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Existence and Boundedness Results for Solutions of Integral Dynamic Equations on Time Scales

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Abstract

In this study, we apply the concept of resolvent that was developed in our paper [Adıvar, M. and Raffoul, Y. N., Appl. Math. Comput., (273), 258–266, 2016] for vector Volterra integral dynamic equations on time scales and show the existence and boundedness of solutions. The resolvent is an abstract term that makes it difficult, if not impossible, to make efficient use of it. However, with the help of Lyapunov functionals and variation of parameters, we are able to verify all the conditions that are related to the resolvent. This paper does provide not only generalized results over the time scales but also completely new results even for well-known particular time scales. All results of this paper are new for the summation equations, while some results are new for integral equations in the continuous case.

Keywords: Integral dynamic equation, Lyapunov, resolvent, time scales, Volterra.
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1 Introduction and Preliminaries

In this research, we apply the concept of resolvent that was obtained in [2] for vector Volterra integral dynamic equations on time scales and show the existence of a maximal

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solution, the continuation of solutions and boundedness of solutions. The notion of resolvent is an abstract term , and hence satisfying all conditions requires good care and a deep understanding of the use of Lyapunov functionals. Thus we consider the integral dynamic equation

$$x(t) = f(t) + \int_0^t C(t,s)x(s)\Delta s,\ t \in [0,\infty)_{\mathbb{T}} =: [0,\infty) \cap \mathbb{T} \tag{1.1}$$

and various forms of it, where x and f are n-vectors ($n \geq 1$), f is continuous on $[0,\infty)_{\mathbb{T}}$, while C is an $n \times n$ matrix valued function that is continuous on $\{(s,t) : 0 \leq s \leq t < \infty\} \cap (\mathbb{T} \times \mathbb{T})$.

For $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm of x. For any $n \times n$ matrix A, define the norm of A by $|A| = \sup\{|Ax| : |x| \leq 1\}$. Let $\mathcal{M}$ denote the vector space of bounded and continuous functions $\phi : [0,\infty)_{\mathbb{T}} \to \mathbb{R}^n$ with the norm

$$\|\phi\| := \sup_{s \in [0,\infty)_{\mathbb{T}}} |\phi(s)|.$$

One can easily see that $(\mathcal{M}, ||\cdot||)$ is a Banach space.

In [2] the authors showed that the resolvent equation associated with (1.1) is given by

$$r(t,s) = -C(t,s) + \int_{\sigma(s)}^t r(t,u)C(u,s)\Delta u. \tag{1.2}$$

and consequently, the solution of (1.1) is given by the variation of parameters formula

$$x(t) = f(t) - \int_0^t r(t,u)f(u)\Delta u. \tag{1.3}$$

In [1] the authors use the topological degree method and Lyapunov functionals on time scales to show existence of periodic solutions of integro-dynamic equations on time scales. It is worth mentioning here that in [1] Lyapunov **functionals** were the first time ever used to analyze solutions of any dynamic equations on time scales. However, the use of Lyapunov functionals for analyzing behavior of solution of integral dynamic equation on time scales is in the earliest stage of development and this paper should serve as a foundation for such study. In [7] Messina and Vecchio displayed interesting Lyapunov functionals and studied the stability of the zero solution of Volterra integral dynamic equations under bounded and unbounded perturbations. In their work they derive different but interesting formula for Δ-derivative of absolute valued functions.

In [8], the second author considered a discrete form of (1.1) and used fixed point theory along with Lyapunov type functionals to prove the existence of an asymptotically periodic solution and boundedness on all solutions. Results of this paper are totally new in the case the time scales is the set of integers; $\mathbb{T} = \mathbb{Z}$.

The authors assume the reader is familiar with time scales, and for a complete treatment of time scales, we refer to [3]. In this paper, we assume that $0 \in \mathbb{T}$, and hence,

$[0, \infty)_{\mathbb{T}} \neq \emptyset$, and it should cause no confusion to start the integral in (1.1) from any initial time t_0 instead of 0. Thus, if the time scale does not include zero, then any initial time $t_0 \in \mathbb{T}$ will do the job.

This paper is organized as follows. In Section 2, we consider the linear integral dynamic equation and use the notion of resolvent coupled with fixed point theory to prove the boundedness of solutions. In Section 3, we use fixed point theory to prove the existence and uniqueness of the solution of nonlinear integral dynamic equations. In Section 4, we employ new and exotic Lyapunov functionals to prove the continuation of solutions for all time of linear and nonlinear dynamic equations and provide examples to particular time scales.

2 Existence and Uniqueness. The Linear Case

Now we use the resolvent equation and the corresponding variation of parameters formula, along with fixed point theory to show the existence and uniqueness of the solution of the linear Volterra integral dynamic equation (1.1).

Theorem 2.1. *Assume the existence of two positive constants K and $\alpha \in (0,1)$ such that for $f \in \mathcal{M}$ we have*

$$|f(t)| \le K \text{ and } \sup_{t \in [0,\infty)_{\mathbb{T}}} \int_0^t |C(t,s)| \Delta s \le \alpha, \tag{2.1}$$

then there is a unique bounded solution of (1.1).

Proof. Define a mapping $\mathcal{M} \to \mathcal{M}$, by

$$(D\phi)(t) = f(t) + \int_0^t C(t,s)\phi(s)\Delta s.$$

Now for $\phi \in \mathcal{M}$, with $\|\phi\| \le q$ for a positive constant q we have that

$$\|(D\phi)\| \le K + \alpha q.$$

Thus $D : \mathcal{M} \to \mathcal{M}$. Left to show that D defines a contraction mapping on $\mathcal{M}$. Let $\phi, \varphi \in \mathcal{M}$. Then

$$\begin{aligned} \|(D\phi) - (D\varphi)\| &\le \sup_{t \in [0,\infty)_{\mathbb{T}}} \int_0^t |C(t,s)| \Delta s \|\phi - \varphi\| \\ &\le \alpha \|\phi - \varphi\|. \end{aligned}$$

Hence, D is a contraction and by the contraction mapping principle it has a unique solution in $\mathcal{M}$ that solves (1.1). This completes the proof. □

In the next theorem, we extend Perron's theorem for an integral equation over the reals to an arbitrary time scale. Its application is essential for the proof of our next results. For the continuous case of Perron's theorem, we refer the reader to [4, page 114]. The proof of the next theorem can be found in [2].

