \text{if } t \in [\tilde{t},1]; \end{cases}$$

$$|g_2(t,u,v)| \leq \begin{cases} a_2|u| + b_2|v| + c_2, & \text{if } t \in [0,\tilde{t}]; \\ A_2|u| + B_2|v| + C_2, & \text{if } t \in [\tilde{t},1], \end{cases}$$

for all $u, v \in \mathbb{R}$.

(H_2) h_1, h_2 satisfy the Lipschitz conditions

$$|h_1(t,u,v) - h_1(t,\bar{u},\bar{v})| \leq \bar{a}_1|u-\bar{u}| + \bar{a}_2|v-\bar{v}|$$

and

$$|h_2(t,u,v) - h_2(t,\bar{u},\bar{v})| \leq \bar{b}_1|u-\bar{u}| + \bar{b}_2|v-\bar{v}|,$$

for all (u,v),$(\bar{u},\bar{v}) \in \mathbb{R}^2$ and $t \in J$. Here for $i = 1,2$, $\bar{a}_i,\ \bar{b}_i$ are nonnegative numbers.

(H_3) There exist d_k, $\bar{d}_k$,D_k and $\bar{D}_k \in \mathbb{R}^+$ such that for every $v \in \mathbb{R}$ we have

$$|I_k(v)| \leq d_k|u| + \bar{d}_k,\ k = 1,2 \tag{3.4}$$

and

$$|J_k(v)| \leq D_k|u| + \bar{D}_k,\ k = 1,2. \tag{3.5}$$

Define square matrices

$$\tilde{M} = \begin{pmatrix} a_1\tilde{t}A_\alpha\bar{A} & b_1\tilde{t}B_\beta\bar{B} \\ a_2\tilde{t}A_\alpha\bar{A} & b_2\tilde{t}B_\beta\bar{B} \end{pmatrix}, \quad \bar{M} = \begin{pmatrix} \bar{a}_1 & \bar{a}_2 \\ \bar{b}_1 & \bar{b}_2 \end{pmatrix}, \text{ and } M_1 = \begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix},$$

where

$$\bar{A} = 1 + |1-\alpha[1]|^{-1}\left(d_1 + d_2 + d_2d_1|1-\alpha[1]|^{-1}\right),$$
$$\bar{B} = 1 + |1-\beta[1]|^{-1}\left(D_1 + D_2 + D_2D_1|1-\beta[1]|^{-1}\right)$$

and

$$A_\alpha = 1 + |1-\alpha[1]|^{-1}\|\alpha\|, \quad B_\beta = 1 + |1-\beta[1]|^{-1}\|\beta\|.$$

Now we define

$$D = \{(u,v) \in PC(J,\mathbb{R}) \times PC(J,\mathbb{R}) : \ \|(u,v)\|_{PC\times PC} \leq R\},$$

with $R = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$, $R_1 \geq 0$, $R_2 \geq 0$ and $R \geq (I - \bar{M} - \tilde{M} - \frac{1}{\tau}M_1)^{-1}(P+K)$,
where

$$P = \|H(0,0)\|_{PC\times PC}, \ \ K = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}$$

with

$$H(0,0) = \begin{bmatrix} H_1(0,0) \\ H_2(0,0) \end{bmatrix}, \ \ K_1 = c_1 + a_1\tilde{C} + b_1\bar{C} \ \text{ and } K_2 = c_2 + a_2\tilde{C} + b_2\bar{C}$$

where

$$\bar{C} = \bar{d}_1 + \bar{d}_2 + \bar{d}_1d_2|1-\alpha[1]|^{-1}, \quad \tilde{C} = \bar{D}_1 + \bar{D}_2 + \bar{D}_1D_2|1-\beta[1]|^{-1}.$$

It is obvious that D is a nonempty, bounded, closed and convex subset of $PC(J,\mathbb{R}) \times PC(J,\mathbb{R})$.

Theorem 3.1. *Suppose that the hypotheses* (H_1)–(H_3) *hold. If the spectral radius of the matrix* $\bar{M}+\tilde{M}$ *is strictly less than one, then the problem* (1.1) *has a least one solution.*

Proof. Clear that we can transform the problem (1.1) into a fixed point problem, of the operator T defined in (3.3). We shall show that T satisfies all the assumptions of Theorem 2.6.

Step 1

Firstly, we show that H is a generalized contraction mapping. Indeed, Indeed, for all $(u,v),\ (\bar{u},\bar{v})\in PC(J,\mathbb{R})\times PC(J,\mathbb{R})$ using (H_3) and (H_3) , for $t\in[0,\tilde{t}]$, we have

$$\begin{aligned}|H_1(u,v)(t)-H_1(\bar{u},\bar{v})(t)| &= |h_1(t,u(t),v(t))-h_1(t,\bar{u}(t),\bar{v}(t))|\\ &\leq \bar{a}_1|u(t)-\bar{u}(t)|+\bar{b}_1|v(t)-\bar{v}(t)|\\ &\leq \bar{a}_1\|u-\bar{u}\|_{[0,\tilde{t}]}+\bar{b}_1\|v-\bar{v}\|_{[0,\tilde{t}]}.\end{aligned}$$

Taking supremum, we obtain that

$$\|H_1(u,v)-H_1(\bar{u},\bar{v})\|_{[0,\tilde{t}]}\leq \bar{a}_1\|u-\bar{u}\|_{[0,\tilde{t}]}+\bar{b}_1\|v-\bar{v}\|_{[0,\tilde{t}]}. \tag{3.6}$$

