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On Initial Value Problems for Caputo–Hadamard Fractional Differential Inclusions with Nonlocal Multi-Point Conditions in Banach Spaces

Nassim Guerraiche
Université Constantine 2
Département de Technologies des Logiciels
et des Systémes d'Informations
Nouvelle Ville Ali Mendjeli, BP: 67A, Constantine, Algeria
nassim.guerraiche@univ-constantine2.dz

Samira Hamani
Université de Mostaganem
Département de Mathématiques
B.P. 227, 27000, Mostaganem, Algeria
hamani_samira@yahoo.fr

Dedicated to Johnny Henderson on the occasion of his 70th birthday.

Abstract

In this paper, we are concerned to prove the existence of solutions to an initial value problem for Caputo–Hadamard fractional differential inclusion in a Banach space. For this, we use the set-valued analog of Mönch's fixed point theorem combined with the technique of measure of noncompactness. Also, we present an example to illustrate our main results.

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1 Introduction

This paper deals with the existence of solutions to initial value problems (IVP for short) with multipoint boundary conditions for fractional differential inclusions. In particular, we consider the initial value problem

$$ {}^{H}_{C}D^{r}y(t) \in F(t, y(t)), \quad \text{for a.e., } \ t \in J = [1, T], 1 < r \leq 2, \tag{1.1}$$

$$ y(1) = y_1, \qquad y'(1) - \sum_{j=1}^{m} \xi_j y(t_j) = 0, \tag{1.2}$$

where ${}^{H}_{C}D^{r}$ is the Caputo–Hadamard fractional derivative, $(E, |\cdot|)$ is a Banach space, $\mathcal{P}(E)$ is the family of all nonempty subsets of E, $F : [1, T] \times E \to \mathcal{P}(E)$ is a multivalued map, $y_1 \in E$, t_j, $j = 1, \cdots, m$, are given points such that $1 < t_1 \leq \cdots \leq t_m < T$, and $\xi_j \in E$ with $1 - \sum_{j=1}^{m} \xi_j \log t_j \neq 0$. This problem was inspired by the paper of Boucherif et al. [9], so here we present a new problem with efficient techniques to resolve it.

In recent years, the study of differential equations and inclusions supplied with a variety of boundary conditions and including fractional derivatives has known a huge development; this is due to the fact that the operators of fractional derivation are nonlocal which allowed to model many phenomena in various fields of science and engineering. Indeed, there are numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetism, etc. In the monographs of Hilfer [17], Kilbas et al. [19], Podlubny [23], and Momani et al. [21], we can find the background mathematics and various applications of fractional calculus of fractional order.

The Caputo left-sided fractional derivative of order r, is defined by

$$ ({}^{c}D^{r}_{a+}h)(t) = \frac{1}{\Gamma(n-r)} \int_a^t (t-s)^{n-r-1} h^{(n)}(s) ds, $$

where $r > 0$, $n = [r] + 1$ and $[r]$ denotes the integer part of r. This derivative is very useful in many applied problems, because it satisfies its initial data which contains $y(0)$, $y'(0)$, etc., as well as the same data for boundary conditions. The Hadamard fractional derivative was introduced by Hadamard in 1892 [15], this derivative has the properties that its kernel contains a logarithmic function of arbitrary exponent and the Hadamard derivative of a constant does not equal to 0. The Caputo–Hadamard fractional derivative given by Jarad et al. [18] is a modified Hadamard fractional derivative, but this fractional derivative kept the characteristic property of the Caputo fractional derivative which is the derivative of a constant is 0.

In this paper, we present existence results for the problem (1.1)–(1.2), when the right hand side is convex valued, by using the set-valued analog of Mönch's fixed point theorem combined with the technique of measure of noncompactness. Recently, this has proved to be a valued tool in solving fractional differential equations and inclusions in Banach spaces; for details, see the papers of Lasota et al. [20], Agarwal et al. [1], Benchohra et al. [7, 8] and Graef et al. [13].

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this paper. Let $(E, |\cdot|)$ be a Banach space, and we set $C(J,E)$ as the Banach space of all continuous functions from J into E with the norm

$$\|y\|_\infty = \sup\{|y(t)| : t \in J\},$$

and $L^1(J,E)$ as the Banach space of Bochner integrable functions $y : J \longrightarrow E$ with the norm

$$\|y\|_{L^1} = \int_J |y(t)|dt.$$

The space $AC(J,E)$ is the space of functions $y : J \to E$ that are absolutely continuous.

