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A Polar Representation of the Hilger Complex Plane

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Abstract

In this short paper, we examine a polar representation of Hilger's complex plane and show how this can be used to simplify the calculation of inversion integrals for Laplace transforms on time scales. This polar form is also useful in analyzing the regions of convergence of Laplace transforms on certain time scales.

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1 Introduction

Over the last two decades, there has been a rapid development of time scales theory as originally developed by Hilger in his 1988 dissertation [10] and subsequent seminal

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paper [9]. Time scale versions of integral transforms that operationalize the time scales calculus in the same ways that their classical continuous and discrete counterparts on $\mathbb{R}$ and $h\mathbb{Z}$ do and have been of particular interest. For example, Hilger [11] originally defined the time scale Fourier transform as

$$\mathcal{F}(f)(\omega) = \int_{-\infty}^{\infty} e_{\overset{o}{\imath}\omega}(t,0)f(t)\Delta t = \int_{-\infty}^{\infty} e^{i\omega t} f(t)\Delta t.$$

Marks et al. [14] investigated properties of this transform including the inversion integral and convolution. In contrast, Hilger also defined a version of the time scales Laplace transform as

$$\mathcal{L}\{f\}(z) = \int_0^{\infty} e_{\ominus z}(t,0)f(t)\Delta t.$$

Hilger's definition of these two transforms unified known results on $\mathbb{R}$ and $h\mathbb{Z}$, but they lacked the extension aspect of operationalizing the time scales calculus on other time scales. To fill this gap, Bohner and Peterson [5, 6] showed that defining the (unilateral) Laplace transform as

$$\mathcal{L}\{f\}(z) = \int_0^{\infty} e_{\ominus z}(\sigma(t),0)f(t)\Delta t \tag{1.1}$$

creates a transform in which derivatives and integrals are transformed into multiplication by powers of z. Davis et al. [7, 8], Ahrendt [1, 2], Bohner et al. [3, 4], and Jackson and Davis [13] have all examined both unilateral and bilateral versions of (1.1) and their inversions in their corresponding regions of convergence (ROCs).

To begin, recall the following facts regarding the Hilger complex plane [5]. The *Hilger real part* of $z \in \mathbb{C}$ is defined as

$$\mathrm{Re}_\mu(z) := \lim_{\tau\to\mu} \frac{|1+\tau z| - 1}{\tau},$$

while the *Hilger imaginary part* of z is given by

$$\mathrm{Im}_\mu(z) := \lim_{\tau\to\mu} \frac{\mathrm{Arg}(z\tau+1)}{\tau},$$

where the argument is the principal argument satisfying $-\pi < \mathrm{Arg}(z) \le \pi$. The purely imaginary Hilger number is given as

$$\overset{o}{\imath}\,\omega := \lim_{\tau\to\mu} \frac{e^{i\omega\tau} - 1}{\tau}.$$

Under the circle plus operation $\oplus$ defined by $a \oplus b = a + b + \mu ab$, every $z \neq -\dfrac{1}{\mu}$ can be written as

$$z = \mathrm{Re}_\mu(z) \oplus \overset{o}{\imath}\,\mathrm{Im}_\mu(z).$$

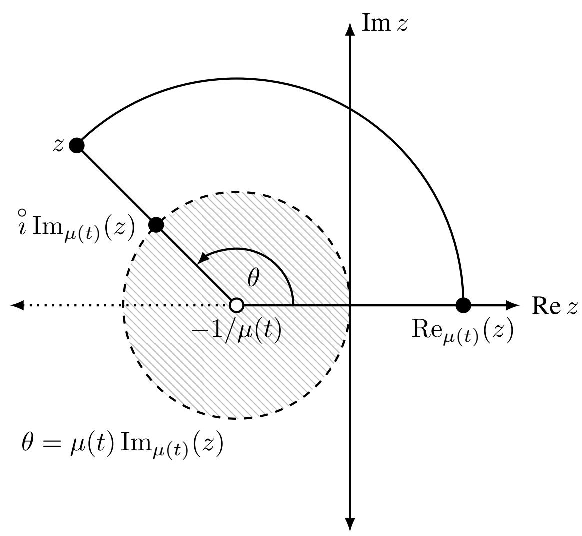

Figure 1.1: Hilger's Complex Plane. z inside the circle have negative Hilger real part, while points on the circle have zero Hilger real part. z outside the circle have positive Hilger real part.

Hilger's decomposition is shown in Figure 1.1.

