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Two-Point Boundary Value Problems for Finite Difference Equations, Uniqueness of Solutions Implies Existence of Solutions

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Abstract

We consider a family of two-point $n-1, 1$ conjugate boundary value problems for nth order nonlinear finite forward difference equations. We first obtain conditions in terms of uniqueness of solutions implies existence of solutions, employing shooting methods. Then we assume appropriate monotonicity properties on the nonlinear terms in the finite difference equation that imply the uniqueness of solutions.

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1 Introduction

In a seminal paper, [28], Lasota and Opial proved that for second order ordinary differential equations, global existence and uniqueness of solutions of initial value problems and uniqueness of solutions of two-point conjugate (Dirichlet) boundary value problems imply the existence of solutions of two-point conjugate boundary value problems. Lasota and Opial [28] initiated a vast study of problems referred to as uniqueness implies existence for nonlinear problems. In the case of boundary value problems for ordinary differential equations, we refer the reader to [3, 4, 8, 10, 16, 18, 24, 26, 27, 29].

Henderson [12] initiated the study of related results for finite difference equations and again, a significant literature, led by Henderson (see, [13–15, 17] for example), has

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developed. Chyan, Henderson and Yin, and others also initiated analogous studies on time scales; we refer the reader to [1, 2, 19–23], for example.

Recently, these types of results were gathered in the monograph [5].

Lasota and Opial [28] consider a low order equation with two Dirichlet or conjugate type boundary conditions. Due to the low order and to the nature of the boundary conditions, they can assume unique solvability and continuous dependence on parameters of initial value problems, unique solvability of the two-point boundary value problems and then employ a shooting method to solve the two-point boundary value problem. In the literature cited above, the order of the equation is typically high and the boundary conditions are complicated. It is standard in these types of results to assume a universal nonlinear disconjugacy condition that n-point conjugate boundary value problems are uniquely solvable. For authoritative accounts related to the unique solvability of n-point conjugate boundary value problems, we refer the reader to [8, 27]. Recently, Eloe and Henderson [6] returned to the original arguments produced by Lasota and Opial [28] and obtained uniqueness implies existence results for two-point problems in which they assumed the unique solvability of only the initial value problem and the two-point problems.

In this paper, we shall carry the recent work of [6] over to finite difference equations. There are two primary objectives in this paper. First, in contrast to the foundational work in [12–14] where it is assumed that the large family of n-point conjugate boundary value problems are uniquely solvable, we shall assume the initial value problems and the class of two-point problems are uniquely solvable. Second, a monotonicity assumption will be imposed on the nonlinear term which implies the unique solvability of the family of two-point boundary value problems.

In what follows, we shall state the specific nth order two-point problem we consider and introduce the notation and preliminary results related to finite difference equations in Section 2. In Section 3, we extend the original Lasota and Opial [28] argument to apply to the nth order two point problem and employ a shooting method to obtain an abstract uniqueness implies existence result. Then in Section 4, we assume an appropriate monotonicity property on the nonlinear term and prove the uniqueness of solutions.

We point out that assuming the unique solvability of a family of boundary value problems realistically implies a local type uniqueness implies existence result. The introduction of the monotonicity property implies that the main result given in Section 4, Corollary 4.2, is a global uniqueness implies existence result.

2 Preliminaries

Let $n \geq 2$ denote an integer, let $a \in \mathbb{R}$ and let $\mathbb{N}_a := \{a, a+1, \dots\}$. Let $M_1, M_2 \in \mathbb{N}_a$ such that $a \leq M_1 < M_2$. We shall employ an interval notation, $[M_1, M_2]_a$ to denote $[M_1, M_2] \cap \mathbb{N}_a$. For $y : \mathbb{N}_a \to \mathbb{R}$ define the usual forward finite differences by

$$\Delta^0 y(m) = y(m), \quad \Delta^1 y(m) = \Delta y(m) = y(m+1) - y(m),$$

$$\Delta^i y(m) = \Delta(\Delta^{i-1} y)(m), \quad i = 2, 3, ...$$

We shall study the family of two-point $n-1, 1$ conjugate boundary value problems of the type