Let $\delta = t\dfrac{d}{dt}$, and then we set

$$AC^n_\delta(J,E) = \{y : J \longrightarrow E, \delta^{n-1}y(t) \in AC(J,E)\}.$$

Let $AC^1(J,E)$ be the space of functions $y : J \to E$ that are absolutely continuous and have an absolutely continuous first derivative. For any Banach space $(X, \|\cdot\|)$, we set

$$P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\},$$

$$P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\},$$

$$P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\},$$

$$P_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact and convex}\}.$$

For a given set V of functions $u : J \to E$, we set

$$V(t) = \{u(t) : u \in V\}, t \in J,$$

and

$$V(J) = \{u(t) : u \in V(t), t \in J\}.$$

A multivalued map $G : X \to \mathcal{P}(X)$ is convex (closed) valued if $G(X)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \cup_{x\in B}G(x)$ is bounded in X for all $B \in P_b(X)$ (i.e., $\sup_{x\in B}\{\sup\{|y| : y \in G(x)\}\} < \infty$). G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and for each open set N of X containing $G(x_0)$, there exists an open neighborhood N_0 of x_0 such that $G(N_0) \subset N$. G is said to be completely continuous if $G(B)$ is relatively compact for every $B \in P_b(X)$. If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e., $x_n \to x_*, y_n \to y_*, y_n \in G(x_n)$ imply $y_* \in G(x_*)$). G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$. A multivalued map $G : J \to P_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \to d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable.

Definition 2.1. A multivalued map $F : J \times E \to \mathcal{P}(E)$ is said to be Carathéodory if:

(1) $t \to F(t, u)$ is measurable for each $u \in E$,

(2) $u \to F(t, u)$ is upper semicontinuous for almost all $t \in J$.

For each $y \in C(J, E)$, define the set of selections of F by

$$S_{F,y} = \{v \in L^1([1, T], E) : v(t) \in F(t, y(t)) \text{ a.e. } t \in [1, T]\}.$$

Let (X, d) be a metric space induced from the normed space $(X, |\cdot|)$. The function $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{\infty\}$ given by

$$H_d(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\right\}$$

is known as the Hausdorff–Pompeiu metric. For more details on multivalued maps see the books of Aubin and Cellina [4], Aubin and Frankowska [5], Deimling [11] and Castaing and Valadier [10].

For convenience, we first recall the definitions of the Kuratowski measure of noncompactness and summarize the main properties of this measure.

Definition 2.2 (See [3,6])**.** Let E be a Banach space and let Ω_E be the bounded subsets of E. The Kuratowski measure of noncompactness is the map $\rho : \Omega_E \to [0, \infty)$ defined by

$$\rho(B) = \inf\{\epsilon > 0 \,:\, B \subset \bigcup_{j=1}^{m} B_j \text{ and } \operatorname{diam}(B_j) \leq \epsilon\}; \text{ here } B \in \Omega_E.$$

Properties: The Kuratowski measure of noncompactness satisfies the following properties (for more details see [3,6])

(1) $\rho(B) = 0 \Leftrightarrow \overline{B}$ is compact (B is relatively compact).

(2) $\rho(B) = \rho(\overline{B})$.

(3) $A \subset B \Rightarrow \rho(A) \leq \rho(B)$.

(4) $\rho(A+B) \leq \rho(A) + \rho(B)$.

(5) $\rho(cB) = |c|\rho(B), c \in \mathbb{R}$.

(6) $\rho(\text{conv}B) = \rho(B)$.

Here $\overline{B}$ and conv$B$ denote the closure and the convex hull of the bounded set B, respectively.

Theorem 2.3 (See [16]). *Let E be a Banach space and C be a countable subset of $L^1(J,E)$ such that there exists $h \in L^1(J,\mathbb{R}_+)$ with $|u(t)| \leq h(t)$ for a.e. $t \in J$ and every $u \in C$. Then, the function $\varphi(t) = \rho(C(t))$ belongs to $L^1(J,\mathbb{R}_+)$ and satisfies*

$$\rho\left(\left\{\int_1^T u(s)ds : u \in C\right\}\right) \leq 2\int_1^T \rho(C(s))ds.$$

Definition 2.4 (See [19]). The Hadamard fractional integral of order r for a function $h : [1,+\infty) \to \mathbb{R}$ is defined as

$$^H I^r h(t) = \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1}\frac{h(s)}{s}ds, \ r > 0,$$

provided the integral exists.

Definition 2.5 (See [19]). For a function h given on the interval $[1,\infty)$, the r-Hadamard fractional-order derivative of h, is defined by

$$(^H D^r h)(t) \quad = \quad \frac{1}{\Gamma(n-r)}\left(t\frac{d}{dt}\right)^n \int_1^t \left(\log\frac{t}{s}\right)^{n-r-1}\frac{h(s)}{s}ds.$$

Here $n = [r]+1$, $[r]$ denotes the integer part of r and $\log(\cdot) = \log_e(\cdot)$.

Definition 2.6 (See [18]). For a given function h which belongs to $AC^n_\delta([a,b],E)$, such that $a > 0$, we define the Caputo-type modification of the left-sided Hadamard fractional derivative by

$$^H_C D^r y(t) = {^H D^r}\left[y(s) - \sum_{k=0}^{n-1}\frac{\delta^k y(a)}{k!}\left(\log\frac{s}{a}\right)^k\right](t),$$

where $\text{Re}(r) \geq 0$ and $n = [\text{Re}(r)]+1$.