Theorem 2.2 (Perron)**.** *Let* $r : [0,\infty)_{\mathbb{T}} \times [0,\infty)_{\mathbb{T}} \to \mathbb{R}$ *be a continuous real valued function on* $0 \le s \le t < \infty$*. If* $\int_0^t r(t,s) f(s) \Delta s \in \mathcal{M}$ *for each* $f \in \mathcal{M}$*, then there exists a positive constant* K *such that* $\int_0^t |r(t,s)| \, \Delta s < K$*, for all* $t \in [0,\infty)_{\mathbb{T}}$*.*

Theorem 2.3. *Suppose* $r(t,s)$ *satisfies* (1.2) *and that* $f \in \mathcal{M}$*. Then every solution* x *of* (1.1) *is bounded if and only if*

$$\sup_{t \in [0,\infty)_{\mathbb{T}}} \int_0^t |r(t,s)| \Delta s < \infty \tag{2.2}$$

holds.

Proof. Suppose (2.2) holds. Then, using (1.3), it is trivial to show that $x(t)$ is bounded. If $x(t)$ and $f(t)$ are bounded, then from (1.3), we have

$$\int_0^t |r(t,s)||f(s)| \Delta s \le \beta$$

for some positive constant β, and the proof follows from Theorem 2.2. □

Theorem 2.4. *Let* C *be an* $n \times n$ *matrix. Assume the existence of a constant* $\alpha \in (0,1)$ *such that*

$$\sup_{t \in [0,\infty)_{\mathbb{T}}} \int_0^t |C(t,s)| \Delta s \le \alpha. \tag{2.3}$$

Then we have the following:

(i) If $f \in \mathcal{M}$*, then so is the solution* x *of* (1.1).

(ii) Suppose, in addition that the time scale $\mathbb{T}$ *is unbounded above and that for each* $T > 0$ *we have* $\int_0^T |C(t,s)| \Delta s \to$ *as* $t \to \infty$*. If* $f(t) \to 0$ *as* $t \to \infty$, *then so does* $x(t)$ *and* $\int_0^t r(t,s) f(s) \Delta s$*.*

(iii) $\int_0^t |r(t,s)| \Delta s \le \dfrac{\alpha}{1-\alpha}$.

Proof. The proof of (i) is the same as the proof of Theorem 2.1. For the proof of (ii) we define the set

$$M = \{\phi : [0,\infty)_{\mathbb{T}} \to \mathbb{R}^n \mid |\phi(t)| \to 0, \text{ as } t \to \infty\}.$$

For $\phi \in M$ define the mapping Q by

$$\big(Q\phi\big)(t) = f(t) + \int_0^t C(t,s)\phi(s)\Delta s.$$

Then

$$\big|\big(Q\phi\big)(t)\big| \le |f(t)| + \int_0^t |C(t,s)\phi(s)|\Delta s.$$

We already know that $f(t) \to 0$ as $t \to \infty$. Given an $\varepsilon > 0$ and $\phi \in M$, find T such that $|\phi(t)| < \varepsilon$ if $t \ge T$ and find d with $|\phi(t)| \le d$ for all $t \ge T$. For this fixed T, find $\eta > T$ such that $t \ge \eta$ implies that $\int_0^T |C(t,s)|\Delta s \le \frac{\varepsilon}{d}$. Then $t \ge \eta$ implies that

$$\begin{aligned}\int_0^t |C(t,s)|\Delta s &\le \int_0^T |C(t,s)|\Delta s + \int_T^t |C(t,s)|\Delta s \\ &\le (d\varepsilon)/d + \alpha\varepsilon < 2\varepsilon.\end{aligned}$$

Thus, $Q : M \to M$ and the fixed point satisfies $x(t) \to 0$, as $t \to \infty$, for every vector function $f \in M$. Using (1.3) we have

$$\int_0^t r(t,s)f(s)\Delta s = x(t) - f(t) \to 0, \text{ as } t \to \infty.$$

This completes the proof of (ii). Using (1.2) and (2.3) we have by changing of order of integrals

$$\begin{aligned}\int_0^t |r(t,s)||\Delta s &\le \int_0^t |C(t,s)|\Delta s + \int_0^t \int_{\sigma(s)}^t |r(t,u)||C(u,s)|\Delta u\Delta s \\ &\le \int_0^t |C(t,s)|\Delta s + \int_0^t |r(t,u)|\Delta u \int_0^t |C(u,s)|\Delta s \\ &\le \alpha + \alpha \int_0^t |r(t,u)|\Delta u.\end{aligned}$$

Therefore,

$$(1-\alpha)\int_0^t |r(t,s)|\Delta s \le \alpha.$$

That is,

$$\sup_{t\in[0,\infty)_{\mathbb{T}}} \int_0^t |r(t,s)|\Delta s \le \frac{\alpha}{1-\alpha}.$$

The proof is complete. $\square$

Example 2.5. Suppose there is a function $d : [0, \infty)_{\mathbb{T}} \to (0, 1]$, with $d(t) \downarrow 0$ with

$$\sup_{t \in [0,\infty)_{\mathbb{T}}} \int_0^t |C(t, s)|(d(s)/d(t))\Delta s \leq \alpha, \ \alpha \in (0, 1) \tag{2.4}$$

and

$$|f(t)| \leq kd(t) \tag{2.5}$$

for some positive constant k. Then the unique solution x of (1.1) is bounded and goes to zero as t approaches infinity. Moreover, $\int_0^t r(t, s)f(s)\Delta s \to 0, \text{ as } t \to \infty.$

Proof. Let

$$\mathcal{M} = \left\{ \phi : [0, \infty)_{\mathbb{T}} \to \mathbb{R}^n \mid |\phi|_d \leq \sup_{t \in [0,\infty)_{\mathbb{T}}} \frac{|\phi(t)|}{|d(t)|} < \infty \right\}.$$

