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Infinitely Many Solutions for Anisotropic Discrete Boundary Value Problems of Kirchhoff Type

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Dedicated to Johnny Henderson on the occasion of his 70th birthday.

Abstract

This paper establishes several criteria for the existence of infinitely many solutions to an anisotropic discrete boundary value problem of $p(k)$-Kirchhoff type in a T-dimensional Hilbert space. The approach is based on the critical point theory. Some recent results are extended and improved. An example is included to show the applicability of the theorems.

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1 Introduction

In this paper, we study the existence of infinitely many solutions to the anisotropic discrete boundary value problem of $p(k)$-Kirchhoff type

$$\begin{cases} -M(\rho(u))\Delta(\phi_{p(k-1)}(\Delta u(k-1)) = \lambda f(k, u(k)), \quad k \in \mathbb{Z}[1, T], \\ u(0) = u(T+1) = 0, \end{cases} \qquad (P_\lambda^f)$$

where $M : [0, \infty) \to \mathbb{R}$ is a continuous function for which there exist two positive constants m_0 and m_1 such that $m_0 \leq M(t) \leq m_1$ for all $t \in [0, \infty)$, $T \geq 2$ is a fixed integer, $\mathbb{Z}[a, b] = \{a, a+1, \ldots, b\}$ for any a, $b \in \mathbb{Z}$ with $a < b$, $p : \mathbb{Z}[0, T] \to [2, \infty)$, $\phi_{p(k)}(t) = |t|^{p(k)-2}t$ for every $(k, t) \in \mathbb{Z}[0, T] \times \mathbb{R}$, $\Delta u(k-1) = u(k) - u(k-1)$ is the forward difference operator, $\lambda > 0$ is a parameter, $f : \mathbb{Z}[1, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and the functional $\rho : \mathbb{R} \to [0, \infty)$ is defined by

$$\rho(u) = \sum_{k=1}^{T+1} \frac{|\Delta u(k-1)|^{p(k-1)}}{p(k-1)}.$$

As usual, a function $u : \mathbb{Z}[0, T+1] \to \mathbb{R}$ is called a *solution* of the problem (P_λ^f) if u satisfies both the equation and the boundary conditions in (P_λ^f). Throughout the paper, we use the notations

$$p^- = \min_{k \in \mathbb{Z}[0,T]} p(k) \quad \text{and} \quad p^+ = \max_{k \in \mathbb{Z}[0,T]} p(k).$$

In a recent paper [8], the present authors studied the existence of at least one nontrivial solution of the problem (P_λ^f) with $\lambda = 1$. This work continues the study in [8]. In this paper, we focus on our study on the existence of infinitely many solutions of the problem (P_λ^f). Specifically, under some suitable assumptions on the behavior of the nonlinearity f at infinity, we prove the existence of a definite interval for λ in which the problem (P_λ^f) has an unbounded sequence of solutions (see Theorem 3.1 below). We also derive some consequences of this theorem. Furthermore, by replacing the conditions at infinity on the nonlinear terms by a similar one at zero, we obtain that the problem (P_λ^f) has a sequence of pairwise distinct solutions converging to zero (Theorem 3.7). We also present an example to illustrate the applicability of our results.

In recent years, many authors have studied the existence of solutions for various discrete boundary value problems because of their wide applications. We refer the reader to [2, 4, 5, 11–14, 16] and the references therein for some recent work. Likewise, problems of Kirchhoff-type have received extensive attention from many researchers; see, for example, [1, 5, 6, 9, 10, 15]. We wish to point out that the existence of solutions for anisotropic discrete boundary value problems of $p(k)$-Kirchhoff type have also been recently studied in the literature, for example, in [8, 18, 19]. The work in this paper continues the study on this last topic.

The remainder of the paper is organized as follows. In Section 2, we recall some definitions and notations. In Section 3, we state and prove the main results in the paper.

2 Preliminary Results

The proofs of our theorems are based on Lemma 2.1 below. This Lemma is a refinement of the variational principle of Ricceri [17, Theorem 2.5] as proved by Bonanno and Molica Bisci in [3, Theorem 2.1].