Lemma 2.7 (See [18]). *Let y belong to $AC^n_\delta([a,b],E)$ or to C^n_δ and $r \in \mathbb{C}$. Then*

$$^H I^r (^H_C D^r) y(t) = y(t) - \sum_{k=0}^{n-1}\frac{\delta^k y(a)}{k!}\left(\log\frac{t}{a}\right)^k.$$

Let us now recall Mönch's fixed point theorem.

Theorem 2.8 (See [22]). *Let K be a closed, convex subset of a Banach space E, U a relatively open subset of K, and $N : \overline{U} \to \mathcal{P}(K)$. Assume graph$N$ is closed, N maps compact sets into relatively compact sets, and for some $x_0 \in U$, the following two conditions are satisfied:*

- $M \subset \overline{U}$, $M \subset \operatorname{conv}(x_0 \cup N(M))$, $\overline{M} = \overline{U}$ *with* C *a countable subset of* M *implies* $\overline{M}$ *is compact,*
- $x \notin (1-\lambda)x_0 + \lambda N(x)$ *for all* $x \in \overline{U}/U$, $\lambda \in (0,1)$.

Then, there exists $x \in \overline{U}$ *with* $x \in N(x)$.

3 Main Results

Definition 3.1. A function $y \in AC^2_\delta([1,T],E)$ is said to be a solution of (1.1)–(1.2), if there exists a function $v \in L^1([1,T],E)$ with $v(t) \in F(t,y(t))$, for a.e., $t \in [1,T]$, such that ${}^H_C D^r y(t) = v(t)$ and the function y satisfies the conditions $y(1) = y_1$ and $y'(1) - \sum_{j=1}^{m} \xi_j y(t_j) = 0$.

Lemma 3.2. *Assume that* $1 - \sum_{j=1}^{m} \xi_j \log t_j \neq 0$. *Let* $h \in L^1([1,T],E)$. *For* $r \in (1,2]$, *a function* y *is a solution of the fractional equation*

$$\begin{aligned} y(t) \;=\; & \frac{\log t}{1-\sum_{j=1}^{m}\xi_j \log t_j}\left[\sum_{j=1}^{m}\left(y_1\xi_j + \frac{\xi_j}{\Gamma(r)}\int_1^{t_j}\left(\log\frac{t_j}{s}\right)^{r-1} h(s)\frac{ds}{s}\right)\right] \\ & + y_1 + \frac{1}{\Gamma(r)}\int_1^{t}\left(\log\frac{t}{s}\right)^{r-1} h(s)\frac{ds}{s} \end{aligned} \tag{3.1}$$

if and only if y *is a solution of the nonlinear fractional initial value problem*

$${}^H_C D^r y(t) = h(t) \;\; \text{for a.e. } t \in J = [1,T], \tag{3.2}$$

$$y(1) = y_1,\; y'(1) - \sum_{j=1}^{m} \xi_j y(t_j) = 0. \tag{3.3}$$

Proof. Applying the Hadamard fractional integral of order r to both sides of (3.2), and by using Lemma 2.7, we find

$$y(t) = c_1 + c_2 \log t + {}^H I^r h(t). \tag{3.4}$$

Then, by using the conditions (3.3), we get

$$c_1 = y_1,$$

and

$$\sum_{j=1}^{m} \xi_j y(t_j) = \sum_{j=1}^{m} y_1 \xi_j + c_2 \sum_{j=1}^{m} \xi_j \log t_j + \sum_{j=1}^{m} \frac{\xi_j}{\Gamma(r)} \int_1^{t_j} \left(\log \frac{t_j}{s}\right)^{r-1} h(s) \frac{ds}{s}.$$

Hence, we have

$$c_2 = \frac{\sum_{j=1}^{m} \left(y_1 \xi_j + \frac{\xi_j}{\Gamma(r)} \int_1^{t_j} \left(\log \frac{t_j}{s}\right)^{r-1} h(s) \frac{ds}{s} \right)}{1 - \sum_{j=1}^{m} \xi_j \log t_j}.$$

Finally, by substitution, we obtain the solution (3.1). Conversely, it is clear that if y satisfies equation (3.1), then equations (3.2) and (3.3) hold. □

In the sequel, we set

$$\Lambda = 1 - \sum_{j=1}^{m} \xi_j \log t_j,$$

with $1 - \sum_{j=1}^{m} \xi_j \log t_j \neq 0$.