Then $\big(\mathcal{M}, |\cdot|_r\big)$ is a Banach space. For $\phi \in \mathcal{M}$, define the mapping Q by

$$\big(Q\phi\big)(t) = f(t) + \int_0^t C(t, s)\phi(s)\Delta s.$$

Then,

$$\begin{aligned}
\big|\big(Q\phi\big)(t)\big|/d(t) &\leq |f(t)|/d(t) + \int_0^t |C(t, s)|(d(s)/d(t))|\phi(s)|/d(s)\Delta s \\
&\leq k + |\phi|_d \int_0^t |C(t, s)|(d(s)/d(t))\Delta s \\
&\leq k + \alpha|\phi|_d,
\end{aligned}$$

which shows that $Q\phi \in \mathcal{M}$. Let $\phi, \eta \in \mathcal{M}$, then we readily have that

$$\Big|\big(Q\phi\big)(t)\big) - \big(Q\eta\big)(t)\Big|/d(t) \leq \alpha|\phi - \eta|_d$$

and so we have Q is a contraction on $\mathcal{M}$ and therefore it has a unique fixed point $x(t)$ in $\mathcal{M}$ that solves (1.1). Moreover, $\sup_{t \in [0,\infty)} \frac{|x(t)|}{|d(t)|} < \infty$ implies that $|x(t)| \leq k^* d(t) \to 0$, as $t \to \infty$. Also by (2.5) we have $|f(t)| \to 0$ as $t \to \infty$ and hence using (1.3) we have

$$\int_0^t |r(t, s)f(s)|\Delta s \leq |f(t)| + |x(t)| \to 0 \text{ as } t \to \infty.$$

This completes the proof. □

3 Existence and Uniqueness of Nonlinear Volterra Dynamic Equations

In this section, we consider the system of nonlinear Volterra dynamic equations of the form

$$x(t) = f(t) + \int_0^t g(t, s, x(s))\Delta s \tag{3.1}$$

in which x is an n vector, $f : [0, \infty)_{\mathbb{T}} \to \mathbb{R}^n$, and $g : \pi \times \mathbb{R}^n \to \mathbb{R}^n$ is C_{rd} (rd-continuous) in t and s and continuous in x, where $\pi = \big\{(t, s) \in [0, \infty)_{\mathbb{T}} \times [0, \infty)_{\mathbb{T}} : 0 \le s \le t < \infty\big\}$. Due to the nonlinearity of the function g, the resolvent method for (1.1) is not applicable to (3.1). Instead, we will use the contraction mapping principle and show the existence of solutions of (3.1) over a short interval, say $[0, T]_{\mathbb{T}} \subset [0, \infty)_{\mathbb{T}}$.

Theorem 3.1. *Suppose there are positive constants a, b, and $\alpha \in (0, 1)$. Suppose*
(a) f is continuous on $[0, a]_{\mathbb{T}} \subset [0, \infty)_{\mathbb{T}}$,
(b) g is continuous on

$$U = \big\{(t, s, x) : (t, s) \in [0, \infty)_{\mathbb{T}} \times [0, \infty)_{\mathbb{T}}, 0 \le s \le t < \infty \text{ , and } |x - f(t)| \le b\big\},$$

(c) g satisfies a Lipschitz condition with respect to x on U

$$\big|g(t, s, x) - g(t, s, y)\big| \le L|x - y|$$

for $(t, s, x),\ (t, s, y) \in U$.
Then there is a unique solution of (3.1) on $[0, T]_{\mathbb{T}} \subset [0, \infty)_{\mathbb{T}}$, where

$$T := \min\{a, b/\widetilde{M}, c\},$$

$\widetilde{M} := \max_U \big|g(t, s, x)\big|$*, and $c := \alpha/L$ for fixed α.*

Proof. Let Ω_b denote the space of C_{rd} functions $\phi : [0, T]_{\mathbb{T}} \to \mathbb{R}^n$, such that

$$||\phi - f|| = \max_{t \in [0,T]_{\mathbb{T}}} \{|\phi(t) - f(t)|\} \le b,$$

where for $\Psi \in \Omega_b$ and the norm $|| \cdot ||$ is taken to be and $\|\Psi\| = \max_{t \in [0,T]_{\mathbb{T}}} \{|\Psi_i(t)|\}$. Clearly, the norm defines the metric ρ. Let $\phi \in \Omega_b$ and define an operator $D : \Omega_b \to \Omega_b$, by

$$D(\phi)(t) = f(t) + \int_0^t g(t, s, \phi(s))\Delta s.$$

Since ϕ is C_{rd} we have that $D(\phi)$ is C_{rd} too and

$$||D(\phi) - f|| = \max_{t \in [0,T]_{\mathbb{T}}} \Big| \int_0^t g(t, s, \phi(s))\Delta s\Big|$$

$$\leq \widetilde{M}T \leq b.$$

This shows that D maps Ω_b into itself. For the contraction part, we let $\phi, \psi \in \Omega_b$. Then

$$\begin{aligned} ||D(\phi) - D(\psi)|| &= \max_{t\in[0,T]_{\mathbb{T}}} \Big| \int_0^t g(t,s,\phi(s))\Delta s - \int_0^t g(t,s,\psi(s))\Delta s \Big| \\ &\leq \max_{t\in[0,T]_{\mathbb{T}}} \int_0^t \Big| g(t,s,\phi(s)) - g(t,s,\psi(s)) \Big| \Delta s \\ &\leq \max_{t\in[0,T]_{\mathbb{T}}} L \int_0^t \Big| \phi(s)) - \psi(s)) \Big| \Delta s \\ &\leq T \max_{t\in[0,T]_{\mathbb{T}}} L \Big| \phi(s)) - \psi(s)) \Big| \\ &= TL||\phi - \psi|| \leq cL||\phi - \psi|| = \alpha||\phi - \psi||. \end{aligned}$$

Thus, by the contraction mapping principle there is a unique function $x \in \Omega_b$ with

$$D(x)(t) = x(t) = f(t) + \int_0^t g(t,s,x(s))\Delta s.$$

□

4 Continuation of Solutions

In this section we define the concept of maximal solution of nonlinear Volterra integral dynamic equations and use a combination of Gronwall's inequality and Lyapunov functionals to show solutions can be continued for all time $t \in [0,\infty)_{\mathbb{T}}$. We begin with a definition on maximal solution.