Lemma 2.1. *Let X be a reflexive real Banach space, let $\Phi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable, coercive, and sequentially weakly lower semicontinuous functional, $\Psi : X \to \mathbb{R}$ be a sequentially weakly upper semicontinuous and continuously Gâteaux differentiable functional, and let λ be a positive parameter. For every $y > \inf\limits_X \Phi$, let*

$$\varphi(y) := \inf_{u\in\Phi^{-1}(-\infty,y)} \frac{\sup_{v\in\Phi^{-1}(-\infty,y)} \Psi(v) - \Psi(u)}{y - \Phi(u)},$$

$$\gamma := \liminf_{y\to\infty} \varphi(y), \quad \textit{and} \quad \zeta := \liminf_{y\to(\inf_X \Phi)^+} \varphi(y).$$

Then:

(a) For every $y > \inf\limits_X \Phi$ and every $\lambda \in (0, 1/\varphi(y))$, the restriction of the functional $I_\lambda := \Phi - \lambda\Psi$ to $\Phi^{-1}(-\infty, y)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.

(b) If $\gamma < \infty$, then for each $\lambda \in (0, 1/\gamma)$, the following alternative holds: either

(b_1) I_λ possesses a global minimum, or

(b_2) there is a sequence $\{u_l\}$ of critical points (local minima) of I_λ such that

$$\lim_{l\to\infty} \Phi(u_l) = \infty.$$

(c) If $\zeta < \infty$, then for each $\lambda \in (0, 1/\zeta)$, the following alternative holds: either

(c_1) there is a global minimum of Φ which is a local minimum of I_λ, or

(c_2) there is a sequence $\{u_l\}$ of pairwise distinct critical points (local minima) of I_λ which converges weakly to a global minimum of Φ.

In the sequel, we let X be the vector space

$$X = \{u : \mathbb{Z}[0, T+1] \to \mathbb{R} : \ u(0) = u(T+1) = 0\}.$$

Then, equipped with the norm

$$\|u\| = \left(\sum_{k=1}^{T+1} |\Delta u(k-1)|^2\right)^{1/2},$$

X is a T-dimensional Banach space. We also introduce the family of norms on X

$$|u|_m = \left(\sum_{k=1}^{T} |u(k)|^m\right)^{1/m} \quad \text{for all } u \in X \text{ and } m \geq 2.$$

It is easy to verify that (see [16])

$$T^{\frac{2-m}{2m}}|u|_2 \leq |u|_m \leq T^{\frac{1}{m}}|u|_2 \quad \text{for all } u \in X \text{ and } m \geq 2.$$

Lemmas 2.2–2.4 below are taken from [8], and their proofs can be found in [7,8,16].

Lemma 2.2. *The following assertions hold:*

(d_1) *For every* $u \in X$ *with* $\|u\| > 1$,

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} \geq T^{\frac{2-p^-}{2}}\|u\|^{p^-} - T.$$

(d_2) *For every* $u \in X$ *with* $\|u\| \leq 1$,

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} \geq T^{\frac{p^+-2}{2}}\|u\|^{p^+}.$$

(d_3) *For every* $u \in X$ *and for any* p, $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *we have*

$$\|u\|_\infty = \max_{k \in \mathbb{Z}[1,T]} |u(k)| \leq (T+1)^{\frac{1}{q}} \left(\sum_{k=1}^{T+1} |\Delta u(k-1)|^p\right)^{\frac{1}{p}}.$$

For any $\lambda > 0$, let Φ, Ψ, $I_\lambda : X \to \mathbb{R}$ be the functionals defined by

$$\Phi(u) = \widehat{M}(\rho(u)), \tag{2.1}$$

$$\Psi(u) = \sum_{k=1}^{T} F(k, u(k)), \tag{2.2}$$

and

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u), \tag{2.3}$$

where $F : \mathbb{Z}[1,T] \times \mathbb{R} \to \mathbb{R}$ and $\widehat{M} : [0,\infty) \to \mathbb{R}$ are functions defined by

$$F(k,t) = \int_0^t f(k,\xi)d\xi \quad \text{for all } (k,t) \in \mathbb{Z}[1,T] \times \mathbb{R}$$

and

$$\widehat{M}(t) = \int_0^t M(\xi)d\xi \quad \text{for all } t \in [0,\infty),$$

with f and M being given in the problem (P_λ^f).