For $t\in[\tilde{t},1]$, we obtain

$$\begin{aligned}|H_1(u,v)(t)-H_1(\bar{u},\bar{v})(t)| &\leq \bar{a}_1|u(t)-\bar{u}(t)|+\bar{b}_1|v(t)-\bar{v}(t)|\\ &= \bar{a}_1|u(t)-\bar{u}(t)|e^{-\tau(t-\eta)}e^{\tau(t-\eta)}\\ &\quad +\bar{b}_1|v(t)-\bar{v}(t)|e^{-\tau(t-\eta)}e^{\tau(t-\eta)}\\ &\leq \bar{a}_1e^{\tau(t-\eta)}\|u-\bar{u}\|_{[\tilde{t},1]}+\bar{b}_1e^{\tau(t-\eta)}\|v-\bar{v}\|_{[\tilde{t},1]}.\end{aligned}$$

Dividing by $e^{\tau(t-\eta)}$ and taking supremum when $t\in[\tilde{t},1]$, we obtain

$$\|H_1(u,v)-H_1(\bar{u},\bar{v})\|_{[\tilde{t},1]}\leq \bar{a}_1\|u-\bar{u}\|_{[\tilde{t},1]}+\bar{b}_1\|v-\bar{v}\|_{[\tilde{t},1]}. \tag{3.7}$$

The inequalities (3.6) and (3.7) will imply that

$$\|H_1(u,v)-H_1(\bar{u},\bar{v})\|_{PC}\leq \bar{a}_1\|u-\bar{u}\|_{PC}+\bar{b}_1\|v-\bar{v}\|_{PC}. \tag{3.8}$$

Similarly, we obtain

$$\|H_2(u,v)-H_2(\bar{u},\bar{v})\|_{PC}\leq \bar{a}_2\|u-\bar{u}\|_{PC}+\bar{b}_2\|v-\bar{v}\|_{PC}. \tag{3.9}$$

Using the (3.8) and (3.9), we get

$$\|H(U)-H(\bar{U})\|_{PC\times PC}\leq \bar{M}\|U-\bar{U}\|_{PC\times PC}. \tag{3.10}$$

for $U=(u,v),\ \bar{U}(\bar{u},\bar{v})$, according to $\rho(\tilde{M}+\bar{M})<1$ and $\bar{M}<\tilde{M}+\bar{M}$, we have $\rho(\bar{M})<1$. Hence H is generalized contraction.

Step 2

G is continuous. Let (u_n, v_n) be a sequence such that $(u_n, v_n) \to (u, v)$ in $PC(J, \mathbb{R}) \times PC(J, \mathbb{R})$, then for each $t \in [0, \tilde{t}]$

$$\begin{aligned}|G_1(u_n, v_n)(t) - G_1(u, v)(t)| \leq \Big\| g_1\Big(\cdot, \frac{\alpha[h_{1,n}]}{1-\alpha[1]} + h_{1,n}(\cdot), \frac{\beta[h_{2,n}]}{1-\beta[1]} + h_{2,n}(\cdot)\Big) \\ - g_1\Big(\cdot, \frac{\alpha[h_1]}{1-\alpha[1]} + h_1(\cdot), \frac{\beta[h_2]}{1-\beta[1]} + h_2(\cdot)\Big)\Big\|_{[0,\tilde{t}]}.\end{aligned}$$

Observe that

$$\begin{aligned}|h_{1,n}(t) - h_1(t)| \leq \sum_{k=1}^{2} \Bigg| I_k\left(\frac{\sum\limits_{0<t_i<t_k} I_i(x_n(t_i)) + \int_0^{t_k} u_n(s)ds}{1-\alpha[1]}\right) \\ - I_k\left(\frac{\sum\limits_{0<t_i<t_k} I_i(x(t_i)) + \int_0^{t_k} u(s)ds}{1-\alpha[1]}\right)\Bigg| + \tilde{t}\|u_n - u\|_{[0,\tilde{t}]}.\end{aligned}$$

Similarly, we have that

$$\begin{aligned}|h_{2,n}(t) - h_2(t)| \leq \sum_{k=1}^{2} \Bigg| J_k\left(\frac{\sum\limits_{0<t_i<t_k} J_i(y_n(t_i)) + \int_0^{t_k} v_n(s)ds}{1-\alpha[1]}\right) \\ - J_k\left(\frac{\sum\limits_{0<t_i<t_k} J_i(y(t_i)) + \int_0^{t_k} v(s)ds}{1-\alpha[1]}\right)\Bigg| + \tilde{t}\|v_n - v\|_{[0,\tilde{t}]}.\end{aligned}$$

Hence

$$h_{i,n} \to h_i \text{ as } n \to \infty, \quad i = 1, 2.$$

So, we obtain

$$\|G_1(u_n, v_n) - G_1(u, v)\|_{[0,\tilde{t}]} \to 0 \quad \text{as } n \to \infty.$$

For $[\tilde{t}, 1]$, and any $\tau > 0$, we have

$$\begin{aligned}&e^{-\tau(t-\eta)}\,|G_1(u_n, v_n)(t) - G_1(u, v)(t)| \\ &\leq \left\| g_1\left(\cdot, \frac{\alpha[h_{1,n}]}{1-\alpha[1]} + h_{1,n}(\cdot), \frac{\beta[h_{2,n}]}{1-\alpha[1]} + h_{2,n}(\cdot)\right) \right. \\ &\quad \left. - g_1\left(\cdot, \frac{\alpha[h_1]}{1-\alpha[1]} + h_1(\cdot), \frac{\beta[h_2]}{1-\alpha[1]} + h_2(\cdot)\right)\right\|_{[\tilde{t},1]}.\end{aligned}$$