Definition 4.1. Let $f : [0,\infty)_{\mathbb{T}} \to \mathbb{R}$ be rd-continuous and for

$$\Upsilon := \big\{(t,s,x) : (t,s) \in [0,\infty)_{\mathbb{T}} \times [0,\infty)_{\mathbb{T}} : 0 \leq s \leq t < \infty \text{ and } x \in \mathbb{R}\big\},$$

let $g : \Upsilon \to \mathbb{R}$. Let x be an rd-continuous solution of the scalar Volterra integral dynamic equation

$$x(t) = f(t) + \int_0^t g(t,s,x(s))\Delta s \tag{4.1}$$

on $[0,A]_{\mathbb{T}} \subset [0,\infty)_{\mathbb{T}}$ with the property that if y is any other solution, then as long as $y(t)$ exists and $t \leq A$ we have $y(t) \leq x(t)$. Then x is called the maximal solution of (4.1). Minimal solutions are defined by asking $y(t) \geq x(t)$.

Of course, if solutions are unique, then the unique solution is the maximal and the minimal solution.

Theorem 4.2. *Let x be the maximal solution of the scalar equation*

$$x(t) = B + \int_0^t p(s, x(s))\Delta s$$

on $[0, A]_{\mathbb{T}} \subset [0, \infty)_{\mathbb{T}}$ where B is a constant, and let $p : [0, A]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ be rd-continuous and $p(t, \cdot)$ is nondecreasing in it's second argument when $t \in [0, A]_{\mathbb{T}}$. If y is an rd-continuous scalar function on $[0, A]_{\mathbb{T}} \subset [0, \infty)_{\mathbb{T}}$ satisfying

$$y(t) \leq B + \int_0^t p(s, y(s))\Delta s, \quad y_0 = y(0) \leq B,$$

then $y(t) \leq x(t)$ on $[0, A]_{\mathbb{T}} \subset [0, \infty)_{\mathbb{T}}$.

Proof. Let

$$Y(t) = y_0 + \int_0^t p(s, y(s))\Delta s,$$

so that $y(t) \leq Y(t)$. Then we have

$$Y^{\Delta}(t) = p(t, y(t)) \leq p(t, Y(t)),$$

by monotonicity. Hence

$$Y^{\Delta}(t) \leq p(t, Y(t)), \quad Y(0) = y_0.$$

From this we conclude that $Y(t) \leq x(t)$ according to [6, Theorem 2.1] or [9, Theorem 6.1]. This completes the proof. □

In the next theorem we utilize Gronwall's inequality to show solutions of a nonlinear Volterra dynamic equation are can be continued for all time.

Theorem 4.3. *Let $g : [0, \infty)_{\mathbb{T}} \times [0, \infty)_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ be rd-continuous, and suppose that for each $T > 0$ there is an rd-continuous scalar function $P(\cdot, T)$ such that*

$$P(\cdot, T) \in \mathcal{R}^{+} \text{(positively regressive)}$$

and

$$|g(t, s, x)| \leq P(s, T)\big(1 + |x|\big) \text{ for } 0 \leq s \leq t \leq T.$$

Let $f : [0, \infty)_{\mathbb{T}} \to \mathbb{R}$ be rd-continuous. If x is a solution of

$$x(t) = f(t) + \int_0^t g(t, s, x(s))\Delta s \tag{4.2}$$

on some interval $[0, \alpha]_{\mathbb{T}}$, then it is bounded, and, hence, is continuable to all of $[0, \infty)_{\mathbb{T}}$ provided that

$$|f(t)| + \int_0^{\alpha} P(s, \alpha)\Delta s \leq Q \tag{4.3}$$

where $\alpha \in [0, \infty)_{\mathbb{T}}$.

Proof. Let x be a solution on $[0,\alpha)_{\mathbb{T}}$. Then for $0 \le t < \alpha$, we have

$$\begin{aligned} |x(t)| &\le |f(t)| + \int_0^t P(s,\alpha)(1+|x(s)|)\Delta s \\ &\le Q + \int_0^t P(s,\alpha)|x(s)|\Delta s. \end{aligned}$$

Thus, by Gronwall's inequality [3] and (4.3) we have

$$|x(t)| \le Q e_P(t,0) < \infty.$$

This completes the proof. □

In the next result, we provide some sufficient conditions for the continuation of the solutions of the equation

$$x(t) = f(t) + \int_0^t C(t,s)\gamma(x(s))\Delta s. \tag{4.4}$$

Theorem 4.4. *Let $f : [0,\infty)_{\mathbb{T}} \to \mathbb{R}$ be rd-continuous, and $C(t,s)$ be a scalar continuous function on $[0,\infty)_{\mathbb{T}} \times [0,\infty)_{\mathbb{T}}$. Let f be a continuously Δ-differentiable function. In addition, we assume $C_t(t,s)$ exists and is continuous. Let $\zeta \in (0,1)$ be a constant. Suppose that $\gamma : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying*

$$|\gamma(x(t))| \le \zeta |x(t)| \tag{4.5}$$

for each fixed x that solves (4.4) *and for all $t \in \mathbb{T}$. Define the function $\beta : \mathbb{T} \to \mathbb{R}$ by*

$$\beta(t) := \begin{cases} -\lambda - \dfrac{2}{\mu(t)} + \zeta|C(\sigma(t),t)| & \text{if } t \in \mathbb{T}_- \\ -\lambda + \zeta|C(\sigma(t),t)| & \text{if } t \in \mathbb{T}_+ \end{cases}, \tag{4.6}$$

where

$$\begin{aligned} \mathbb{T}_- &:= \{s \in [0,\infty)_{\mathbb{T}} : x(s)\, x^\sigma(s) < 0\}, \\ \mathbb{T}_+ &:= \{s \in [0,\infty)_{\mathbb{T}} : x(s)\, x^\sigma(s) \ge 0\}. \end{aligned}$$