Lemma 2.3. *The functionals Φ, Ψ, and I_λ satisfy the conditions:*

(a) Φ is continuously Gâteaux differentiable, sequentially weakly lower semicontinuous, strongly continuous, and its Gâteaux differential at $u \in X$ is given by

$$\Phi'(u)(v) = M(\rho(u)) \sum_{k=1}^{T+1} \phi_{p(k-1)}(\Delta u(k-1))\Delta v(k-1) \quad \textit{for every } v \in X.$$

(b) Ψ is continuously Gâteaux differentiable, sequentially weakly upper semicontinuous, and its Gâteaux differential at $u \in X$ is given by

$$\Psi'(u)(v) = \sum_{k=1}^{T} f(k, u(k))v(k) \quad \textit{for every } v \in X.$$

(c) I_λ is continuously Gâteaux differentiable and its Gâteaux differential at $u \in X$ is given by

$$\begin{aligned} I'_\lambda(u)(v) =& M(\rho(u)) \sum_{k=1}^{T+1} \phi_{p(k-1)}(\Delta u(k-1))\Delta v(k-1) \\ & - \lambda \sum_{k=1}^{T} f(k, u(k))v(k) \quad \textit{for every } v \in X. \end{aligned}$$

Lemma 2.4. *If $u \in X$ is a critical point of the functional I_λ, then u is a solution of the problem (P^f_λ).*

3 Main Results

For any nonnegative constant γ, let

$$\sigma(\gamma) = \sqrt{T+1}\, \sqrt[p^-]{\frac{p^+}{m_0\sqrt{T^{2-p^-}}}\widehat{M}(\gamma) + \sqrt{T^{p^-}}}.$$

Then, $\sigma(\gamma) > 0$.

We formulate our first existence result as follows.

Theorem 3.1. *Assume that there exist two nonnegative sequences $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ with*

$$\lim_{n\to\infty} b_n = \infty \tag{3.1}$$

such that

(A_1) $\widehat{M}\left(\frac{2a_n^{p_n^+}}{p^-}\right) < \widehat{M}(b_n)$ *for every* $n \in \mathbb{N}$*, where*

$$p_n^+ = \begin{cases} p^+, & a_n \geq 1, \\ p^-, & a_n < 1; \end{cases}$$

(A_2) $\mathcal{A}^\infty < \frac{m_0}{m_1}\mathcal{B}^\infty$*, where*

$$\mathcal{A}^\infty = \liminf_{n\to\infty} \frac{\sum_{k=1}^T \max_{|t|\leq\sigma(b_n)} F(k,t) - \sum_{k=1}^T F(k,a_n)}{\widehat{M}(b_n) - \widehat{M}\left(\frac{2a_n^{p_n^+}}{p^-}\right)},$$

and

$$\mathcal{B}^\infty = \limsup_{\xi\to\infty} \frac{\sum_{k=1}^T F(k,\xi)}{\widehat{M}\left(\frac{2\xi^{p^+}}{p^-}\right)}.$$

Then, for each $\lambda \in \left(\frac{m_1}{m_0\mathcal{B}^\infty}, \frac{1}{\mathcal{A}^\infty}\right)$*, the problem* (P_λ^f) *has an unbounded sequence of solutions.*

The proof of Theorem 3.1 is based on an application of Lemma 2.1(b).