The sign of the Hilger real part of z is determined by its relation to the *Hilger imaginary disk*. Points on the circle have zero Hilger real part, while points interior have negative Hilger real part and points exterior have positive Hilger real part. As the time scale function $e_z(t,0)$ is the continuous exponential e^{zt} on $\mathbb{R}$ and $(1+z)^t$ on $\mathbb{Z}$, for these two homogeneous time scales where μ is constant, a necessary and sufficient condition for stability is having the Hilger real part of remain negative for all $t \in \mathbb{T}$. In general, however, Pötzsche, Siegmund, and Wirth [15] show that the Hilger real part need only be negative "on average". Analogously, in [13], Jackson and Davis show that in terms of inverse Laplace transforms, the boundary of the ROC need not consist of points for which $\mathrm{Re}_m(z)$ equals c for all time as occurs on $h\mathbb{Z}$ and $\mathbb{R}$, but rather just equal to c "on average" in some sense. They denote the collection of such c by $\overline{\mathrm{Re}_m(z)} = c$.

2 A Polar Representation

Let $t \in \mathbb{T}$ be fixed. Consider the locus of points $\Gamma := \{z \in \mathbb{C} : \mathrm{Re}_\mu(z) = R\}$. For $z \in \Gamma$, this leads to the following polar form of z as shown in Figure 2.1:

$$z = -\frac{1}{\mu} + \left(R + \frac{1}{\mu}\right) e^{i\omega\mu}, \quad -\pi/\mu \le \omega \le \pi/\mu. \tag{2.1}$$

Note that $z = x + iy$ where $x = -\dfrac{1}{\mu} + \left(R + \dfrac{1}{\mu}\right)\cos(\omega\mu)$ and $y = \left(R + \dfrac{1}{\mu}\right)\sin(\omega\mu)$.

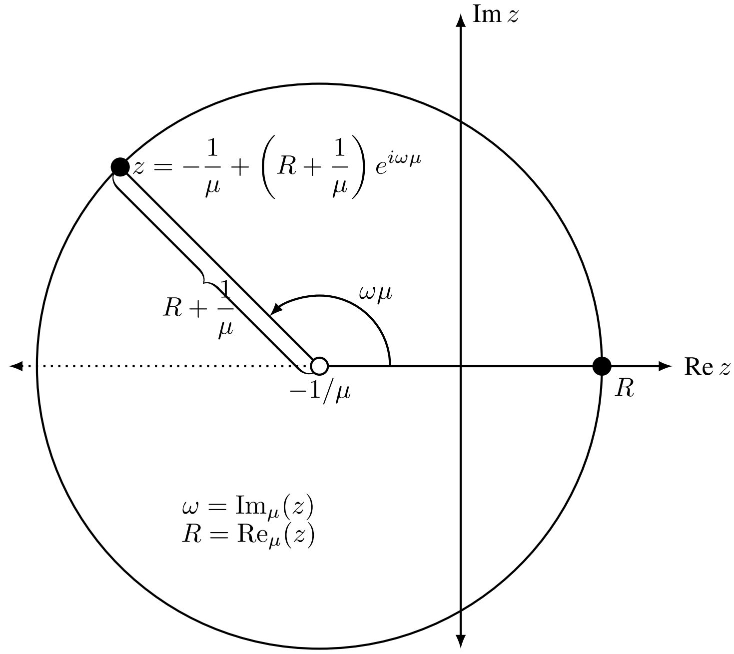

Figure 2.1: The generalized time scale polar form for the discrete case. Notice the transformation is determined associating each point in the complex plane with its Hilger real part R and a translated form of the Hilger imaginary part given by $\mu\omega$.

A surprisingly powerful consequence of this polar form is how it can simplify certain calculations involving time scale exponential functions in a particularly useful way. For example,

$$\begin{aligned} e_z(\sigma(t),0) &= (1+\mu z)e_z(t,0) \\ &= \left[1+\mu\left(-\frac{1}{\mu}+\left(R+\frac{1}{\mu}\right)e^{i\omega\mu}\right)\right]e_z(t,0) \\ &= (1+\mu R)e^{i\omega\mu}e_z(t,0). \end{aligned} \tag{2.2}$$