Note that

$$|h_{1,n}(t) - h_1(t)| \le \sum_{k=1}^{2} \left| I_k\left(\frac{\sum_{0<t_i<t_k} I_i(x_n(t_i)) + \int_0^{t_k} u_n(s)ds}{1-\alpha[1]}\right)\right.$$
$$\left. - I_k\left(\frac{\sum_{0<t_i<t_k} I_i(x(t_i)) + \int_0^{t_k} u(s)ds}{1-\alpha[1]}\right)\right| + \tilde{t}\|u_n - u\|_{[0,\tilde{t}]} + \frac{e^{\tau(1-\eta)}}{\tau}\|u_n - u\|_{[\tilde{t},1]}.$$

Similarly, we have that

$$|h_{2,n}(t) - h_2(t)| \le \sum_{k=1}^{2} \left| J_k\left(\frac{\sum_{0<t_i<t_k} J_i(y_n(t_i)) + \int_0^{t_k} v_n(s)ds}{1-\alpha[1]}\right)\right.$$
$$\left. - J_k\left(\frac{\sum_{0<t_i<t_k} J_i(y(t_i)) + \int_0^{t_k} v(s)ds}{1-\alpha[1]}\right)\right| + \tilde{t}\|v_n - v\|_{[0,\tilde{t}]} + \frac{e^{\tau(1-\eta)}}{\tau}\|v_n - v\|_{[\tilde{t},1]}.$$

Then

$$\|G_1(u_n, v_n) - G_1(u, v)\|_{[\tilde{t},1]} \to 0 \text{ as } n \to \infty.$$

Furthermore:

$$\|G_1(u_n, v_n) - G_1(u, v)\|_{PC} \to 0 \text{ as } n \to \infty.$$

Similarly, we can obtain

$$\|G_2(u_n, v_n) - G_2(u, v)\|_{PC} \to 0 \text{ as } n \to \infty.$$

Step 3

G maps bounded sets into bounded sets in D. It suffices to show that for any $r > 0$ there exists a positive constant l such that for each $(u, v) \in D$, we have

$$\|G(u, v)\|_{PC\times PC} \le l = (l_1, l_2).$$

For each $t \in [0, \tilde{t}]$, we find that

$$\begin{aligned}
&|G_1(u, v)(t)| = \left|g_1(t, (1-\alpha[1])^{-1}\alpha[h_1] + h_1(t), (1-\beta[1])^{-1}\beta[h_2] + h_2(t))\right| \\
&\le a_1 \left|(1-\alpha[1])^{-1}\alpha[h_1] + h_1(t)\right| + b_1 \left|(1-\beta[1])^{-1}\beta[h_2] + h_2(t))\right| + c_1 \\
&\le a_1 \left|1-\alpha[1]\right|^{-1} \|\alpha\|\|h_1\|_{[0,\tilde{t}]} + a_1\|h_1\|_{[0,\tilde{t}]} \\
&+ b_1 \left|1-\beta[1]\right|^{-1} \|\beta\|\|h_2\|_{[0,\tilde{t}]} + \|h_2\|_{[0,\tilde{t}]} + c_1.
\end{aligned}$$

Then

$$\begin{aligned} |G_1(u,v)(t)| &\le a_1\left(1+|1-\alpha[1]|^{-1}\|\alpha\|\right)\|h_1\|_{[0,\tilde{t}]} \\ &\quad +b_1\left(1+|1-\beta[1]|^{-1}\|\beta\|\right)\|h_2\|_{[0,\tilde{t}]}+c_1. \end{aligned} \tag{3.11}$$

Note that

$$\begin{aligned} |h_1(t)| &= \left|\sum_{0<t_k<t} I_k\left((1-\alpha[1])^{-1}\left(\sum_{0<t_i<t_k} I_i(x(t_i))+\int_0^{t_k} u(s)ds\right)\right)+\int_0^t u(s)ds\right| \\ &\le \sum_{k=1}^{2}\left|I_k\left((1-\alpha[1])^{-1}\left(\sum_{0<t_i<t_k} I_i(x(t_i))+\int_0^{t_k} u(s)ds\right)\right)\right|+\int_0^{\tilde{t}}|u(s)|ds \\ &\le d_1\left|(1-\alpha[1])^{-1}\int_0^{t_1} u(s)ds\right| \\ &\quad +d_2\left|(1-\alpha[1])^{-1}\left(I_1(x(t_1))+\int_0^{t_2} u(s)ds\right)\right|+\bar{d}_1+\bar{d}_2+\tilde{t}\|u\|_{[0,\tilde{t}]} \\ &\le d_1|1-\alpha[1]|^{-1}\int_0^{t_1}|u(s)|ds \\ &\quad +d_2|1-\alpha[1]|^{-1}\left(|I_1(x(t_1))|+\int_0^{t_2}|u(s)|ds\right)+\bar{d}_1+\bar{d}_2+\tilde{t}\|u\|_{[0,\tilde{t}]} \\ &\le \tilde{t}d_1|1-\alpha[1]|^{-1}\|u\|_{[0,\tilde{t}]}+d_2|1-\alpha[1]|^{-1}\left(d_1|x(t_1)|+\bar{d}_1+\tilde{t}\|u\|_{[0,\tilde{t}]}\right) \\ &\quad +\tilde{t}\|u\|_{[0,\tilde{t}]}+\bar{d}_1+\bar{d}_2 \\ &\le \tilde{t}d_1|1-\alpha[1]|^{-1}\|u\|_{[0,\tilde{t}]}+\tilde{t}\|u\|_{[0,\tilde{t}]}+\bar{d}_1+\bar{d}_2 \\ &\quad +|1-\alpha[1]|^{-1}d_2\left(d_1|1-\alpha[1]|^{-1}\int_0^{t_1}|u(s)|ds+\bar{d}_1+\tilde{t}\|u\|_{[0,\tilde{t}]}\right) \\ &\le \tilde{t}\left(1+|1-\alpha[1]|^{-1}\left(d_1+d_2+d_2d_1|1-\alpha[1]|^{-1}\right)\right)\|u\|_{[0,\tilde{t}]} \\ &\quad +\bar{d}_1+\bar{d}_2+\bar{d}_1d_2|1-\alpha[1]|^{-1}. \end{aligned}$$