$$\Delta^n y(m) = f(t, y(m), y(m+1), \ldots, y(m+n-1)), \quad m \in \mathbb{N}_a, \tag{2.1}$$

with the boundary conditions

$$\Delta^{i-1} y(M_1) = a_i, \quad i = 1, \ldots, n-1, \quad y(M_2 + n) = a_n, \tag{2.2}$$

for any $M_1 < M_2 \in \mathbb{N}_a$, $a_i \in \mathbb{R}$, $i = 1, \ldots, n$, where $f : \mathbb{N}_a \times \mathbb{R}^n \to \mathbb{R}$. In particular, we shall show that under suitable conditions on f, and for any $M_1 < M_2 \in \mathbb{N}_a$, $a_i \in \mathbb{R}$, $i = 1, \ldots, n$, the uniqueness of solutions of (2.1), (2.2) implies the existence of solutions of (2.1), (2.2).

Remark 2.1. Since $\Delta^l y(m) = \sum_{j=0}^{l} (-1)^{l-j} \binom{l}{j} y(m+j)$, initial value problems are commonly expressed in the form

$$y(m+n) = g(m, y(m), \ldots, y(m+n-1)), \quad y(m_0 + j) = b_j, \quad j = 0, \ldots, n-1.$$

We work with the delta difference notation because algebraically this will better serve our purposes in Section 4.

For arguments related to uniqueness implies existence results, it is common to assume that solutions of initial value problems exist, are continuous with respect to initial conditions, and extend to all of $\mathbb{N}_a$. The following assumption, assumed throughout, provides the existence and continuity we shall require from initial value problems.

(A) $f(m, y_1, \ldots, y_n) : \mathbb{N}_a \times \mathbb{R}^n \to \mathbb{R}$ is continuous, and the equation

$$r_n = f(m, r_0, \ldots, r_{n-1}) - \sum_{j=0}^{n-1} (-1)^{n-j} \binom{n}{j} r_j$$

can be solved for explicitly r_0 as a continuous function of $r_1, \ldots, r_n$ for each $m \in \mathbb{N}_a$.

Given $m_0 \in \mathbb{N}_a$, $a_1, \ldots, a_n \in \mathbb{R}$, an initial value problem for (2.1) consists of the equation (2.1) on $\mathbb{N}_a$ along with the initial conditions

$$\Delta^{i-1} y(m_0) = a_i, \quad i = 1, \ldots, n. \tag{2.3}$$

Since one can solve (2.1) explicitly for $y(m+n)$, then one can calculate $y(m_0+n)$ and proceed inductively to the right; with the assumption that one can solve (2.1) explicitly for $y(m)$, one can solve (2.1) explicitly for $y(m_0)$, and then proceed inductively to the left. We shall state the following immediate consequence of Condition (A) as a lemma.

Lemma 2.2. *Assume that with respect to* (2.1)*, Condition (A) is satisfied. Then solutions of initial value problems for* (2.1) *exist and are unique on* $\mathbb{N}_a$*, and solutions of initial value problems depend continuously on the initial conditions.*

We introduce the concept of generalized zero of a map $y : \mathbb{N}_a \to \mathbb{R}$ as defined by Hartman [9]. The point $m_0 = a$ is a generalized zero of y if $y(a) = 0$, and $m_0 > a$ is a generalized zero of y if $y(m_0) = 0$ or there is an integer $j \geq 1$ such that $(-1)^j y(m_0 - j)y(m_0) > 0$ and if $j > 1$, $y(m_0 - j + 1) = \cdots = y(m_0 - 1) = 0$. Hartman [9] also proved the following discrete version of Rolle's theorem.

Lemma 2.3. *Assume* $y : \mathbb{N}_a \to \mathbb{R}$ *and assume* y *has generalized zeros at* $m_1 < m_2$*,* $m_1, m_2 \in \mathbb{N}_a$*. Then* $\Delta y : \mathbb{N}_a \to \mathbb{R}$ *has a generalized zero in* $[m_1, m_2 - 1]_a$*.*

The uniqueness assumption that is assumed through Section 3 takes the following form.