Theorem 3.3. *Let $R > 0$, $B = \{x \in E : \|x\| \leq R\}$, $U = \{x \in C(J, E) : \|x\|_\infty < R\}$, and assume that:*

(H1) $F : [1, T] \times E \to \mathcal{P}_{cp,c}(E)$ *is a Carathéodory multi-valued map.*

(H2) For each $R > 0$, there exists a function $p \in L^1(J, \mathbb{R}^+)$ such that

$$\|F(t, y)\|_{\mathcal{P}} = \sup\{|v| \ : \ v(t) \in F(t, y)\} \leq p(t)$$

for each $(t, y) \in J \times E$ with $|y| \leq R$, and

$$\liminf_{R \to \infty} \frac{\int_1^T p(t)dt}{R} = \delta_1 < \infty.$$

(H3) There exists $l \in L^1(J, \mathbb{R}^+)$, with $I^\alpha l < \infty$, such that

$$H_d(F(t, y), F(t, \bar{y})) \leq l(t)|y - \bar{y}| \quad \text{for every } y, \bar{y} \in E,$$

and

$$d(0, F(t,0)) \leq l(t), \ \ a.e. \ \ t \in J.$$

(H4) There exists a Carathéodory function $\psi : J \times [1, 2R] \to \mathbb{R}_+$ *such that*

$$\rho(F(t,M)) \geq \psi(t, \rho(M)) \ \ a.e. \ t \in J \text{ and each } M \subset B.$$

(H5) The function $\varphi = 0$ *is the unique solution in* $C(J, [1, 2R])$ *of the inequality*

$$\begin{aligned} \varphi(t) \ &\leq \ \frac{2\log t}{\Lambda}\left(\sum_{j=1}^{m}\left[\frac{\xi_j}{\Gamma(r)}\int_1^{t_j}\left(\log\frac{t_j}{s}\right)^{r-1}\psi(s,\rho(M(s))\frac{ds}{s}\right]\right) \\ &\quad + \frac{2}{\Gamma(r)}\int_1^t\left(\log\frac{t}{s}\right)^{r-1}\psi(s,\rho(M(s))\frac{ds}{s} \end{aligned}$$

for $t \in J$.

Then, the IVP (1.1)–(1.2) *has at least one solution* y *in* $C(J, B)$*, provided that*

$$\delta_1 < \frac{\Gamma(r)}{(\log T)^{r-1}}\left[1 + \frac{\log T\sum_{j=1}^m \xi_j}{|\Lambda|}\right]^{-1}. \tag{3.5}$$

Proof. Transform the problem (1.1)–(1.2) into a fixed point problem. Consider the multivalued operator,

$$N(y) = \left\{ h \in C([1,T], E) : \begin{array}{rcl} h(t) &=& \frac{\log t}{\Lambda}\sum_{j=1}^{m}\frac{\xi_j}{\Gamma(r)}\int_1^{t_j}\left(\log\frac{t_j}{s}\right)^{r-1}v(s)\frac{ds}{s} \\ && +\frac{\log t}{\Lambda}\left(\sum_{j=1}^{m} y_1\xi_j\right) + y_1 \\ && +\frac{1}{\Gamma(r)}\int_1^t\left(\log\frac{t}{s}\right)^{r-1}v(s)\frac{ds}{s}, v \in S_{F,y} \end{array} \right\}.$$

Clearly, from Lemma 3.2, the fixed points of N are solutions to (1.1)–(1.2). We shall show that N satisfies the assumptions of Mönch's fixed point theorem. The proof will be given in several steps. First note $\overline{U} = C(J, B)$.

Step 1

$N(y)$ is convex for each $y \in C(J, B)$. Indeed, if h_1, h_2 belong to $N(y)$, then there exist $v_1, v_2 \in S_{F,y}$ such that for each $t \in J$ we have

$$\begin{aligned} h_i(t) \ &= \ \frac{\log t}{\Lambda}\left[\sum_{j=1}^{m}\left(y_1\xi_j + \frac{\xi_j}{\Gamma(r)}\int_1^{t_j}\left(\log\frac{t_j}{s}\right)^{r-1}v_i(s)\frac{ds}{s}\right)\right] \\ &\quad + y_1 + \frac{1}{\Gamma(r)}\int_1^t\left(\log\frac{t}{s}\right)^{r-1}v_i(s)\frac{ds}{s}, \end{aligned}$$

for $i = 1, 2$. Let $0 \le d \le 1$. Then, for each $t \in J$, we have

$$\begin{aligned}
&(dh_1 + (1-d)h_2)(t) \\
&= \frac{\log t}{\Lambda}\left[\sum_{j=1}^{m} \frac{\xi_j}{\Gamma(r)} \int_1^{t_j} \left(\log \frac{t_j}{s}\right)^{r-1} [dv_1 + (1-d)v_2](s)\frac{ds}{s}\right] \\
&+\frac{\log t}{\Lambda}\left(\sum_{j=1}^{m} y_1\xi_j\right) + y_1 \\
&+\frac{1}{\Gamma(r)} \int_1^{t} \left(\log \frac{t}{s}\right)^{r-1} [dv_1 + (1-d)v_2](s)\frac{ds}{s}.
\end{aligned}$$