Proof. Let the functionals Φ, Ψ, and I_λ be given by (2.1)–(2.3), respectively. To show Φ and Ψ satisfy all the regularity assumptions in Lemma 2.1, by Lemma 2.3, we see that the only thing we need to verify is that Φ is coercive. In fact, in view of (2.1), we have

$$\Phi(u) \geq \frac{m_0}{p^+} \sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)}. \tag{3.2}$$

Then, from part (d_1) of Lemma 2.2, it yields that

$$\Phi(u) \geq \frac{m_0}{p^+}\left(T^{\frac{2-p^-}{2}}\|u\|^{p^-} - T\right) \to \infty \quad \text{as } \|u\| \to \infty,$$

i.e., Φ is coercive.

Let

$$r_n = \widehat{M}(b_n) \quad \text{for all } n \in \mathbb{N}. \tag{3.3}$$

Then, $r_n > 0$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} r_n = \infty$ by (3.1). Let $u \in E$ be such that $u \in \Phi^{-1}(-\infty, r_n)$. Then, from parts (d_1) and (d_2) of Lemma 2.2 and (3.2), it follows that

$$\|u\|^{p^-} < \frac{r_n p^+}{m_0\sqrt{T^{2-p^-}}} + \sqrt{T^{p^-}} \quad \text{if } \|u\| > 1 \tag{3.4a}$$

and

$$\|u\|^{p^+} < \frac{r_n p^+}{m_0\sqrt{T^{p^+-2}}} \quad \text{if } \|u\| \le 1. \tag{3.4b}$$

Note that

$$\frac{r_n p^+}{m_0\sqrt{T^{2-p^-}}} + \sqrt{T^{p^-}} > 1 \quad \text{and} \quad \frac{r_n p^+}{m_0\sqrt{T^{p^+-2}}} < \frac{r_n p^+}{m_0\sqrt{T^{2-p^-}}} + \sqrt{T^{p^-}}.$$

Then, from (3.4), we have

$$\begin{aligned}\|u\| &< \max\left\{\left(\frac{r_n p^+}{m_0\sqrt{T^{p^+-2}}}\right)^{\frac{1}{p^+}}, \left(\frac{r_n p^+}{m_0\sqrt{T^{2-p^-}}} + \sqrt{T^{p^-}}\right)^{\frac{1}{p^-}}\right\} \\ &= \left(\frac{r_n p^+}{m_0\sqrt{T^{2-p^-}}} + \sqrt{T^{p^-}}\right)^{\frac{1}{p^-}}.\end{aligned}$$

Thus, from part (d_3) of Lemma 2.2 with $p = q = 2$, we obtain that

$$|u(k)| \le \sqrt{T+1}\|u\| \le \sqrt{T+1}\left(\frac{r_n p^+}{m_0\sqrt{T^{2-p^-}}} + \sqrt{T^{p^-}}\right)^{\frac{1}{p^-}} = \sigma(b_n)$$

for all $k \in \mathbb{Z}[1,T]$. Hence,

$$\Psi(u) = \sum_{k=1}^{T} F(k, u(k)) \le \sum_{k=1}^{T} \max_{|t|\le\sigma(b_n)} F(k,t) \quad \text{for every } u \in E \text{ with } \Phi(u) < r_n,$$

or equivalently,

$$\sup_{u\in\Phi^{-1}(-\infty,r_n)} \Psi(u) \le \sum_{k=1}^{T} \max_{|t|\le\sigma(b_n)} F(k,t). \tag{3.5}$$

Now, for each $n \in \mathbb{N}$, let $w_n : \mathbb{Z}[0, T+1] \to \mathbb{R}$ be defined by

$$w_n(k) = \begin{cases} a_n, & k \in \mathbb{Z}[1,T], \\ 0, & k = 0, T+1. \end{cases}$$

Then, $w_n \in X$ and

$$\sum_{k=1}^{T+1} \frac{|\Delta w_n(k-1)|^{p(k-1)}}{p(k-1)} \le \frac{2a_n^{p_n^+}}{p^-}.$$

This, together with (2.1), (A_1), and (3.3), implies that

$$\Phi(w_n) \le \widehat{M}\left(\frac{2a_n^{p_n^+}}{p^-}\right) < \widehat{M}(b_n) = r_n \quad \text{for any } n \in \mathbb{N}. \tag{3.6}$$