(B) Solutions of the two-point boundary value problems (2.1), (2.2) are unique if they exist.

This uniqueness assumption is in contrast to the uniqueness assumption employed by Henderson [12], for example. There it is assumed that solutions of n-point conjugate problems are uniquely solvable if they exist and takes the following form.

($\bar{B}$) Given

$$m_1 < m_2 < \cdots < m_n, \quad m_1, m_2, \ldots, m_n \in \mathbb{N}_a,$$

if y and z are solutions of (2.1) such that $y(m_1) = z(m_1)$, and $y - z$ has a generalized zero at m_i, $i = 2, \ldots, n$, then $y(m) = z(m)$ on $[m_1, m_n]_a$.

Note that if one assumes Condition (A) and Condition ($\bar{B}$) then the conclusion of ($\bar{B}$) is that $y(m) = z(m)$ on $\mathbb{N}_a$.

3 Uniqueness of Solutions implies Existence of Solutions

Recently, Eloe and Henderson [6] extended the original arguments due to Lasota and Opial [28] to apply to two-point boundary value problems for higher order ordinary differential equations. In this section, we develop a discrete analogue of that method produced in [6]. The compactness arguments for boundary value problems for finite difference equation are far more simple than the corresponding arguments for boundary value problems for ordinary differential equations due to the finiteness of the domain of the independent variable.

We state the first lemma without proof. It follows immediately from Condition (A) and finite induction. See [12].

Lemma 3.1. *Assume that with respect to* (2.1)*, Condition (A) is satisfied. If there exists* $m_0 \in \mathbb{N}_a$*, a sequence of solutions* $\{y_k(m)\}$ *of* (2.1)*, and* $M > 0$ *such that* $|y_k(m)| \leq M$ *for all* $m \in [m_0, m_0 + n - 1]_a$ *for all* k*, then there exists a subsequence* $\{y_{k_j}(m)\}$ *that converges pointwise to a solution of* (2.1) *on* $\mathbb{N}_a$*.*

For the sake of self-containment, we shall also state the Brouwer invariance of domain theorem.

Theorem 3.2. *If* $\mathcal{U} \subset \mathbb{R}^k$ *is open,* $\phi : \mathcal{U} \to \mathbb{R}^k$ *is one-to-one and continuous on* $\mathcal{U}$*, then* ϕ *is a homeomorphism and* $\phi(\mathcal{U})$ *is open in* $\mathbb{R}^k$*.*

We now provide sufficient conditions such that solutions of (2.1), (2.2) depend continuously on boundary points and boundary conditions. The application of the Brouwer invariance of domain theorem is standard and for applications to discrete problems we refer the reader to [7].

Theorem 3.3. *Assume that with respect to* (2.1)*, Conditions (A) and (B) are satisfied.*

(i) Given any $a \leq M_1 < M_2$, $M_1, M_2 \in \mathbb{N}_a$ *and any solution* y *of* (2.1)*, there exists* $\epsilon > 0$ *such that if*

$$|\Delta^{i-1}y(M_1) - y_{i1}| < \epsilon, \quad i = 1, \ldots, n-1, \quad \text{and } |y(M_2 + n) - y_{n1}| < \epsilon,$$

then there exists a solution z *of* (2.1) *such that*

$$\Delta^{i-1}z(M_1) = y_{l1}, \quad i = 1, \ldots, n-1, \quad z(M_2 + n) = y_{n1}.$$

(ii) If $y_{ik} \to y_i$, $i = 1, \ldots, n$ *and* z_k *is a sequence of solutions of* (2.1) *satisfying* $\Delta^{i-1}z_k(M_1) = y_{ik}$, $i = 1, \ldots, n-1$, $\Delta z(M_2 + n) = y_{nk}$*, then* z_k *converges pointwise to* y *on* $\mathbb{N}_a$*.*