Since $S_{F,y}$ is convex (because F has convex values), we have

$$dh_1 + (1-d)h_2 \in N(y).$$

Step 2

$N(M)$ *is relatively compact for each compact* $M \subset \overline{U}$. Let $M \subset \overline{U}$ be a compact set and let (h_n) by any sequence of elements of $N(M)$. We show that (h_n) has a convergent subsequence by using the Arzelà–Ascoli criterion of compactness in $C(J, B)$. Since $h_n \in N(M)$ there exist $y_n \in M$ and $v_n \in S_{F,y_n}$ such that

$$\begin{aligned}
h_n(t) &= \frac{\log t}{1 - \sum_{j=1}^{m} \xi_j \log t_j}\left[\sum_{j=1}^{m}\left(y_1\xi_j + \frac{\xi_j}{\Gamma(r)} \int_1^{t_j} \left(\log \frac{t_j}{s}\right)^{r-1} v_n(s)\frac{ds}{s}\right)\right] \\
&+y_1 + \frac{1}{\Gamma(r)} \int_1^{t} \left(\log \frac{t}{s}\right)^{r-1} v_n(s)\frac{ds}{s}
\end{aligned}$$

for $n \ge 1$. Using Theorem 2.3 and the properties of the measure of noncompactness of Kuratowski, we have

$$\begin{aligned}
\rho(\{h_n(t)\}) &\le \frac{2\log t}{\Lambda}\left[\sum_{j=1}^{m} \frac{\xi_j}{\Gamma(r)} \int_1^{t_j} \rho\left(\left\{\left(\log \frac{t_j}{s}\right)^{r-1} v_n(s)\frac{ds}{s} : n \ge 1\right\}\right)\right] \\
&+\frac{2}{\Gamma(r)} \int_1^{t} \rho\left(\left\{\left(\log \frac{t}{s}\right)^{r-1} v_n(s)\frac{ds}{s} : n \ge 1\right\}\right).
\end{aligned} \tag{3.6}$$

On the other hand, since $M(s)$ is compact in E, the set $\{v_n(s) : n \ge 1\}$ is compact. Consequently, $\rho(v_n(s) : n \ge 1) = 0$ for a.e. $s \in J$. Furthermore,

$$\rho\left(\left\{\left(\log \frac{t}{s}\right)^{r-1} v_n(s)\frac{ds}{s}\right\}\right) = \left(\log \frac{t}{s}\right)^{r-1} \frac{1}{s}\rho(\{v_n(s) : n \ge 1\}) = 0,$$

and

$$\rho\left(\left\{\left(\log\frac{t_j}{s}\right)^{r-1}v_n(s)\frac{ds}{s}\right\}\right)=\left(\log\frac{t_j}{s}\right)^{r-1}\frac{1}{s}\rho(\{v_n(s):n\geq 1\})=0,$$

for a.e. $t,s\in J$. Now (3.6) implies that $\{h_n(t):n\geq 1\}$ is relatively compact in B, for each $t\in J$. In addition, for each t_1 and t_2 from J, $t_1<t_2$, we have

$$\begin{aligned}
&|h_n(t_2)-h_n(t_1)|\\
&= \Big|\frac{(\log t_2-\log t_1)}{\Lambda}\sum_{j=1}^{m}\left[y_1\xi_j+\frac{\xi_j}{\Gamma(r)}\int_1^{t_j}\left(\log\frac{t_j}{s}\right)^{r-1}v_n(s)\frac{ds}{s}\right]\\
&\quad+\frac{1}{\Gamma(r)}\int_1^{t_1}\left[\left(\log\frac{t_2}{s}\right)^{r-1}-\left(\log\frac{t_1}{s}\right)^{r-1}\right]v_n(s)\frac{ds}{s}\\
&\quad+\frac{1}{\Gamma(r)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{r-1}v_n(s)\frac{ds}{s}\Big|\\
&\leq \frac{(\log t_2-\log t_1)}{\Lambda}\sum_{j=1}^{m}\left|y_1\xi_j+\frac{\xi_j}{\Gamma(r)}\int_1^{t_j}\left(\log\frac{t_j}{s}\right)^{r-1}v_n(s)\frac{ds}{s}\right|\\
&\quad+\frac{1}{\Gamma(r)}\int_1^{t_1}p(s)\left[\left(\log\frac{t_2}{s}\right)^{r-1}-\left(\log\frac{t_2}{s}\right)^{r-1}\right]\frac{ds}{s}\\
&\quad+\frac{1}{\Gamma(r)}\int_{t_1}^{t_2}p(s)\left(\log\frac{t_2}{s}\right)^{r-1}\frac{ds}{s}.
\end{aligned}$$

As $t_1\to t_2$, the right-hand side of the above inequality tends to zero. This shows that$\{h_n:n\geq 1\}$ is equicontinuous. Consequently, $\{h_n:n\geq 1\}$ is relatively compact in $C(J,B)$.