To appreciate the subtle utility of (2.2), consider the Laplace transform, $F(z)$, of $f(t)$ given by (1.1). In [13], Jackson and Davis show that the inverse Laplace transform of $F(z)$ can be computed via

$$\mathcal{L}^{-1}\{F(z)\}(t) = \frac{1}{2\pi i}\int_{\overline{\mathrm{Re}_m(z)=c}} e_z(t,0)F(z)\,dz, \tag{2.3}$$

where the orientation here is taken in the clockwise direction. It may seem surprising that the inverse Laplace transformation does not have a shift in its argument even though the Laplace transformation does. In what follows, we will verify (2.3) with a novel argument involving the polar form for discrete $\mathbb{T}$. Along the way, we will show that by converting the line integral to a real integral, that a shift does indeed appear in the argument, via (2.2). We begin by explicitly writing the integrand in its polar form, which we make explicit by denoting $z \equiv z_p$:

$$\frac{1}{2\pi i}\int_{\overline{\mathrm{Re}_m(z)=c}} e_z(t,0)F(z)\,dz = \frac{1}{2\pi i}\int_{\overline{\mathrm{Re}_m(z_p)=c}} e_{z_p}(t,0)F(z_p)\,dz_p,$$

where

$$z_p = -\frac{1}{\mu} + \left(R + \frac{1}{\mu}\right)e^{i\omega\mu}.$$

Now, $dz_p = i(1+\mu R)e^{i\omega\mu}\,d\omega$, so the contour integral becomes the real integral

$$\frac{1}{2\pi i}\int_{-\pi/\mu}^{\pi/\mu} e_{z_p}(t,0)\int_0^\infty e_{\ominus z_p}(\sigma(x),0)f(x)\,\Delta x\, i(1+\mu R)e^{i\omega\mu}\,d\omega$$

and applying Fubini's theorem we have

$$\frac{1}{2\pi i}\int_0^\infty f(x)\int_{-\pi/\mu}^{\pi/\mu} e_{z_p}(t,0)e_{\ominus z_p}(\sigma(x),0)i(1+\mu R)e^{i\omega\mu}\,d\omega\,\Delta x. \tag{2.4}$$

The Jacobian gives us the exact multiplier needed to apply (2.2) and introduce the shift. Hence, (2.4) becomes

$$\frac{1}{2\pi}\int_0^\infty f(x)\int_{-\pi/\mu}^{\pi/\mu} e_{z_p}(\sigma(t),0)e_{\ominus z_p}(\sigma(x),0)\,d\omega\,\Delta x,$$

which can be rewritten as

$$\frac{1}{2\pi}\int_0^\infty f(x)\int_{-\pi/\mu}^{\pi/\mu} e_{\ominus z_p}(\sigma(x),\sigma(t))\,d\omega\,\Delta x. \tag{2.5}$$

Now consider two cases: first, if $x = t$, then $e_{\ominus z_p}(\sigma(x),\sigma(t)) \equiv 1$ and thus the inner integral in (2.5) reduces to

$$\int_{-\pi/\mu}^{\pi/\mu} d\omega = 2\pi/\mu.$$

On the other hand, if $x \neq t$, then converting (2.4) back to the line integral form, we obtain

$$\frac{1}{2\pi i}\int_0^\infty f(x)\int_{\overline{\mathrm{Re}_m(z)=c}} e_z(t,0)e_{\ominus z}(\sigma(x),0)\,dz\,\Delta x$$
$$= \frac{1}{2\pi i}\int_0^\infty f(x)\int_{\overline{\mathrm{Re}_m(z)=c}} e_z(t,\sigma(x))\,dz\,\Delta x. \tag{2.6}$$

However,

$$\int_{\overline{\mathrm{Re}_m(z)=c}} e_z(t,\sigma(x))\,dz = 0$$

since $e_z(t,\sigma(x))$ is analytic on the interior of $\overline{\mathrm{Re}_m(z)} = c$ as well as its boundary, and thus the inversion integral in this case vanishes.

Together, these two cases show that

$$\begin{aligned}\frac{1}{2\pi i}\int_{\overline{\mathrm{Re}_m(z)=c}} e_z(t,0)F(z)\,dz &= \frac{1}{2\pi}\int_0^{\infty} f(x)\int_{-\pi/\mu}^{\pi/\mu} e_{\ominus z_p}(\sigma(x),\sigma(t))\,d\omega\,\Delta x\\ &= \frac{1}{2\pi}\int_0^{\infty}\frac{2\pi}{\mu(t)} f(x)\delta(x-t)\,\Delta x = f(t),\end{aligned}$$

as desired. Note then that the preceding computations show that we may use the complex-valued complex line integral in (2.3) to invert the transform or we may use the complex-valued real integral given by

$$\mathcal{L}^{-1}\{F(z)\}(t) = \frac{1}{2\pi}\int_{-\pi/\mu}^{\pi/\mu} e_{z_p}(\sigma(t),0)F(z_p)\,d\omega. \tag{2.7}$$

This fact is advantageous computationally since for different choices of F, there will be times when the line integral is more efficient to compute and other times when the real integral is easier to compute.