This implies that

$$\|h_1\|_{[0,\tilde{t}]} \le \tilde{t}\bar{A}\|u\|_{[0,\tilde{t}]}+\bar{C}. \tag{3.12}$$

Similarly, we have that

$$\|h_2\|_{[0,\tilde{t}]} \le \tilde{t}\bar{B}\|v\|_{[0,\tilde{t}]}+\tilde{C}. \tag{3.13}$$

Now, using (3.12) and (3.13), we obtain

$$\|G_1(u,v)\|_{[0,\tilde{t}]} \le a_1A_\alpha\bar{A}\tilde{t}\|u\|_{[0,\tilde{t}]}+a_1\bar{C}+b_1B_\beta\bar{B}\tilde{t}\|v\|_{[0,\tilde{t}]}+b_1\tilde{C}+c_1. \tag{3.14}$$

On the other hand, for any $(u, v) \in PC(J, \mathbb{R}) \times PC(J, \mathbb{R})$, $t \in [\tilde{t}, 1]$ and $\tau > 0$, we obtain that

$$\begin{aligned} |G_1(u,v)(t)| &\leq A_1 \left|1-\alpha[1]\right|^{-1} \|\alpha\| \|h_1\|_{[0,\tilde{t}]} + A_1|h_1(t)| \\ &+ B_1 \left|1-\beta[1]\right|^{-1} \|\beta\| \|h_2\|_{[0,\tilde{t}]} + B_1|h_2(t)| + C_1. \end{aligned} \tag{3.15}$$

Clearly,

$$\begin{aligned} |h_1(t)| &\leq \tilde{t}\bar{A}\|u\|_{[0,\tilde{t}]} + \bar{C} + \int_{\tilde{t}}^{t} |u(s)|ds \\ &:= \tilde{t}\bar{A}\|u\|_{[0,\tilde{t}]} + \bar{C} + \int_{\tilde{t}}^{t} e^{\tau(s-\eta)}e^{-\tau(s-\eta)}|u(s)|ds. \end{aligned}$$

Then

$$|h_1(t)| \leq \tilde{t}\bar{A}\|u\|_{[0,\tilde{t}]} + \bar{C} + \frac{e^{\tau(t-\eta)}}{\tau}\|u\|_{[\tilde{t},1]}. \tag{3.16}$$

Similarly, we have that

$$|h_2(t)| \leq \tilde{t}\bar{B}\|v\|_{[0,\tilde{t}]} + \tilde{C} + \frac{e^{\tau(t-\eta)}}{\tau}\|v\|_{[\tilde{t},1]}. \tag{3.17}$$

Substituting (3.12), (3.13), (3.16) and (3.17) to (3.15), we have

$$\begin{aligned} |G_1(u,v)(t)| &\leq A_1\bar{A}A_\alpha\tilde{t}\|u\|_{[0,\tilde{t}]} + B_1\bar{B}B_\beta\tilde{t}\|v\|_{[0,\tilde{t}]} + \frac{A_1 e^{\tau(t-\eta)}}{\tau}\|u\|_{[\tilde{t},1]} \\ &+ \frac{B_1 e^{\tau(t-\eta)}}{\tau}\|v\|_{[\tilde{t},1]} + C_1 + A_1\tilde{C} + B_1\bar{C}. \end{aligned}$$

Dividing by $e^{\tau(t-\eta)}$ and taking the supremum when $t \in [\tilde{t}, 1]$, we obtain

$$\begin{aligned} \|G_1(u,v)\|_{[\tilde{t},1]} &\leq \frac{A_1}{\tau}\|u\|_{[\tilde{t},1]} + \frac{B_1}{\tau}\|v\|_{[\tilde{t},1]} \\ &+ \left(A_1\bar{A}A_\alpha\tilde{t}\|u\|_{[0,\tilde{t}]} + A_1\tilde{C} + B_1\bar{B}B_\beta\tilde{t}\|v\|_{[0,\tilde{t}]} + B_1\bar{C} + C_1 \right) e^{-\tau(\tilde{t}-\eta)}. \end{aligned}$$

Now we can take advantage from the special choice of the norm $\|\cdot\|_{[0,\tilde{t}]}$, more exactly from the choice of $\eta < \tilde{t}$, to assume (choosing large enough $\tau > 0$) that

$$A_1 e^{-\tau(\tilde{t}-\eta)} \leq a_1, \quad B_1 e^{-\tau(\tilde{t}-\eta)} \leq b_1, \quad C_1 e^{-\tau(\tilde{t}-\eta)} \leq c_1.$$

By deduction, we have

$$\begin{aligned} \|G_1(u,v)\|_{[\tilde{t},1]} \leq\ & a_1\tilde{t}\bar{A}A_\alpha\|u\|_{[0,\tilde{t}]} + b_1\tilde{t}\bar{B}B_\beta\|v\|_{[0,\tilde{t}]} \\ & + c_1 + a_1\tilde{C} + b_1\bar{C} \\ & + \frac{B_1}{\tau}\|v\|_{[\tilde{t},1]} + \frac{A_1}{\tau}\|u\|_{[\tilde{t},1]}. \end{aligned} \tag{3.18}$$