Step 3

N has a closed graph. Let $y_n\to y_*$ $h_n\in N(y_n)$ and $h_n\to h_*$. We need to show that $h_*\in N(y_*)$. Note that $h_n\in N(y_n)$ means that there exists $v_n\in S_{F,y_n}$, such that, for each $t\in J$,

$$\begin{aligned}
h_n(t) &= \frac{\log t}{1-\sum_{j=1}^{m}\xi_j\log t_j}\left[\sum_{j=1}^{m}\left(y_1\xi_j+\frac{\xi_j}{\Gamma(r)}\int_1^{t_j}\left(\log\frac{t_j}{s}\right)^{r-1}v_n(s)\frac{ds}{s}\right)\right]\\
&\quad+y_1+\frac{1}{\Gamma(r)}\int_1^{t}\left(\log\frac{t}{s}\right)^{r-1}v_n(s)\frac{ds}{s}.
\end{aligned}$$

We must show that there exists $v_*\in S_{F,y_*}$ such that, for each $t\in J$,

$$\begin{aligned}
h_*(t) &= \frac{\log t}{1-\sum_{j=1}^{m}\xi_j\log t_j}\left[\sum_{j=1}^{m}\left(y_1\xi_j+\frac{\xi_j}{\Gamma(r)}\int_1^{t_j}\left(\log\frac{t_j}{s}\right)^{r-1}v_*(s)\frac{ds}{s}\right)\right]\\
&\quad+y_1+\frac{1}{\Gamma(r)}\int_1^{t}\left(\log\frac{t}{s}\right)^{r-1}v_*(s)\frac{ds}{s}.
\end{aligned}$$

Since $F(t,\cdot)$ is upper semi-continuous, then for every $\epsilon > 0$, there exists a natural number $n_0(\epsilon)$ such that, for every $n \geq n_0$, we have

$$v_n(t) \in F(t, y_n(t)) \subset F(t, y_*(t)) + \epsilon B(0,1), \quad \text{a.e. } t \in J.$$

Since $F(\cdot,\cdot)$ has compact values, then there exists a subsequence $v_{n_m}(\cdot)$ such that

$$v_{n_m}(\cdot) \to v_*(\cdot) \text{ as } m \to \infty,$$

and

$$v_*(t) \in F(t, y_*(t)), \text{ a.e. } t \in J.$$

For every $w \in F(t, y_*(t))$, we have

$$|v_{n_m}(t) - v_*(t)| \leq |v_{n_m}(t) - w| + |w - v_*(t)|.$$

Then,

$$|v_{n_m}(t) - v_*(t)| \leq d(v_{n_m}(t), F(t, y_*(t))).$$

We obtain an analogous relation by interchanging the roles of v_{n_m} and v_*; it follows that

$$|v_{n_m}(t) - v_*(t)| \leq H_d(F(t, y_n(t)), F(t, y_*(t))) \leq l(t)\|y_n - y_*\|_\infty.$$

Then,

$$\begin{aligned}
&|h_n(t) - h_*(t)| \\
&\leq \frac{\log t \sum_{j=1}^m \xi_j}{\Gamma(r)(1 - \sum_{j=1}^m \xi_j \log t_j)} \int_1^{t_j} \left(\log \frac{t_j}{s}\right)^{r-1} |v_n(s) - v_*(s)| \frac{ds}{s} \\
&\quad + \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} |v_n(s) - v_*(s)| \frac{ds}{s} \\
&\leq \frac{\log T \sum_{j=1}^m \xi_j}{\Gamma(r)(1 - \sum_{j=1}^m \xi_j \log t_j)} \int_1^{T} \left(\log \frac{t_j}{s}\right)^{r-1} l(s) \frac{ds}{s} \|y_{n_m} - y_*\|_\infty \\
&\quad + \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{t}{s}\right)^{r-1} l(s) \frac{ds}{s} \|y_{n_m} - y_*\|_\infty.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\|h_n - h_*\|_\infty \\
&\leq \frac{\log T \sum_{j=1}^m \xi_j}{\Gamma(r)(1 - \sum_{j=1}^m \xi_j \log t_j)} \int_1^{T} \left(\log \frac{t_j}{s}\right)^{r-1} l(s) \frac{ds}{s} \|y_{n_m} - y_*\|_\infty \\
&\quad + \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{t}{s}\right)^{r-1} l(s) \frac{ds}{s} \|y_{n_m} - y_*\|_\infty \to 0,
\end{aligned}$$

as $m \to \infty$.