Thus, in view of (3.3), (3.5), and (3.6), we see that

$$\begin{aligned}\varphi(r_n) &= \inf_{u\in\Phi^{-1}(-\infty,r_n)} \frac{\sup_{u\in\Phi^{-1}(-\infty,r_k)}\Psi(u)-\Psi(u)}{r_n-\Phi(u)} \\ &\le \frac{\sup_{u\in\Phi^{-1}(-\infty,r_n)}\Psi(u)-\sum_{k=1}^{T}F(k,w_n(k))}{r_n-\Phi(w_n)} \\ &\le \frac{\sum_{k=1}^{T}\max_{|t|\le\sigma(b_n)}F(k,t)-\sum_{k=1}^{T}F(k,a_n)}{\widehat{M}(b_n)-\widehat{M}\left(\frac{2a_n^{p^+}}{p^-}\right)}.\end{aligned}$$

Therefore, for γ defined in Lemma 2.1, from (A_2), we have $\gamma \le \liminf_{n\to\infty}\varphi(r_n) \le \mathcal{A}^\infty < \infty$. Then, by Lemma 2.1 (b), we see that, for each $\lambda \in \left(0, \frac{1}{\mathcal{A}^\infty}\right)$, one of the following alternatives holds:

(b_1) $I_\lambda = \Phi - \lambda\Psi$ possesses a global minimum, or

(b_2) there is a sequence $\{u_n\}$ of critical points of I_λ such that $\lim_{n\to\infty}\|u_n\| = \infty$.

In the following, we show that if $\lambda \in \left(\frac{m_1}{m_0\mathcal{B}^\infty}, \frac{1}{\mathcal{A}^\infty}\right)$, then (b_1) does not hold, and so (b_2) must hold. From the definition of $\mathcal{B}^\infty$, there exists a sequence $\{c_n\}$ of positive numbers such that $c_n \ge 1$, $\lim_{n\to\infty} c_n = \infty$, and

$$\mathcal{B}^\infty = \lim_{n\to\infty}\frac{\sum_{k=1}^{T}F(k,c_n)}{\widehat{M}\left(\frac{2c_n^{p^+}}{p^-}\right)}. \tag{3.7}$$

For any $n \in \mathbb{N}$, we define $\{y_n\} \in X$ by letting

$$y_n(k) = \begin{cases} c_n, & k \in \mathbb{Z}[1,T], \\ 0, & k = 0, T+1. \end{cases}$$

Then, from (2.1), (3.6), and the fact that $\widehat{M}(t) \le m_1 t$, we have

$$\Phi(y_n) \le \widehat{M}\left(\frac{2c_n^{p^+}}{p^-}\right) \le \frac{2m_1c_n^{p^+}}{p^-}.$$

Thus, by (2.3),

$$I_\lambda(y_n) \le \frac{2m_1c_n^{p^+}}{p^-} - \lambda\sum_{k=1}^{T}F(k,c_n). \tag{3.8}$$

Now, we consider two cases.

Case 1: $\mathcal{B}^\infty = \infty$. Choose $L > 0$ large enough so that

$$L > \frac{m_1}{\lambda m_0}. \tag{3.9}$$

By (3.7), there exists $N_1 \in \mathbb{N}$ such that

$$\sum_{k=1}^{T} F(k, c_n) > L\widehat{M}\left(\frac{2c_n^{p^+}}{p^-}\right) \geq \frac{2m_0 L c_n^{p^+}}{p^-} \quad \text{for all } n \geq N_1.$$

Then, from (3.8), we have

$$I_\lambda(y_n) \leq \frac{2c_n^{p^+}}{p^-}\left(m_1 - \lambda m_0 L\right) \quad \text{for all } n \geq N_1.$$

Hence, from (3.9) and the fact that $\lim_{n\to\infty} c_n = \infty$, we have $\lim_{n\to\infty} I_\lambda(y_n) = -\infty$.