Now (3.14) and (3.18) imply that

$$\begin{aligned}\|G_1(u,v)\|_{PC} \leq & \left(a_1\tilde{t}\bar{A}A_\alpha + \frac{A_1}{\tau}\right)\|u\|_{PC} \\ & + \left(b_1\tilde{t}B_\beta\bar{B} + \frac{B_1}{\tau}\right)\|v\|_{PC} \\ & + c_1 + a_1\tilde{C} + b_1\bar{C} := l_1.\end{aligned} \tag{3.19}$$

Similarly

$$\begin{aligned}\|G_2(u,v)\|_{PC} \leq & \left(a_2\tilde{t}A_\alpha\bar{A} + \frac{A_2}{\tau}\right)\|u\|_{PC} \\ & + \left(b_2\tilde{t}B_\beta\bar{B} + \frac{B_2}{\tau}\right)\|v\|_{PC} \\ & + c_2 + a_2\tilde{C} + b_2\bar{C} := l_2.\end{aligned} \tag{3.20}$$

Step 4

G maps bounded sets into equicontinuous sets of $PC(J,\mathbb{R}) \times PC(J,\mathbb{R})$. Let D be a bounded set. There exists $r \geq 0$ such that for any $(u,v) \in D$, we have

$$\|u\|_{PC} \leq r \quad \text{and} \quad \|v\|_{PC} \leq r.$$

Let $r_1, r_2 \in [0,\tilde{t}]$, $r_1 < r_2$ and $(u,v) \in D$, thus we have

$$|G_1(u,v)(r_2) - G_1(u,v)(r_1)| = |L_1(r_2) - L_1(r_1)|$$

where

$$L_1(r_2) = g_1\left(r_2, \frac{\alpha[h_1]}{1-\alpha[1]} + h_1(r_2), \frac{\beta[h_2]}{1-\beta[1]} + h_2(r_2)\right)$$

and

$$L_1(r_1) = g_1\left(r_1, \frac{\alpha[h_1]}{1-\alpha[1]} + h_1(r_1), \frac{\beta[h_2]}{1-\beta[1]} + h_2(r_1)\right).$$

Observe that

$$\begin{aligned}& |h_1(r_2) - h_1(r_1)| \\ \leq & \sum_{r_1<t_k<r_2}\left|I_k\left((1-\alpha[1])^{-1}\left(I_1(x(t_1)) + \int_0^{t_k} u(s)ds\right)\right)\right| + \int_{r_2}^{r_1}|u(s)|ds \\ \leq & \sum_{r_1<t_k<r_2} d_k\left|(1-\alpha[1])^{-1}\left(I_1(x(t_1)) + \int_0^{t_k} u(s)ds\right)\right| \\ & + \sum_{r_1<t_k<r_2}\bar{d}_k + (r_2-r_1)\|u\|_{[0,\tilde{t}]} \\ \leq & |1-\alpha[1]|^{-1}\sum_{r_1<t_k<r_2} d_k\left(d_1|x(t_1)| + \bar{d}_1 + \tilde{t}\|u\|_{[0,\tilde{t}]}\right)\end{aligned}$$

$$
\begin{aligned}
&+\sum_{r_1<t_k<r_2}\bar{d}_k+(r_2-r_1)\|u\|_{[0,\tilde{t}]}\\
\leq\ &|1-\alpha[1]|^{-1}\tilde{t}r\sum_{r_1<t_k<r_2}d_k\left(|1-\alpha[1]|^{-1}d_1+1\right)\\
&+\sum_{r_1<t_k<r_2}\bar{d}_k+\bar{d}_1|1-\alpha[1]|^{-1}\sum_{r_1<t_k<r_2}d_k+(r_2-r_1)r.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
|h_2(r_2)-h_2(r_1)|\leq |1-\alpha[1]|^{-1}\tilde{t}r&\sum_{r_1<t_k<r_2}D_k\left(|1-\alpha[1]|^{-1}D_1+1\right)\\
&+\sum_{r_1<t_k<r_2}\bar{D}_k+\bar{D}_1|1-\alpha[1]|^{-1}\sum_{r_1<t_k<r_2}D_k+(r_2-r_1)r.
\end{aligned}
$$

Then

$$|h_1(r_2)-h_1(r_1)|\to 0 \text{ as } r_2\to r_1$$

and

$$|h_2(r_2)-h_2(r_1)|\to \text{ as } r_2\to r_1.$$

This follows from the continuity of g_1, g_2, we have

$$|G_1(u,v)(r_2)-G_1(u,v)(r_1)|\to 0 \text{ as } r_2\to r_1 \text{ for } t\in[0,\tilde{t}].$$

Similarly,

$$|G_2(u,v)(r_2)-G_2(u,v)(r_1)|\to 0 \text{ as } r_2\to r_1 \text{ for } t\in[0,\tilde{t}].$$

Secondly, for $r_1,\ r_2\in[\tilde{t},1]$, $r_1<r_2$ and $(u,v)\in D$, we can obtain

$$
\begin{aligned}
&\left|e^{-\tau(r_2-\eta)}G_1(u,v)(r_2)-e^{-\tau(r_1-\eta)}G_1(u,v)(r_1)\right|\\
=\ &\left|e^{-\tau(r_2-\eta)}g_1\left(r_2,\frac{\alpha[h_1]}{1-\alpha[1]}+h_1(r_2),\frac{\beta[h_2]}{1-\beta[1]}+h_2(r_2)\right)\right.\\
&\left.-e^{-\tau(r_1-\eta)}g_1\left(r_1,\frac{\alpha[h_1]}{1-\alpha[1]}+h_1(r_1),\frac{\beta[h_2]}{1-\beta[1]}+h_2(r_1)\right)\right|.
\end{aligned}
$$