Proof. Define $\mathcal{U} \subset \mathbb{R}^n$ to be the open set

$$\mathcal{U} = \{(c_1, \ldots, c_n) : c_i \in \mathbb{R}, i = 1, \ldots, n\}.$$

Let $m_0 \in \mathbb{N}_a$. Define $\phi : \mathcal{U} \to \mathbb{R}^n$ by

$$\phi(c_1, \ldots, c_n) = (y(M_1), \Delta y(M_1), \ldots, \Delta^{(n-2)}y(M_1), y(M_2 + n)),$$

where y is the unique solution of (2.1) satisfying the initial conditions $\Delta^{(i-1)}y(m_0) = c_i$, $i = 1, \ldots, n$. Then by Lemma 3.1, ϕ is continuous on $\mathcal{U}$.

To see that ϕ is a $1-1$ map on $\mathcal{U}$ let

$$(c_1, \ldots, c_n), (d_1, \ldots, d_n) \in \mathcal{U}$$

and assume

$$\phi(c_1, \ldots, c_n) = \phi(d_1, \ldots, d_n).$$

It follows by Condition (B) that $c_i = d_i$, $i = 1, \dots, n$, since if y, z are solutions of (2.1) and $\Delta^{(i-1)}y(M_1) = \Delta^{(i-1)}z(m_1)$, $i = 1, \dots, n-1$, $y(M_2) = z(M_2)$, then $y \equiv z$ on $\mathbb{N}_a$; in particular, $c_i = \Delta^{(i-1)}y(m_0) = \Delta^{(i-1)}z(m_0) = d_i$, $i = 1, \dots, n$. Apply Brouwer's invariance of domain theorem to obtain that $\phi(\mathcal{U})$ is open in $\mathbb{R}^n$ which proves (i), and to obtain that ϕ^{-1} is continuous on $\mathcal{U}$ which proves (ii). □

With the continuous dependence of solutions of (2.1) on the boundary conditions (2.2), we can state the analogue of Lemma 3.1.

Lemma 3.4. *Assume that with respect to* (2.1)*, Conditions (A) and (B) are satisfied. If there exist* $m_1, m_2 \in \mathbb{N}_a$*,* $m_1 < m_2$*, a sequence of solutions* $\{y_k(m)\}$ *of* (2.1)*, and* $M > 0$ *such that* $|y_k(m)| \le M$ *for all* $m \in [m_1, m_1+n-2]_a \cup \{m_2+n\}$ *for all k, then there exists a subsequence* $\{y_{k_j}(m)\}$ *that converges pointwise to a solution of* (2.1) *on* $\mathbb{N}_a$*.*

We only outline the proof of Lemma 3.4 since the domain $[m_1, \dots m_2+n]_a$ is finite. Obtain the convergence of a sequence on the finite set $\{m_1, \dots m_2+n\} \cap \mathbb{N}_a$; then there is a solution of an initial value problem with initial value m_1. Apply Condition (A) or apply Lemma 3.1.

We are now in a position to adapt the method of Lasota and Opial [28] and show that the uniqueness of solutions of the boundary value problem (2.1), (2.2) implies the existence of solutions of the boundary value problem (2.1), (2.2).

Theorem 3.5. *Assume that with respect to* (2.1)*, Conditions (A) and (B) are satisfied. Then for each* $M_1 < M_2 \in \mathbb{N}_a$*,* $a_i \in \mathbb{R}$*,* $i = 1, \dots, n$*, the two point boundary value problem* (2.1)*,* (2.2) *has a unique solution.*

Proof. Let $\alpha \in \mathbb{R}$ and denote by $y(m;\alpha)$ the solution of the initial value problem (2.1), with initial conditions

$$\Delta^{i-1}y(M_1) = a_i, \quad i = 1, \dots, n-1, \quad \Delta^{n-1}y(M_1) = \alpha.$$

Let

$$\Omega = \{p \in \mathbb{R} : \text{ there exists } \alpha \in \mathbb{R} \text{ with } y(M_2+n;\alpha) = p\}.$$

The theorem is proved by showing $\Omega = \mathbb{R}$. By Lemma 2.2, $\Omega \neq \emptyset$, so the theorem is proved by showing Ω is opened and closed. Theorem 3.3 implies that Ω is open.