Suppose that for each $T \in [0,\infty)_{\mathbb{T}}$ and for all $t \in \mathbb{T}$, the following inequality holds

$$\beta(t) + \int_{\sigma(t)}^T |C_u(u,t)|\Delta u \le 0. \tag{4.7}$$

Then each solution x of (4.4) *can be continued for all future times.*

Proof. We show that if a solution x is defined on $[0,T)_{\mathbb{T}}$, it is bounded. Let λ be a positive constant such that $|f^{\Delta}(t)| \leq \lambda$ on $[0,T)_{\mathbb{T}}$. Define the functional $V(t) = V(t,x(\cdot))$ by

$$V(t) = e_{\ominus\lambda}(t,0)\Big[1 + |x(t)| + \int_0^t \int_t^T |C_u(u,s)|\Delta u|\gamma(x(s))|\Delta s\Big]. \tag{4.8}$$

From [1, Lemma 5] we have that

$$|x\left(t\right)|^{\Delta} = \begin{cases} \dfrac{x\left(t\right)}{|x\left(t\right)|}x^{\Delta}\left(t\right) & \text{if } t \in \mathbb{T}_+ \\ -\dfrac{2}{\mu\left(t\right)}\left|x\left(t\right)\right| - \dfrac{x\left(t\right)}{|x\left(t\right)|}x^{\Delta}\left(t\right) & \text{for } t \in \mathbb{T}_- \end{cases}. \tag{4.9}$$

Thus for $t \in \mathbb{T}_+$, using the product rules, we get that

$$\begin{aligned} \dot{V}(t,x) &= \ominus\lambda e_{\ominus\lambda}(t,0)\Big[1 + |x(t)| + \int_0^t \int_t^T |C_u(u,s)|\Delta u|\gamma(x(s))|\Delta s\Big] \\ &+ e_{\ominus\lambda}(\sigma(t),0)\Big[|x(t)|^{\Delta} + \int_{\sigma(t)}^T |C_u(u,t)|\Delta u|\gamma(x(t))| \\ &- \int_0^t |C_t(t,s)||\gamma(x(s))|\Delta s\Big]. \end{aligned} \tag{4.10}$$

Rearranging the expression (4.10) gives

$$\begin{aligned} \dot{V}(t,x) &= e_{\ominus\lambda}(t,0)\Big[\ominus\lambda + \ominus\lambda|x| + \ominus\lambda \int_0^t \int_t^T |C_u(u,s)|\Delta u|\gamma(x(s))|\Delta s\Big] \\ &+ e_{\ominus\lambda}(\sigma(t),0)\Big[\frac{x}{|x|}\Big(f^{\Delta}(t) + C(\sigma(t),t)\gamma(x) + \int_0^t C_t(t,s)\gamma(x(s))\Delta s\Big) \\ &+ \int_{\sigma(t)}^T |C_u(u,s)|\Delta u|\gamma(x(t))| - \int_0^t |C_t(t,s)||\gamma(x(s))|\Delta s\Big] \\ &\leq e_{\ominus\lambda}(t,0)\Big[\ominus\lambda + \ominus\lambda|x|\Big] \\ &+ e_{\ominus\lambda}(\sigma(t),0)\Big[|f^{\Delta}(t)| + \zeta|C(\sigma(t),t)||x(t)| + \zeta\int_{\sigma(t)}^T |C_u(u,t)|\Delta u|x(t)|\Big] \\ &\leq e_{\ominus\lambda}(t,0)\Big[\ominus\lambda + \ominus\lambda|x|\Big] \\ &+ e_{\ominus\lambda}(\sigma(t),0)\Big[\lambda + \zeta|C(\sigma(t),t)||x(t)| + \zeta\int_{\sigma(t)}^T |C_u(u,t)|\Delta u|x(t)|\Big], \end{aligned} \tag{4.11}$$

where (4.9) is used to calculate the Δ-derivative of $|x(t)|$. Since we have

$$e_{\ominus\lambda}(\sigma(t),0) = \Big(1 + \mu(t)\ominus\lambda\Big)e_{\ominus\lambda}(t,0)$$

$$\begin{aligned} &= \Big(1+\frac{\mu(t)(-\lambda)}{1+\mu(t)\lambda}\Big)e_{\ominus\lambda}(t,0) \\ &= \frac{1}{1+\mu(t)\lambda}e_{\ominus\lambda}(t,0), \end{aligned} \tag{4.12}$$

the expression (4.11) reduces to

$$\begin{aligned} \dot{V}(t,x) &\leq \frac{-\lambda}{1+\mu(t)\lambda}e_{\ominus\lambda}(t,0)|x(t)| \\ &+ \frac{1}{1+\mu(t)\lambda}e_{\ominus\lambda}(t,0)\Big(\zeta|C(\sigma(t),t)|+\zeta\int_{\sigma(t)}^{T}|C_u(u,t)|\Delta u\Big)|x(t)| \\ &\leq \frac{1}{1+\mu(t)\lambda}e_{\ominus\lambda}(t,0)\Big(\beta(t)+\zeta\int_{\sigma(t)}^{T}|C_u(u,t)|\Delta u\Big)|x(t)| \\ &\leq 0. \end{aligned}$$