Note that

$$
\begin{aligned}
|h_1(r_2)-h_1(r_1)|\leq |1-\alpha[1]|^{-1}\frac{e^{\tau(r_2-\eta)}}{\tau}r&\sum_{r_1<t_k<r_2}d_k\left(|1-\alpha[1]|^{-1}d_1+1\right)\\
&+\sum_{r_1<t_k<r_2}\bar{d}_k+\bar{d}_1|1-\alpha[1]|^{-1}\sum_{r_1<t_k<r_2}d_k+\frac{e^{\tau(r_2-\eta)}-e^{\tau(r_1-\eta)}}{\tau}r.
\end{aligned}
$$

Similarly, we have

$$|h_2(r_2) - h_2(r_1)| \leq |1-\alpha[1]|^{-1}\frac{e^{\tau(r_2-\eta)}}{\tau} r \sum_{r_1<t_k<r_2} D_k \left(|1-\alpha[1]|^{-1}D_1 + 1\right)$$
$$+ \sum_{r_1<t_k<r_2} \bar{D}_k + \bar{D}_1|1-\alpha[1]|^{-1} \sum_{r_1<t_k<r_2} D_k + \frac{e^{\tau(r_2-\eta)} - e^{\tau(r_1-\eta)}}{\tau} r.$$

This implies that

$$|h_1(r_2) - h_1(r_1)| \to 0 \text{ as } r_2 \to r_1$$

and

$$|h_2(r_2) - h_2(r_1)| \to 0 \text{ as } r_2 \to r_1.$$

Since g_1 and g_2 are continuous functions, then we get

$$\left|e^{-\tau(r_2-\eta)}G_1(u,v)(r_2) - e^{-\tau(r_1-\eta)}G_1(u,v)(r_1)\right| \to 0 \text{ as } r_2 \to r_1 \text{ for } t \in [\tilde{t}, 1].$$

Similarly,we have

$$\left|e^{-\tau(r_2-\eta)}G_2(u,v)(r_2) - e^{-\tau(r_1-\eta)}G_2(u,v)(r_1)\right| \to 0 \text{ as } r_2 \to r_1 \text{ for } t \in [\tilde{t}, 1].$$

So by Steps 2–4, we deduce that G is completely continuous.

Step 5

We look for nonempty, bounded, closed and convex subset D of $PC(J,\mathbb{R}) \times PC(J,\mathbb{R})$ such that

$$G(D) + H(D) \subseteq D.$$

The inequalities (3.19) and (3.20) imply that

$$\begin{bmatrix} \|G_1(u,v)\|_{PC} \\ \\ \|G_2(u,v)\|_{PC} \end{bmatrix} \leq \left(\tilde{M} + \frac{1}{\tau}M_1\right) \begin{bmatrix} \|u\|_{PC} \\ \\ \|v\|_{PC} \end{bmatrix} + \begin{bmatrix} K_1 \\ \\ K_2 \end{bmatrix},$$

where

$$K_1 = c_1 + a_1\tilde{C} + b_1\bar{C}$$

and

$$K_2 = c_2 + a_2\tilde{C} + b_1\bar{C}.$$

Using the vector-valued norm, equivalently,

$$\|G(u,v)\|_{PC\times PC} \leq \left(\tilde{M} + \frac{1}{\tau}M_1\right) \|(u,v)\|_{PC\times PC} + K, \tag{3.21}$$

where

$$K = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}.$$

On the other hand, it follows from (3.10) we get

$$\|H(u,v)\|_{PC\times PC} \le \bar{M}\|(u,v)\|_{PC\times PC} + P, \quad (u,v) \in PC \times PC, \tag{3.22}$$

where

$$P = \|H(0,0)\|_{PC\times PC}.$$

Now we look for $R = (R_1, R_2) \in \mathbb{R}_+^2$ such that

$$\|H(u,v) + G(u,v)\|_{PC\times PC} \le R \text{ for } (u,v) \in PC(J,\mathbb{R}) \times PC(J,\mathbb{R})$$

with $\|(u,v)\|_{PC\times PC} \le R$. To this end, due to (3.21) and (3.22), it is sufficient that

$$\left(I - \bar{M} - \tilde{M} - \frac{1}{\tau}M_1\right) R + P + K \le R.$$

Or equivalently

$$P + K \le \left(I - \bar{M} - \tilde{M} - \frac{1}{\tau}M_1\right) R. \tag{3.23}$$

Since $\rho(\bar{M} + \tilde{M}) < 1$ and the entries of $\frac{1}{\tau}$ are as small as desired for $\tau > 0$ large enough, according to Lemma 2.5, we can choose τ such that

$$\rho\left(\bar{M} + \tilde{M} + \frac{1}{\tau}M_1\right) < 1.$$

Then, $I - \bar{M} - \frac{1}{\tau}M_1 - \tilde{M}$ is invertible and its inverse $\left(I - \bar{M} - \frac{1}{\tau}M_1 - \tilde{M}\right)^{-1}$ is a nonnegative matrix, (3.23) is equivalent to

$$R \ge \left(I - \bar{M} - \tilde{M} - \frac{1}{\tau}M_1\right)^{-1} (P + K). \tag{3.24}$$