To show Ω is closed, let p_0 denote a limit point of Ω and let p_k denote a sequence of reals in Ω converging to p_0. Assume $y(M_2+n;\alpha_k) = p_k$ for each $k \in \mathbb{N}_1$. Set

$$M = \max\left\{ |a_1|, \max_{1=1,\dots n-2}(|a_{i+1}| + \sum_{l=0}^{i-1} \binom{i}{l} |a_{l+i}|, |p_0| + 1 \right\}.$$

Then $|y(m;\alpha_k)| \le M$ for all $m \in [M_1, M_1+n-2]_a \cup \{M_2+n\}$ for all k and by Lemma 3.4 there exists a subsequence of $\{y(m;\alpha_k)\}$ converging pointwise to a solution y of (2.1) on $\mathbb{N}_a$. In particular, $y(M_2) = p_0$ and $p_0 \in \Omega$. The theorem is proved. □

Remark 3.6. It is interesting to note, see [12] for example, that in the literature cited above, it is standard that $\{p_k\}$ converges monotonically to p_0 and then the sequences $y(m, \alpha_k)$ are monotone as well due to Condition (B) (or Condition $(\bar{B})$). Then the proof that Ω is closed is commonly obtained by obtained contradiction. Here, as in [28] or [6], the proof that Ω is closed is direct, and unlike [28] or [6]; however, the proof is independent of monotonic behavior due to the finiteness of the interval.

4 Global Uniqueness of Solutions implies Existence of Solutions

In this section we continue to assume Condition (A). We now assume $f(m, y_1, \dots, y_n)$ is monotone nondecreasing in each y_j, and strictly increasing in at least one y_j. We show that this monotonicity condition implies that Condition (B) holds. In particular, the monotonicity condition coupled with Condition (A) implies first, the uniqueness of solutions of (2.1), (2.2), and then the existence of solutions of (2.1), (2.2).

Theorem 4.1. *Assume that with respect to* (2.1)*, Condition (A) is satisfied. Assume in addition, that for each* $j \in \{1, \dots, n\}$,

$$\frac{\partial f}{\partial y_j}(m, y_1, \dots, y_n) = f_{y_j}(m, y_1, \dots, y_n)$$

exists and assume $f_{y_j} \geq 0$ *on* $\mathbb{N}_a \times \mathbb{R}^n$*. Assume that for at least one* j *the inequality* $f_{y_j} > 0$ *is strict. Then solutions of the boundary value problem* (2.1)*,* (2.2) *are unique if they exist.*

Proof. Assume that y and z are distinct solutions of the boundary value problem (2.1), (2.2). We first argue that there exists $M_3 \in [M_1 + n - 1, \dots, M_2 + n - 1]_a$ such that $y - z$ has a generalized zero at M_3. So, for the sake of contradiction, assume $y - z$ is of constant sign on $[M_1 + n - 1, \dots, M_2 + n - 1]_a$ and without loss of generality assume $(y - z))(m) > 0$ for $m \in [M_1 + n - 1, M_2 + n - 1]_a$. Set $u(m) = (y - z)(m)$. Then

$$\Delta^n u(m) = f(m, y(m), \dots, y(m+n-1)) - f(m, z(m), \dots, z(m+n-1)) > 0, \quad (4.1)$$

$m \in [M_1+n-1, M_2]_a$. This implies $\Delta^{n-1}u(m)$ is increasing on $[M_1, M_2+1]_a$. Repeated applications of Rolle's theorem (Lemma 2.3) implies $\Delta^{n-1}u(m)$ has a generalized zero in $[M_1, M_2 + 1]_a$ and (4.1) implies there is at most one generalized zero of $\Delta^{n-1}u(m)$ in $[M_1, M_2 + 1]_a$. Thus, $\Delta^{n-1}u(M_1) \leq 0$. We assume y and z are distinct, and so it follows that $\Delta^{n-1}u(M_1) < 0$ since if $\Delta^{n-1}u(M_1) = 0$, y and z would then satisfy the same initial conditions at M_1. Conclude that u satisfies