On the other hand, for $t\in\mathbb{T}_-$, we have

$$\begin{aligned} \dot{V}(t,x) &= e_{\ominus\lambda}(t,0)\Big[\ominus\lambda+\ominus\lambda|x|+\ominus\lambda\int_0^t\int_t^T|C_u(u,s)|\Delta u|\gamma(x(s))|\Delta s\Big] \\ &+ e_{\ominus\lambda}(\sigma(t),0)\Big[-\frac{2}{\mu(t)}|x(t)|-\frac{x(t)}{|x(t)|}\Big(f^{\Delta}(t)+C(\sigma(t),t)\gamma(x(t)) \\ &+ \int_0^t C_t(t,s)\gamma(x(s))\Delta s\Big) \\ &+ \int_{\sigma(t)}^{T}|C_u(u,t)|\Delta u|\gamma(x(t))|-\int_0^t|C_t(t,s)||\gamma(x(s))|\Delta s\Big)\Big] \\ &\leq e_{\ominus\lambda}(t,0)\Big[\ominus\lambda+\ominus\lambda|x|\Big] \\ &+ e_{\ominus\lambda}(\sigma(t),0)\Big[-\frac{2}{\mu(t)}|x(t)|+|f^{\Delta}(t)|+|C(\sigma(t),t)||\gamma(x)| \\ &+ \int_{\sigma(t)}^{T}|C_u(u,s)|\Delta u|\gamma(x(t))|\Big)\Big] \\ &\leq e_{\ominus\lambda}(t,0)\Big[\ominus\lambda+\ominus\lambda|x|\Big] \\ &+ \frac{1}{1+\mu(t)\lambda}e_{\ominus\lambda}(t,0)\Big[-\frac{2}{\mu(t)}|x(t)|+\lambda+\zeta|C(\sigma(t),t)||x(t)| \\ &+ \zeta\int_{\sigma(t)}^{T}|C_u(u,t)|\Delta u|x(t)|\Big] \\ &= e_{\ominus\lambda}(t,0)\Big[\ominus\lambda+\ominus\lambda|x(t)|-\frac{2}{\mu(t)}\frac{1}{1+\mu(t)\lambda}|x(t)|+\frac{\lambda}{1+\mu(t)\lambda} \\ &+ \frac{1}{1+\mu(t)\lambda}\Big(\zeta|C(\sigma(t),t)|+\zeta\int_{\sigma(t)}^{T}|C_u(u,s)|\Delta u\Big)|x(t)|\Big] \end{aligned}$$

$$
\begin{aligned}
&= \frac{e_{\ominus\lambda}(t,0)}{1+\mu(t)\lambda}\Big(-\lambda - \frac{2}{\mu(t)} + \zeta|C(\sigma(t),t)| + \zeta\int_{\sigma(t)}^{T}|C_u(u,t)|\Delta u\Big)|x(t)| \\
&= \frac{e_{\ominus\lambda}(t,0)}{1+\mu(t)\lambda}\Big(\beta(t) + \int_{\sigma(t)}^{T}|C_u(u,t)|\Delta u\Big)|x(t)| \\
&\leq 0,
\end{aligned}
$$

where we use (4.9) to calculate the Δ-derivative of $|x(t)|$. This shows that V is decreasing and hence for all $t \in [0,T)_{\mathbb{T}}$, we have that

$$
\begin{aligned}
V(t) &\leq V(0) \\
&\leq 1 + |x(0)|.
\end{aligned}
$$

This shows that $V(t)$ is bounded on $[0,T)_{\mathbb{T}}$. Now using (4.8) and for $t \in [0,T)_{\mathbb{T}}$ we arrive at

$$
\begin{aligned}
|x(t)| &\leq e_{\lambda}(t,0)V(t) \\
&\leq e_{\lambda}(t,0)V(0) \\
&\leq e_{\lambda}(t,0)\big(1+|x(0)|\big) \\
&\leq e_{\lambda}(T,0)\big(1+|x(0)|\big) < \infty.
\end{aligned}
$$

This shows x is bounded on $[0,T)_{\mathbb{T}}$ and since T is arbitrary, solutions can be continued for all future times. This completes the proof. □

Remark 4.5. Theorem 4.4 is new for the continuous case, $\mathbb{T} = \mathbb{R}$, and for the discrete case, $\mathbb{T} = \mathbb{Z}$. Note that if $\mathbb{T} = \mathbb{R}$, then the set $\mathbb{T}_{-}$ is empty (since $x^{\sigma}(t) = x(t)$ for $t \in \mathbb{R}$) and the function β in (4.6) turns into the following

$$
\beta(t) := -\lambda + \zeta|C(\sigma(t),t)|. \tag{4.13}
$$

We illustrate the results of Theorem 2.1, Theorem 2.4, and Theorem 4.4 in the following example.

Example 4.6. Let $\mathbb{T} = 2^{\mathbb{N}_0} = \{2^n : n = 0,1,2,\ldots\}$ and $t_0 = 1$. Then for $t = 2^n$, we have $\sigma(t) = 2^{n+1}$ and $\mu(t) = 2^n$. Setting $f(t) = \dfrac{1}{t} = 2^{-n}$, $s = 2^m$, and $C(t,s) = \dfrac{s}{t^2} = 2^{m-2n}$, the equation (1.1) turns into

$$
x(2^n) = 2^{-n} - \sum_{m=1}^{n-1} 2^{2(m-n)} x\left(2^m\right),\ n = 0,1,\ldots, \tag{4.14}
$$

(see [3, Theorem 1.79 (ii)]). Since $f(t)$ and $\displaystyle\sum_{m=1}^{n-1} 2^{2m-2n} = 1 - 2^{-2n+2}$ are bounded we have that by Theorem 2.1 that solutions of (4.14) are bounded. Similarly, for $T = 2^k \in$

$2^{\mathbb{N}_0}$, we have that

$$\sum_{m=1}^{k-1} 2^{2(m-n)} = \frac{1}{3}2^{-2n}(2^{2k}-4) \to 0, \text{ as } n \to \infty.$$

Thus by (ii.) of Theorem 2.4, the solution x of (4.14) approaches zero as $t \to \infty$. In addition, we have that $\int_0^t r(t,s)f(s)\Delta s = \sum_{m=1}^{n-1} r(2^n, 2^m) \to 0$, as $t \to \infty$. Next we apply Theorem 4.4. If we set $\lambda = 1$, $\zeta = 1$, and $\gamma(x) = x$, then we have by (4.6) that

$$\beta(t) = \begin{cases} -1 - 7(2^{-n-2}) & \text{for } t \in \mathbb{T}_- \\ -1 + 2^{-n-2} & \text{for } t \in \mathbb{T}_+ \end{cases}. \tag{4.15}$$

Setting $u = 2^m$ and $t = 2^n$ one can find

$$|C_u(u,t)| = \frac{t(u+\sigma(u))}{u^2\sigma(u)^2} = \frac{3}{4}\frac{2^n}{2^{3m}}.$$