Step 4

M is relatively compact in $C(J,B)$. Suppose $M\subset\overline{U}$, $M\subset \operatorname{conv}(\{0\}\cup N(M))$, and $\overline{M}=\overline{C}$ for some countable set $C\subset M$. Using an estimation similar to the one used in Step 2, we can see that $N(M)$ is equicontinuous. Then, from $M\subset \operatorname{conv}(\{0\}\cup N(M))$, we deduce that M is equicontinuous as well. To apply the Arzelà–Ascoli theorem, it remains to show that $M(t)$ is relatively compact in E for each $t\in J$. Since $C\subset M\subset \operatorname{conv}(\{0\}\cup N(M))$ and C is countable, we can find a countable set $H=\{h_n : n\geq 1\}\subset N(M)$ with $C\subset \operatorname{conv}(\{0\}\cup H)$. Then, there exist $y_n\in M$ and $v_n\in S_{F,y_n}$ such that

$$\begin{aligned} h_n(t) &= \frac{\log t}{1-\sum_{j=1}^m \xi_j \log t_j}\left[\sum_{j=1}^m\left(y_1\xi_j+\frac{\xi_j}{\Gamma(r)}\int_1^{t_j}\left(\log\frac{t_j}{s}\right)^{r-1}v_n(s)\frac{ds}{s}\right)\right] \\ &\quad +y_1+\frac{1}{\Gamma(r)}\int_1^t\left(\log\frac{t}{s}\right)^{r-1}v_n(s)\frac{ds}{s}. \end{aligned}$$

From $\overline{M}\subset\overline{C}\subset\overline{\operatorname{conv}}(\{0\}\cup H))$, and according to Theorem 2.3, we have

$$\rho(M(t))\leq \rho(\overline{C(t)}\leq \rho(H(t))=\rho(\{h_n((t):n\geq 1\}).$$

Using (3.6) and the fact that $v_n(s)\in M(s)$, we obtain

$$\begin{aligned} \rho(M(t)) &\leq \frac{2\log t}{\Lambda}\sum_{j=1}^m\left[\frac{\xi_j}{\Gamma(r)}\int_1^{t_j}\rho\left(\left\{\left(\log\frac{t_j}{s}\right)^{r-1}v_n(s)\frac{ds}{s}:n\geq 1\right\}\right)\right] \\ &\quad +\frac{2}{\Gamma(r)}\int_1^t\rho\left(\left\{\left(\log\frac{t}{s}\right)^{r-1}v_n(s)\frac{ds}{s}:n\geq 1\right\}\right) \\ &\leq \frac{2\log t}{\Lambda}\sum_{j=1}^m\left(\frac{\xi_j}{\Gamma(r)}\int_1^{t_j}\left(\log\frac{t_j}{s}\right)^{r-1}\rho(M(s))\frac{ds}{s}\right) \\ &\quad +\frac{2}{\Gamma(r)}\int_1^t\left(\log\frac{t}{s}\right)^{r-1}\rho(M(s))\frac{ds}{s} \\ &\leq \frac{2\log t}{\Lambda}\left[\sum_{j=1}^m\frac{\xi_j}{\Gamma(r)}\int_1^{t_j}\left(\log\frac{t_j}{s}\right)^{r-1}\psi(s,\rho(M(s))\frac{ds}{s}\right] \\ &\quad +\frac{2}{\Gamma(r)}\int_1^t\left(\log\frac{t}{s}\right)^{r-1}\psi(s,\rho(M(s))\frac{ds}{s}. \end{aligned}$$

Also, the function φ given by $\varphi(t)=\rho(M(t))$ belongs to $C(J,[1,2R])$. Consequently, by $(H5)$, $\varphi=0$; that is, $\rho(M(t))=0$ for all $t\in J$. Now, by the Arzelà–Ascoli theorem, M is relatively compact in $C(J,B)$.

Step 5

$N(\overline{U}) \subset \overline{U}$. If this were not the case, let $h \in N(y)$ with $y \in U$. Then, by (H2), there exists a function $v \in S_{F,y}$ and $p \in L^1(J, \mathbb{R}^+)$ such that

$$\begin{aligned} h(t) &= \frac{\log t}{1 - \sum_{j=1}^m \xi_j \log t_j} \left[\sum_{j=1}^m \left(y_1 \xi_j + \frac{\xi_j}{\Gamma(r)} \int_1^{t_j} \left(\log \frac{t_j}{s} \right)^{r-1} v(s) \frac{ds}{s} \right) \right] \\ &\quad + y_1 + \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s} \right)^{r-1} v(s) \frac{ds}{s}, \end{aligned}$$

and

$$\begin{aligned} R &\leq \|N(y)\|_{\mathcal{P}} \\ &\leq \frac{\log T}{|\Lambda|} \left(\sum_{j=1}^m |y_1 \xi_j| + \sum_{j=1}^m \frac{|\xi_j|}{\Gamma(r)} \int_1^{t_j} \left(\log \frac{t_j}{s} \right)^{r-1} |v(s)| \frac{ds}{s} \right) + |y_1| \\ &\quad + \left| \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s} \right)^{r-1} \frac{v(s)}{s} ds \right| \\ &\leq \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s} \right)^{r-1} \frac{|v(s)|}{s} ds + \frac{\log T}{|\Lambda|} \sum_{j=1}^m |y_1 \xi_j| \\ &\quad + \frac{\log T \sum_{j=1}^m |\xi_j|}{|\lambda| \Gamma(r)} \int_1^T \left(\log \frac{T}{s} \right)^{r-1} \frac{|v(s)|}{s} ds + |y_1| \\ &\leq \frac{(\log T)^{r-1}}{\Gamma(r)} \int_1^T p(s) ds + |y_1| + \frac{(\log T)^r \sum_{j=1}^m |\xi_j|}{|\Lambda| \Gamma(r)} \int_1^T p(s) ds. \end{aligned}$$