Case 2: $\mathcal{B}^\infty < \infty$. Since $\lambda > \dfrac{m_1}{m_0\mathcal{B}^\infty}$, we can fix $\epsilon > 0$ such that

$$\epsilon \in \left(0, \mathcal{B}^\infty - \frac{m_1}{m_0\lambda}\right). \tag{3.10}$$

Then from (3.7), there exists $N_2 \in \mathbb{N}$ such that

$$\sum_{k=1}^{T} F(k, c_n) > (\mathcal{B}^\infty - \epsilon)\widehat{M}\left(\frac{2c_n^{p^+}}{p^-}\right) \geq (\mathcal{B}^\infty - \epsilon)\frac{2m_0 c_n^{p^+}}{p^-} \quad \text{for all } n \geq N_2.$$

From (3.8), we see that

$$I_\lambda(y_n) \leq \frac{2c_n^{p^+}}{p^-}\left(m_1 - \lambda(\mathcal{B}^\infty - \epsilon)m_0\right) \quad \text{for all } n \geq N_2.$$

Thus, by (3.10) and the fact that $\lim_{n\to\infty} c_n = \infty$, we see that $\lim_{n\to\infty} I_\lambda(y_n) = -\infty$.

Combining the above two cases, we see that for $\lambda \in \left(\dfrac{m_1}{m_0\mathcal{B}^\infty}, \dfrac{1}{\mathcal{A}^\infty}\right)$, I_λ is always unbounded from below. This show that (b_1) does not hold, and thus (b_2) must. An application of Lemma 2.4 completes the proof of the theorem. □

Remark 3.2. Let $\{a_n\}$ and $\{b_n\}$ be two real nonnegative sequences such that (3.1) holds and $2a_n^{p_n^+} < p^- b_n$. Assume that $\mathcal{A}^\infty = 0$ and $\mathcal{B}^\infty = \infty$. Then, all the assumptions of Theorem 3.1 are satisfied, and so the problem (P_λ^f) has infinitely many solutions for any $\lambda \in (0, \infty)$.

Theorem 3.3. *Assume that* $\mathcal{C}^\infty < \dfrac{m_0}{m_1}\mathcal{B}^\infty$*, where* $\mathcal{B}^\infty$ *is defined as in Theorem 3.1 and*

$$\mathcal{C}^\infty = \liminf_{\xi\to\infty} \frac{\sum_{k=1}^{T} \max_{|t|\le\sigma(\xi)} F(k,t)}{\widehat{M}(\xi)}.$$

Then, for each $\lambda \in \left(\dfrac{m_1}{m_0\mathcal{B}^\infty}, \dfrac{1}{\mathcal{C}^\infty}\right)$*, the problem* (P_λ^f) *has an unbounded sequence of solutions.*

Proof. Let $a_n = 0$ for every $n \in \mathbb{N}$ and $\{b_n\}$ be a sequence of positive numbers satisfying (3.1) and

$$\mathcal{C}^\infty = \lim_{n\to\infty} \frac{\sum_{k=1}^{T} \max_{|t|\le\sigma(b_n)} F(k,t)}{\widehat{M}(b_n)}.$$

Then, with $\mathcal{A}^\infty = \mathcal{C}^\infty$, conditions (A_1) and (A_2) of Theorem 3.1 are satisfied. Hence, the conclusion follows. □

Now, we give an application of Theorem 3.3.

Example 3.4. Consider the problem

$$\begin{cases} -M(\rho(u))\Delta(\phi_{p(k-1)}(\Delta u(k-1)) = \lambda f(k,u(k)), & k \in \mathbb{Z}[1,4], \\ u(0) = u(5) = 0, \end{cases} \tag{3.11}$$

where $M(t) = 3 + 2\sin t$ for all $t \in [0,\infty)$, $p(k) = 2^{k-1} + 1$ for $k = 0, 1, 2, 3, 4$, and

$$f(k,t) = f(t) = \begin{cases} 0, & \text{if } t \in (-\infty, 0], \\ 9t^8 + 9t^8\sin(\pi e^t) + \pi t^9 e^t \cos(\pi e^t), & \text{if } t \in (0,\infty). \end{cases}$$