Therefore,

$$G(D) + F(D) \subseteq D.$$

Thus problem (1.1) has at least one solution. □

4 An Example

Example 4.1. Consider the problem

$$\begin{cases} x' = \frac{1}{10}(x + \sin y) + f_1(t) + \frac{1}{2}y'\left[1 + e^{-\frac{4}{5}(x'-1)}\right]^{-1}, & t \in J,\ t \neq \frac{1}{3}; \\ y' = \cos\left(\frac{x+y}{4}\right) + f_2(t) + \frac{1}{10}x'\left[1 + e^{-\frac{2}{5}(y'-1)}\right]^{-1},\ t \neq \frac{1}{3}; \\ \Delta x(\frac{1}{3}) = \frac{1}{6}\sin\left(x\left(\frac{1}{3}\right)\right), \\ \Delta x(\frac{1}{3}) = \frac{1}{5}\cos\left(y\left(\frac{1}{3}\right)\right), \\ x(0) = \int_0^{\frac{1}{2}} x(s)ds,\ \ y(0) = \int_0^{\frac{1}{2}} y(s)ds. \end{cases} \tag{4.1}$$

Here $f_1,\ f_2 \in C(J, \mathbb{R})$. This problem can be regarded as the form (1.1). In this case.

$$g_1(t, u, v) = \frac{1}{10}(u + \sin v) + f_1(t),$$

$$g_2(t, u, v) = \cos\left(\frac{u+v}{4}\right) + f_2(t),$$

$$h_1(t, u, v) = \frac{1}{2}v\left[1 + e^{-\frac{4}{5}(u-1)}\right]^{-1},$$

$$h_2(t, u, v) = \frac{1}{10}u\left[1 + e^{-\frac{2}{5}(v-1)}\right]^{-1}$$

$$I_1(u) = \frac{1}{6}\sin\left(x\left(\frac{1}{3}\right)\right),$$

$$J_1(v) = \frac{1}{5}\cos\left(y\left(\frac{1}{3}\right)\right).$$

We have $\tilde{t} = \frac{1}{2}$, we have that

$$\alpha[1] = \beta[1] = \|\alpha\| = \|\beta\| = \frac{1}{2}.$$

Consequently, $A_\alpha = B_\beta = 2$. For any $u, v \in \mathbb{R}$ and $t \in J$:

$$|g_1(t, u, v)| \leq \frac{1}{10}|u| + \frac{1}{10}|v| + |f_1(t)|,$$

$$|g_2(t, u, v)| = \frac{1}{4}|u| + \frac{1}{4}|v| + |f_2(t)|.$$

Hence condition (H_1) is satisfied with

$$\begin{aligned} a_1 = A_1 = \frac{1}{10}, &\quad c_1 = \|f_1\|_{\left[0,\frac{1}{2}\right]}, \\ b_1 = B_1 = \frac{1}{10}, &\quad C_1 = \|f_1\|_{\left[\frac{1}{2},1\right]}, \\ a_2 = A_2 = \frac{1}{4}, &\quad c_2 = \|f_2\|_{\left[0,\frac{1}{2}\right]}, \\ b_2 = B_2 = \frac{1}{4}, &\quad C_2 = \|f_2\|_{\left[\frac{1}{2},1\right]}. \end{aligned}$$

For any $u, v, \bar{u}, \bar{v} \in \mathbb{R}$ and $t \in J$

$$|h_1(t,u,v) - h_1(t,\bar{u},\bar{v})| \le \frac{1}{10}|u-\bar{u}| + \frac{1}{2}|v-\bar{v}|,$$

and

$$|h_2(t,u,v) - h_2(t,\bar{u},\bar{v})| \le \frac{1}{10}|u-\bar{u}| + \frac{1}{10}|v-\bar{v}|.$$

Hence condition (H_3) satisfy with $\bar{a}_1 = \bar{a}_2 = \bar{b}_2 = \dfrac{1}{10}$ and $\bar{b}_1 = \dfrac{1}{2}$, consequently

$$\bar{M} = \begin{pmatrix} \frac{1}{10} & \frac{1}{2} \\ \frac{1}{10} & \frac{1}{10}. \end{pmatrix}.$$

We have for each $u, v \in \mathbb{R}$.

$$|I_1(u)| \le \frac{1}{6}|u| + 1,$$

$$|J_1(u)| \le \frac{1}{5}|u| + 1.$$

Thus condition (H_5) satisfied with

$$d_1 = \frac{1}{6},\ D_1 = \frac{1}{5},\ \bar{d}_1 = \bar{D}_1 = 1.$$

Then we have that

$$\bar{A} = 1 + d_1|1-\alpha[1]|^{-1} = 1 + \left(\frac{1}{6}\right) \times 2 = \frac{4}{3}$$

$$\bar{B} = 1 + d_1|1 - \alpha[1]|^{-1} = 1 + \left(\frac{1}{5}\right) \times 2 = \frac{7}{5}.$$

For this example

$$\tilde{M} = \begin{pmatrix} \frac{2}{15} & \frac{7}{50} \\ \frac{1}{3} & \frac{7}{20} \end{pmatrix}.$$

Then

$$\tilde{M} + \bar{M} = \begin{pmatrix} \frac{7}{30} & \frac{16}{25} \\ \frac{13}{30} & \frac{9}{20} \end{pmatrix},$$

which is convergent to zero because its eigenvalues are $\lambda_1 = 0,88 < 1$, $|\lambda_2| = 0,2 < 1$. From Theorem (3.1), the problem (4.1) has at least one solution.

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References

[1] H. Abdeli, J.R. Graef, H. Kadari, A. Ouahab and A. Oumansour, Existence of solutions to systems of second-order impulsive differential equation with integral boundary condition on the half-line. *Dyn. Contin. Discrete Impuls. Syst., Ser. A, Math. Anal.*, to appear.

[2] M. Benchohra, E.P. Gatsori, L. Górniewicz and S. K. Ntouyas, Defined evolution impulsive differential equations with nonlocal conditions. *Fixed Point Theory*, **4**, (2003), 185–204.

[3] A. Baliki, J. J. Nieto, A. Ouahab and M. L. Sinacer, Random semilinear system of differential equations with impulses. *Fixed Point Theory Appl.* (2017), Paper No. 27, 29 pp.