Dividing both sides by R and taking the lower limits as $R \to \infty$, we conclude that

$$\frac{\delta_1 (\log T)^{r-1}}{\Gamma(r)} \left[1 + \frac{\log T \sum_{j=1}^m |\xi_j|}{|\Lambda|} \right] \geq 1,$$

which contradicts (3.5). Hence $N(\overline{U}) \subset \overline{U}$.

As a consequence of Steps 1–5 and Theorem 2.8, we conclude that N has a fixed point $y \in C(J, B)$ which is a solution of (1.1)–(1.2). This concludes the proof. □

4 An Example

We conclude this paper with an example to illustrate our main results. We apply the Theorem 3.3 to the following fractional differential inclusion

$$ {}^H_C D^r y(t) \in F(t, y(t)), \quad \text{for a.e. } t \in J_1 = [1, e], 1 < r \leq 2, \tag{4.1}$$

$$y(1) = y_1, \, y'(1) - \sum_{j=1}^m \xi_j y(t_j) = 0. \tag{4.2}$$

We set

$$F(t,y)=\{v\in\mathbb{R}: f_1(t,y)\le v\le f_2(t,y)\},$$

where $f_1, f_2 : [1,e]\times\mathbb{R}\to\mathbb{R}$. We assume that for each $t\in[1,e]$, $f_1(t,\cdot)$ is lower semi-continuous (i.e., the set $\{y\in\mathbb{R} : f_1(t,y)>\mu\}$ is open for each $\mu\in\mathbb{R}$), and assume that for each $t\in[1,e]$, $f_2(t,\cdot)$ is upper semi-continuous (i.e., the set $\{y\in\mathbb{R} : f_2(t,y)<\mu\}$ is open for each $\mu\in\mathbb{R}$). Assume that there are $p\in C([1,e],\mathbb{R})$ and $\psi : [0,\infty)\to(0,\infty)$ continuous and nondecreasing such that

$$\begin{aligned}\|F(t,u)\|_{\mathcal{P}} &= \sup\{|v|, v(t)\in F(t,u)\}\\ &= \max(|f_1(t,u)|,|f_2(t,u)|)\le p(t), \quad \text{for each } t\in[1,e], u\in\mathbb{R}.\end{aligned}$$

It is clear that F is compact and convex-valued, and it is upper semi-continuous, furthermore, we assume that for $(t,y)\in J_1\times\mathbb{R}$ with $|y|\le R$, we have

$$\liminf_{R\to\infty}\frac{\int_1^e p(t)dt}{R}=\delta_2<\infty.$$

We also assume that there exists a Carathéodory function $\psi : J_1\times[1,2R]\to\mathbb{R}_+$ with

$$\rho(F(t,M))\le\psi(t,\rho(M)), \text{ a.e. } t\in J_1 \text{ and each } M\subset B=\{x\in\mathbb{R} : |x|\le R\},$$

and $\varphi=0$ is the unique solution in $C(J_1,[1,2R])$ of the inequality

$$\begin{aligned}\varphi(t) \le\ & \frac{2\log t}{\Lambda}\sum_{j=1}^{m}\left(\frac{\xi_j}{\Gamma(r)}\int_1^{t_j}\left(\log\frac{t_j}{s}\right)^{r-1}\psi(s,\rho(M(s))\frac{ds}{s}\right)\\ &+\frac{2}{\Gamma(r)}\int_1^t\left(\log\frac{t}{s}\right)^{r-1}\psi(s,\rho(M(s))\frac{ds}{s},\end{aligned}$$

for $t\in J_1$. Finally, we assume that there exists δ_2 such that

$$\delta_2<\Gamma(r)\left[1+\frac{\sum_{j=1}^m\xi_j}{|\Lambda|}\right]^{-1}.$$

Since all conditions of Theorem 3.3 hold, (4.1)–(4.2) has at least one solution y on J_1.

5 Conclusion

In this paper, we used a topological method, more precisely the set-valued analog of Mönch's fixed point theorem, to resolve a problem of fractional differential inclusions taking into account the convexity of the set-valued map F. One of the difficult things we face in these kinds of problems is to prove that the set of selection $S_{F,y}$ is nonempty, i.e., to prove that the set-valued map F has a measurable selection. In Theorem 3.3 we gave sufficient conditions to prove the existence of solutions to the concerned problem. We would like to mention that finding a Carathéodory function ψ written explicitly is remaining an open problem.

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