For $2^{n+1} = \sigma(t) \leq 2^k = T$, we have

$$\int_{\sigma(t)}^{T} |C_u(u,t)|\Delta u = \frac{3}{4}2^n \sum_{m=n+1}^{k-1} 2^{-2m} = 2^{n-2}(2^{-2n} - 2^{-2k}) = 2^{-n-2} - 2^{-2k-n-2}.$$

Then, the condition (4.7) turns into

$$\beta(t) + \int_{\sigma(t)}^{T} |C_u(u,t)|\Delta u = \begin{cases} -1 - (6 + 2^{-2k})2^{-n-2} & \text{for } t \in \mathbb{T}_- \\ -1 + (2 - 2^{-2k})2^{-n-2} & \text{for } t \in \mathbb{T}_+ \end{cases} < 0\,. \tag{4.16}$$

It can be similarly shown that the condition (4.7) holds for $T < \sigma(t)$. Thus by Theorem 4.4 all solutions of (4.14) remain bounded for all future time.

Theorem 4.4 shows that if γ satisfies (4.5), then the sign of γ has nothing to do with continuation. In the next result we show that if the *signs are right*, then the growth of γ has nothing to do with continuation of the solutions of

$$x(t) = f(t) + \int_0^t C(t,s)\gamma(x(\sigma(s)))\Delta s. \tag{4.17}$$

Thus, we replace the condition (4.5) with $x\gamma(x) \geq 0$.

Theorem 4.7. *Let $f : [0,\infty)_{\mathbb{T}} \to \mathbb{R}$ be rd-continuous, and $C(t,s)$ be a scalar continuous function on $[0,\infty)_{\mathbb{T}} \times [0,\infty)_{\mathbb{T}}$. Let f be a continuously Δ-differentiable function.*

In addition, we assume $C_t(t,s)$ exists and continuous. Let $\gamma : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying

$$x(t)\gamma(x(t)) \geq 0 \textit{ for all } t \in \mathbb{T}. \tag{4.18}$$

Suppose that for each $T \in [0,\infty)_{\mathbb{T}}$ we have

$$C(\sigma(t),t) + \int_{\sigma(t)}^{T} |C_u(u,t)|\Delta u \leq 0 \textit{ for all } t \in \mathbb{T}. \tag{4.19}$$

Then each solution of (4.17) *can be continued for all future times.*

Proof. We show that if a solution x is defined on $[0,T)_{\mathbb{T}}$, it is bounded. Let λ be a positive constant such that $|f^{\Delta}(t)| \leq \lambda$ on $[0,T)_{\mathbb{T}}$. Define the functional $V(t) = V(t,x(\cdot))$ by

$$V(t) = e_{\ominus\lambda}(t,0)\Big[1 + |x(t)| + \int_0^t \int_t^T |C_u(u,s)|\Delta u|\gamma(x^{\sigma}(s))|\Delta s\Big]. \tag{4.20}$$

Thus for $t \in \mathbb{T}_+$, using the product rules we get that

$$\begin{aligned}
\dot{V}(t,x) &= \ominus\lambda e_{\ominus\lambda}(t,0)\Big[1 + |x(t)| + \int_0^t \int_t^T |C_u(u,s)|\Delta u|\gamma(x^{\sigma}(s))|\Delta s\Big] \\
&+ e_{\ominus\lambda}(\sigma(t),0)\Big[|x(t)|^{\Delta} + \int_{\sigma(t)}^{T} |C_u(u,t)|\Delta u|\gamma(x^{\sigma}(t))| \\
&- \int_0^t |C_t(t,s)||\gamma(x^{\sigma}(s))|\Delta s\Big].
\end{aligned} \tag{4.21}$$

Since $x\gamma(x) > 0$ and $x(t)x^{\sigma}(t)$ is non negative on $\mathbb{T}_+$ we have that $\dfrac{x}{|x|} = \dfrac{x^{\sigma}}{|x^{\sigma}|}$, and hence,

$$\frac{x}{|x|}C(\sigma(t),t)\gamma(x^{\sigma}) = C(\sigma(t),t)\frac{x^{\sigma}\gamma(x^{\sigma})}{|x^{\sigma}|} = C(\sigma(t),t)\frac{|x^{\sigma}\gamma(x^{\sigma})|}{|x^{\sigma}|} = C(\sigma(t),t)|\gamma(x^{\sigma})|$$

for all $t \in \mathbb{T}_+$. Rearranging the expression (4.21) gives

$$\begin{aligned}
\dot{V}(t,x) &= e_{\ominus\lambda}(t,0)\Big[\ominus\lambda + \ominus\lambda|x| + \ominus\lambda\int_0^t \int_t^T |C_u(u,s)|\Delta u|\gamma(x^{\sigma}(s))|\Delta s\Big] \\
&+ e_{\ominus\lambda}(\sigma(t),0)\Big[\frac{x}{|x|}\Big(f^{\Delta}(t) + C(\sigma(t),t)\gamma(x^{\sigma}) + \int_0^t C_t(t,s)\gamma(x^{\sigma}(s))\Delta s\Big) \\
&+ \int_{\sigma(t)}^{T} |C_u(u,s)|\Delta u|\gamma(x^{\sigma}(t))| - \int_0^t |C_t(t,s)||\gamma(x^{\sigma}(s))|\Delta s\Big] \\
&\leq e_{\ominus\lambda}(t,0)\Big[\ominus\lambda + \ominus\lambda|x|\Big]
\end{aligned}$$

$$
\begin{aligned}
&+ \quad e_{\ominus\lambda}(\sigma(t),0)\Big[|f^{\Delta}(t)| + C(\sigma(t),t)|\gamma(x^{\sigma}(t))| \\
&+ \quad \int_{\sigma(t)}^{T} |C_u(u,t)|\Delta u|\gamma(x^{\sigma}(t))|\Big] \\
&\leq \quad e_{\ominus\lambda}(t,0)\Big[\ominus\lambda\Big] \\
&+ \quad e_{\ominus\lambda}(\sigma(t),0)\Big[\lambda + C(\sigma(t),t)|\gamma(x^{\sigma}(t))| + \int_{\sigma(t)}^{T} |C_u(u,t)|\Delta u|\gamma(x^{\sigma}(t))|\Big].
\end{aligned} \tag{4.22}
$$