Example 2.1. Consider the time scale $\mathbb{P}_{a,b}$. In [13], it is shown that the boundary of the region of convergence for the Laplace transform on this $\mathbb{T}$ is given by

$$f(x,y) = e^{2a(-x+c)}\cdot\frac{(1+bc)^2}{(1+bx)^2+(1+by)^2} = 1,$$

where a and b are the parameters in the $\mathbb{P}_{ab}$ time scale and c is the exponential order of the function being transformed.

To see how the polar form (2.1) aids in the computation of the ROC, let $z = x+iy$ where

$$x = -\frac{1}{b}+\left(R+\frac{1}{b}\right)\cos(\omega b). \qquad y = \left(R+\frac{1}{b}\right)\sin(\omega b), \qquad R = \frac{|1+bz|-1}{b} \tag{2.8}$$

Since the boundary of the ROC occurs when $f(x,y)=1$, this occurs when

$$(x+1/b)^2+y^2 = 1/b^2(1+bc)^2e^{2a(-x+c)} = (R+1/b)^2.$$

Solving the latter equation for x, we have

$$x = c-\frac{1}{a}\ln\left(\frac{1+bR}{1+bc}\right), \tag{2.9}$$

Setting this equal to the polar form for x in (2.8), we obtain a parameterization for R in terms of ω:

$$R = \frac{b\sec(b\omega)W_0\left(\pm\sqrt{\frac{a^2(bc+1)^2e^{\frac{2a(bc+1)}{b}}\cos^2(b\omega)}{b^2}}\right) - a}{ab},$$

$$R = \frac{b\sec(b\omega)W_{-1}\left(-\sqrt{\frac{a^2(bc+1)^2e^{\frac{2a(bc+1)}{b}}\cos^2(b\omega)}{b^2}}\right) - a}{ab},$$

where W_0 and W_{-1} are branches of the Lambert W function. This explicit representation allows us the freedom to compute the real integral in terms of ω rather than the complex line integral in the inversion integral.

On the other hand, we can also use the polar form to aid in determining when the ROC becomes disconnected. From above, the boundary of the ROC occurs when $f(x, y) = 1$, i.e., when $(x + 1/b)^2 + y^2 = (R + 1/b)^2$. Applying (2.9) and the observation in [13] that the ROC becomes disconnected precisely when $y = 0$, we conclude that disconnection occurs when

$$(R + 1/b)^2 - \left(c + \frac{1}{b} - \frac{1}{a} \cdot \ln\left(\frac{1 + bR}{1 + bc}\right)\right)^2 = 0,$$

which has solutions

$$R = \frac{-b\,W_0\left(\pm\frac{a(bc+1)e^{\frac{a}{b}+ac}}{b}\right) - a}{ab},\ \frac{-b\,W_{-1}\left(-\frac{a(bc+1)e^{\frac{a}{b}+ac}}{b}\right) - a}{ab}.$$

However, the positive argument for W_0 above reduces to $R = c$, which makes sense due to the fact that $(c, 0)$ is always a point on the boundary of the ROC for each μ. Set

$$R_1 = \frac{-b\,W_0\left(-\frac{a(bc+1)e^{\frac{a}{b}+ac}}{b}\right) - a}{ab}, \qquad R_2 = \frac{-b\,W_{-1}\left(-\frac{a(bc+1)e^{\frac{a}{b}+ac}}{b}\right) - a}{ab}.$$

If $R_1 > c > 0$, then the ROC disconnects between R_1 and R_2. We can use these to generate the conditions in terms of a, b, and c under which the ROC disconnects. Since R_1 and R_2 are real and positive when the arguments of W_0 and W_1 above are bounded below by $-1/e$, we solve

$$-\frac{a(bc+1)e^{\frac{a}{b}+ac}}{b} = -\frac{1}{e}$$

to obtain

$$c = \frac{bW_0\left(\frac{1}{e}\right) - a}{ab}.$$

Now, c must be positive, and by the last equation this occurs when $a < b\,W_0(1/e)$. Thus, the ROC is connected when $a < b\,W_0(1/e)$ and $c \geq \dfrac{-a + bW_0(1/e)}{ab}$. The ROC is disconnected when $a < b\,W_0(1/e)$ and $c < \dfrac{-a + bW_0(1/e)}{ab}$.

In a forthcoming paper, we bring these techniques to bear on inverting Fourier transforms, and the utility of the polar representation is even more pronounced.

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