$$\Delta^{i-1}u(M_1) = 0, \quad i = 1, 1, \dots, n-2, \quad u^{n-1}(M_1) < 0,$$

which implies $u(M_1 + n - 1) = \Delta^{n-1}u(M_1) < 0$. This contradicts that $u(m) = (y - z)(m) > 0$ on $[M_1 + n - 1, \dots, M_2 + n - 1]_a$. Thus, there exists $M_3 \in [M_1 + n - 1, \dots, M_2 + n - 1]_a$ such that $y - z$ has a generalized zero at M_3.

Let

$$\mu = \inf\{m \in [M_1 + n - 1, \dots, M_2 + n - 1]_a : (y - z) \text{ has a generalized zero at } m\}.$$

If $\mu > M_1 + n - 1$, apply the argument in the preceding paragraph employing repeated applications of Rolle's theorem and show there exists $M_3 \in [M_1 + n - 1, \dots, \mu - 1]_a$ such that $y - z$ has a generalized zero at M_3. This contradicts the definition of μ so $\mu = M_1 + n - 1$.

Note that $\Delta^{i-1}u(M_1) = 0$, $i = 1, \dots, n - 1$ and u has a generalized zero at $M_1 + n - 1$. This implies that $u(M_1 + n - 1) = 0$ and now, $\Delta^{i-1}u(M_1) = 0$, $i = 1, \dots, n$. So y and z, solutions of (2.1), satisfy the same initial conditions at M_1; Lemma 2.2 implies $y \equiv z$ on $\mathbb{N}_a$ contradicting the original assumption that y and z are distinct solutions of (2.1) on $\mathbb{N}_a$. □

We close the article with two corollaries. The first corollary is the main result of the article.

Corollary 4.2. *Assume that with respect to* (2.1)*, Condition (A) is satisfied. Assume in addition that for each* $j \in \{1, \dots, n\}$,

$$\frac{\partial f}{\partial y_j}(m, y_1, \dots, y_n) = f_{y_j}(m, y_1, \dots, y_n)$$

exists and assume $f_{y_j} \geq 0$ *on* $\mathbb{N}_a \times \mathbb{R}^n$*. Assume that for at least one* j *the inequality* $f_{y_j} > 0$ *is strict. Then for each* $M_1 < M_2 \in \mathbb{N}_a$, $a_i \in \mathbb{R}$, $i = 1, \dots, n$*, the two point boundary value problem* (2.1)*,* (2.2) *has a unique solution.*

Proof. Apply Theorem 4.1 and Condition B is satisfied. Now apply Theorem 3.5. □

In general, uniqueness assumptions such as Condition (B) or Condition $(\bar{B})$ are strong conditions. In the case of ordinary differential equations, the analogues of Condition (B) or Condition $(\bar{B})$ can be verified in the case of Lipschitz equations on small enough intervals; hence, for ordinary differential equations, uniqueness conditions such as Condition (B) or Condition $(\bar{B})$ tend to imply local uniqueness implies existence results [11]. In the case of finite difference equations with step size 1, the local flavor of the analogous results would be contained in controlling the size of Lipschitz coefficients. So it is important to note the Corollary 4.2 is truly a global uniqueness implies existence type result.

To state the second corollary, define the eigenvalue problem

$$\Delta^n y(m) = \lambda \sum_{j=0}^{n-1} c_j y(m + j - 1), \quad m \in [M_1, M_2]_a, \tag{4.2}$$

$$\Delta^{i-1}y(M_1) = 0, \quad i = 1, \dots, n-1, \quad y(M_2 + n) = 0, \tag{4.3}$$

where

$$M_1 < M_2 \in \mathbb{N}_a.$$

Corollary 4.3. *Assume* $c_j \geq 0$, $j = 0, \dots, n-1$, *and assume that for some* $j \in \{0, \dots, n-1\}$, $c_j > 0$. *If* λ *is an eigenvalue of* (4.2), (4.3) *then* $\lambda < 0$.

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