[4] O. Bolojan and R. Precup, Implicit first order differential systems with nonlocal conditions. *Electron. J. Qual. Theory Differ. Equ.*, **69**, 2014, 1–13.

[5] O. Bolojan-Nica, G. Infante and P. Pietramala, Existence results for impulsive systems with initial nonlocal conditions. *Math. Model. Anal.*, **18**, (2013), 599–611.

[6] A. Boucherif, Nonlocal Cauchy problems for first-order multivalued differential equations. *Electron. J. Differential Equations*, **47**, (2002), 9p.

[7] H. Berrezoug, J. Henderson and A. Ouahab, Existence and uniqueness of solutions for a system of impulsive differential equations on the half-line. *J. Nonl. Funct. Anal.*, Vol.2017(2017), Article ID **38**, 1–16.

[8] A. Boucherif and R. Precup, On the nonlocal initial value problem for first order differential equations. *Fixed Point Theory*, **4**, (2003), 205–212.

[9] K. Deng, Exponential decay of solutions of semilinear parabolic equations with nonlocal initial conditions. *J. Math. Anal. Appl.* **179** (1993), 630–637.

[10] A. M. A. El-Sayed, E. M. Hamdallah and Kh. W. Elkadeky, Internal nonlocal and integral condition problems of the differential equation $x' = f(t, x, x')$. *J. Nonlinear Sci. Appl.* **4**, (2003), 193–199.

[11] M. Feng, B. Du and W. Ge, Impulsive boundary value problems with integral boundary conditions and one-dimensional p-Laplacian. *Nonlinear Anal.*, **70**, (2009), 3119–3126.

[12] M. Feng and D. Xie, Multiple positive solutions of multi-point boundary value problem for second-order impulsive differential equations. *J. Comput. Appl. Math.* **223**, (2009), 438–448.

[13] J. R. Graef, J. Henderson, and A. Ouahab, *Topological Methods for Differential Equations and Inclusions*. Monographs and Research Notes in Mathematics Series Profile. Boca Raton, FL: CRC Press, 2019.

[14] J. R. Graef, H. Kadari, A. Ouahab and A. Oumansour, Existence results for systems of second-order impulsive differential equations. *Acta Math. Univ. Comenian. (N.S.)* **88** (2019), 51–66.

[15] J. R. Graef and A. Ouahab, Some existence results for impulsive dynamic equations on time scales with integral boundary conditions. *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.,* **13B** (2006), 11–24.

[16] C. Guendouz, J. E. Lazreg, J. J. Nieto and A. Ouahab, Existence and compactness results for a system of fractional differential equations. *J. Funct. Spaces* 2020, Art. ID 5735140, 12 pp.

[17] G. Infante, F.M. Minhós and P. Pietramala, Non-negative solutions of systems of ODEs with coupled boundary conditions. *Commun. Nonlinear Sci. Numer. Simul.* **17**, (2012), 4952–4960.

[18] G. Infante and P. Pietramala, Eigenvalues and non-negative solutions of a system with nonlocal BCs. *Nonlinear Stud.*, **16**, (2009), 187–196.

[19] G. Infante and P. Pietramala, Nonlocal impulsive boundary value problems with solutions that change sign. *In Mathematical Models in Engineering, Biology and Medicine, volume 1124 of AIP Conf. Proc.* pp. 205–213. *Amer. Inst. Phys., Melville,* NY, 2009.

[20] T. Jankowski, Positive solutions for second order impulsive differential equations involving Stieltjes integral conditions. *Nonlinear Anal.*, **74**, (2011), 3775–3785.

[21] T. Jankowski, Positive solutions to second order four-point boundary value problems for impulsive differential equations. *Appl. Math. Comput.* **202** (2008), 550–561.

[22] G. L. Karakostas and P. Ch. Tsamatos, Existence of multiple positive solutions for a nonlocal boundary value problem. *Topol. Methods Nonlinear Anal.* **19**, (2002), 109-121.

[23] R. Ma. A survey on nonlocal boundary value problems. *Appl. Math. E-Notes,* **7**, (2007), 257–279.

[24] O. Nica, Nonlocal initial value problems for first order differential systems. *Fixed Point Theory*, **13** (2012), 603–612.

[25] O. Nica, Existence results for second order three-point boundary value problems, *Differ. Equ. Appl.* **4**, (2012), 24 pp.

[26] O. Nica, Initial-value problems for first-order differential systems with general nonlocal conditions. *Electron. J. Differential Equations*, **74**, (2012), 15 pp.

[27] O. Nica and R. Precup, On the nonlocal initial value problem for first order differential systems. *Stud. Univ. Babeş-Bolyai Math.* **56**, (2011), 113–125.

[28] S.K. Ntouyas. Nonlocal initial and boundary value problems: a survey. In Handbook of Differential Equations: *Ordinary Differential Equations. Vol. II*, pp. 461-557. Elsevier B. V., Amsterdam, 2005.

[29] R. Precup and D. Trif, Multiple positive solutions of non-local initial value problems for first order differential systems. *Nonlinear Anal.*, **75**, (2012), 5961–5970.

[30] R.S. Varga, *Matrix Iterative Analysis*, 2nd revised and expanded, Springer Series in Computational Mathematics. Springer, Berlin, 2000.

[31] J.R.L. Webb and G. Infante, Positive solutions of nonlocal boundary value problems: a unified approach. *J. London Math. Soc.* (2), **74**, (2006), 673–693.

[32] J.R.L. Webb and G. Infante, Nonlocal boundary value problems of arbitrary order. *J. Lond. Math. Soc.,* **79**, (2009), 238–258.