From the definition of f, we see that

$$F(t) = \begin{cases} 0, & \text{if } t \in (-\infty, 0], \\ t^9(1 + \sin(\pi e^t)), & \text{if } t \in (0,\infty). \end{cases}$$

Clearly, $T = 4$, $p^- = 3/2$, and $p^+ = 9$. Moreover, with $m_0 = 1$ and $m_1 = 5$, we have $m_0 \le M(t) \le m_1$ on $[0,\infty)$. Note that

$$\mathcal{C}^\infty = \liminf_{\xi\to\infty} \frac{\sum_{k=1}^{T} \max_{|t|\le\sigma(\xi)} F(k,t)}{\widehat{M}(\xi)} = 0$$

and

$$\mathcal{B}^\infty = \limsup_{\xi\to\infty} \frac{\sum_{k=1}^{T} F(k,\xi)}{\widehat{M}\left(\frac{2\xi^{p^+}}{p^-}\right)} = 6.$$

Then, all the assumptions of Theorem 3.3 are satisfied. Hence by Theorem 3.3, the problem (3.11) has an unbounded sequence of solutions for each $\lambda \in \left(\frac{5}{6}, \infty\right)$.

Corollaries 3.5 and 3.6 below are direct consequences of Theorems 3.1 and 3.3, respectively.

Corollary 3.5. *Assume that there exist two real nonnegative sequences* $\{a_n\}$ *and* $\{b_n\}$ *such that* (3.1) *holds,* $2a_n^{p_n^+} < p^- b_n$*,* $\mathcal{A}_\infty < 1$*, and* $\mathcal{B}_\infty > \frac{m_1}{m_0}$*. Then, the problem*

$$\begin{cases} -M(\rho(u))\Delta(\phi_{p(k-1)}(\Delta u(k-1)) = f(k, u(k)), & k \in \mathbb{Z}[1, T], \\ u(0) = u(T+1) = 0 \end{cases} \tag{P^f}$$

has an unbounded sequence of solutions.

Corollary 3.6. *Assume that* $\mathcal{B}_\infty > \frac{m_1}{m_0}$ *and* $\mathcal{C}_\infty < 1$*. Then, the problem* (P^f) *has an unbounded sequence of solutions.*

Finally, by using Lemma 2.1 (c), instead of (b), and arguing as in the proof of Theorem 3.1, we can establish the following result.

Theorem 3.7. *Assume that there exist two real nonnegative sequences* $\{d_n\}$ *and* $\{e_n\}$ *with* $\lim_{n\to\infty} e_n = 0$ *such that*

(A_3) $\widehat{M}\left(\frac{2d_n^{\hat{p}_n^+}}{p^-}\right) < \widehat{M}(e_n)$ *for every* $n \in \mathbb{N}$*, where*

$$\hat{p}_n^+ = \begin{cases} p^+, & d_n \geq 1, \\ p^-, & d_n < 1; \end{cases}$$

(A_4) $\mathcal{A}^0 < \frac{m_0}{m_1}\mathcal{B}^0$*, where*

$$\mathcal{A}^0 = \liminf_{n\to\infty} \frac{\sum_{k=1}^T \max_{|t|\leq\sigma(e_n)} F(k,t) - \sum_{k=1}^T F(k, a_n)}{\widehat{M}(e_n) - \widehat{M}\left(\frac{2d_n^{\hat{p}_n^+}}{p^-}\right)},$$

and

$$\mathcal{B}^0 = \limsup_{\xi\to 0} \frac{\sum_{k=1}^T F(k,\xi)}{\widehat{M}\left(\frac{2\xi^{p^+}}{p^-}\right)}.$$

Then, for each $\lambda \in \left(\frac{m_1}{m_0\mathcal{B}^0}, \frac{1}{\mathcal{A}^0}\right)$*, the problem* (P_λ^f) *has a sequence of pairwise distinct solutions which converges to* 0 *in* X*.*

Remark 3.8. Applying Theorem 3.7, results similar to Theorem 3.3 and Corollaries 3.5–3.6 can be formulated without difficulty. The details are omitted.

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