Using (4.12) the expression (4.22) reduces to

$$
\dot{V}(t,x) \leq \frac{1}{1+\mu(t)\lambda} e_{\ominus\lambda}(t,0)\Big(C(\sigma(t),t) + \int_{\sigma(t)}^{T} |C_u(u,t)|\Delta u\Big)|\gamma(x^{\sigma}(t))| \leq 0 \tag{4.23}
$$

On the other hand, for $t \in \mathbb{T}_-$ we have $xx^{\sigma} \leq 0$ and therefore, $\frac{-x}{|x|} = \frac{x^{\sigma}}{|x^{\sigma}|}$. This along with $x\gamma(x) \geq 0$ yield

$$
\frac{-x}{|x|}C(\sigma(t),t)\gamma(x^{\sigma}) = C(\sigma(t),t)\frac{x^{\sigma}\gamma(x^{\sigma})}{|x^{\sigma}|} = C(\sigma(t),t)\frac{|x^{\sigma}\gamma(x^{\sigma})|}{|x^{\sigma}|} = C(\sigma(t),t)|\gamma(x^{\sigma})|.
$$

Thus for $t \in \mathbb{T}_-$ we have

$$
\begin{aligned}
\dot{V}(t,x) &= e_{\ominus\lambda}(t,0)\Big[\ominus\lambda + \ominus\lambda|x| + \ominus\lambda\int_0^t \int_t^T |C_u(u,s)|\Delta u|\gamma(x^{\sigma}(s))|\Delta s\Big] \\
&+ e_{\ominus\lambda}(\sigma(t),0)\Big[-\frac{2}{\mu(t)}|x(t)| - \frac{x(t)}{|x(t)|}\Big(f^{\Delta}(t) + C(\sigma(t),t)\gamma(x^{\sigma}(t)) \\
&+ \int_0^t C_t(t,s)\gamma(x^{\sigma}(s))\Delta s\Big) \\
&+ \int_{\sigma(t)}^{T} |C_u(u,t)|\Delta u|\gamma(x^{\sigma}(t))| - \int_0^t |C_t(t,s)||\gamma(x^{\sigma}(s))|\Delta s\Big)\Big] \\
&\leq e_{\ominus\lambda}(t,0)\Big[\ominus\lambda + \ominus\lambda|x|\Big] \\
&+ e_{\ominus\lambda}(\sigma(t),0)\Big[-\frac{2}{\mu(t)}|x(t)| + |f^{\Delta}(t)| + C(\sigma(t),t)|\gamma(x^{\sigma}(t))| \\
&+ \int_{\sigma(t)}^{T} |C_u(u,s)|\Delta u|\gamma(x^{\sigma}(t))|\Big)\Big] \\
&\leq e_{\ominus\lambda}(t,0)\Big[\ominus\lambda + \ominus\lambda|x|\Big] \\
&+ \frac{1}{1+\mu(t)\lambda} e_{\ominus\lambda}(t,0)\Big[-\frac{2}{\mu(t)}|x(t)| + \lambda + C(\sigma(t),t)|\gamma(x^{\sigma}(t))|
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_{\sigma(t)}^{T} |C_u(u,t)|\Delta u|\gamma(x^\sigma(t))|\Big] \\
&= e_{\ominus\lambda}(t,0)\Big[\ominus\lambda + \ominus\lambda|x(t)| - \frac{2}{\mu(t)}\frac{1}{1+\mu(t)\lambda}|x(t)| + \frac{\lambda}{1+\mu(t)\lambda} \\
&+ \frac{1}{1+\mu(t)\lambda}\Big(C(\sigma(t),t) + \int_{\sigma(t)}^{T} |C_u(u,s)|\Delta u\Big)|\gamma(x^\sigma(t))|\Big] \\
&= \frac{e_{\ominus\lambda}(t,0)}{1+\mu(t)\lambda}\Big(C(\sigma(t),t) + \int_{\sigma(t)}^{T} |C_u(u,t)|\Delta u\Big)|\gamma(x^\sigma(t))| \\
&\le 0,
\end{aligned}
$$

where we use (4.9) to calculate the Δ-derivative of $|x(t)|$. The proof is completed by using the similar arguments in the proof of Theorem 4.4. □

Remark 4.8. In the particular case when $\mathbb{T} = \mathbb{R}$ the set $\mathbb{T}_-$ is empty and Theorem 4.7 covers [5, Theorem 3.3.7]. To the best of our knowledge Theorem 4.7 is new for all other time scales including the discrete case $\mathbb{T} = \mathbb{Z}$.

Example 4.9. Let $\mathbb{T} = \mathbb{Z}$ and $t_0 = 0$. Then for $t \in \mathbb{Z}$, we have $\sigma(t) = t + 1$ and $\mu(t) = 1$. Setting $f(t) = \dfrac{1}{t+1}$, $C(t,s) = -2^{s-2t}$, and $\gamma(x) = x^3$, the equation (1.1) turns into

$$
x(t) = \frac{1}{t+1} - \sum_{s=0}^{t-1} 2^{s-2n} x^3\,(s+1)\,, t = 0, 1, \ldots, \tag{4.24}
$$

(see [3, Theorem 1.79 (ii)]). Since

$$
C(\sigma(t),t) = -2^{-t+1} \text{ and } |C_t(t,s)| = \frac{3}{4}2^{s-2t} = \frac{3}{4}|C(t,s)|,
$$

we have

$$
C(\sigma(t),t) + \sum_{s=t+1}^{T} |C_t(t,s)| = -2^{-t+1} + 2^{-t-2} - 2^{-T+t} \le 0.
$$

This shows that the condition (4.19) holds for $T \ge t+1$. One can similarly show that (4.19) holds for $T < t+1$. From Theorem 4.7 each solution of (4.24) can be continued for all future times.

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