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Denumerably Many Positive Solutions for Iterative Systems of Singular Two-Point Boundary Value Problems on Time Scales

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Abstract

In this paper we consider a dynamical iterative system of two-point boundary value problems with integral boundary conditions, having n singularities and involving an increasing homeomorphism, positive homomorphism operator. By applying Hölder's inequality and Krasnoselskii's cone fixed point theorem in a Banach space, we derive sufficient conditions for the existence of denumerably many positive solutions. Finally we provide an example to check validity of our obtained results.

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1 Introduction

The differential, difference and dynamic equations on time scales are three equations play important role for modelling in the environment. Among them, the theory of dynamic equations on time scales is the most recent and was introduced by Stefan Hilger in his PhD thesis in 1988 [13] with three main features: unification, extension and discretization. Since a time scale is any closed and nonempty subset of the real numbers set. So, by this theory, we can extend known results from continuous and discrete analysis to a more general setting. As a matter of fact, this theory allows us to consider time scales which possess hybrid behaviours (both continuous and discrete). These types of time scales play an important role for applications, since most of the phenomena in the environment are neither only discrete nor only continuous, but they possess both behaviours. Moreover, basic results on this issue have been well documented in the articles Agarwal and Bohner [1], Agarwal et al. [2] and monographs of Bohner and Peterson [6, 7].

The study of turbulent flow through porous media is important for a wide range of scientific and engineering applications such as fluidized bed combustion, compact heat exchangers, combustion in an inert porous matrix, high temperature gas-cooled reactors, chemical catalytic reactors [8] and drying of different products such as iron ore [16]. To study such type of problems, Leibenson [14] introduced the following p-Laplacian equation,

$$\big(\phi_p(\vartheta'(t))\big)' = f\big(t, \vartheta(t), \vartheta'(t)\big),$$

where $\phi_p(\vartheta) = |\vartheta|^{p-2}\vartheta,\ p > 1$, is the p-Laplacian operator its inverse function is denoted by $\phi_q(\tau)$ with $\phi_q(\tau) = |\tau|^{q-2}\tau$, and p, q satisfy $1/p+1/q = 1$. It is well known fact that the p-Laplacian operator and fractional calculus arises from many applied fields such as turbulant filtration in porous media, blood flow problems, rheology, modelling of viscoplasticity, material science, it is worth studying the fractional differential equations with p-Laplacian operator.

In this paper, we consider an operator ϕ called increasing homeomorphism and positive homomorphism operator(IHPHO), which generalizes and improves the p-Laplacian operator for some $p > 1$, and ϕ is not necessarily odd. Liang and Zhang [15] studied countably many positive solutions for nonlinear singular m–point boundary value problems on time scales with IHPHO,

$$\begin{aligned} &\big(\phi(\vartheta^\Delta(t))\big)^\nabla + a(t)f\big(\vartheta(t)\big) = 0,\ t \in [0, T]_{\mathbb{T}} \\ &\vartheta(0) = \sum_{i=1}^{m-2} a_i\vartheta(\xi_i),\ \vartheta^\Delta(T) = 0, \end{aligned}$$

by using the fixed-point index theory and a new fixed-point theorem in cones.

In [9], Dogan considered second order p–Laplacian boundary value problem on time scales,

$$\begin{gathered}\left(\phi_p(\vartheta^{\Delta}(t))\right)^{\nabla} + \omega(t)f\left(t,\vartheta(t)\right) = 0,\ t\in[0,T]_{\mathbb{T}}\\ \vartheta(0)=\sum_{i=1}^{m-2}a_i\vartheta(\xi_i),\ \phi_p(\vartheta^{\Delta}(T))=\sum_{i=1}^{m-2}b_i\phi_p(\vartheta^{\Delta}(\xi_i)),\end{gathered}$$

and established existence of multiple positive solutions by applying fixed-point index theory.

Inspired by aforementioned works, in this paper by applying Hölder's inequality and Krasnoselskii's cone fixed point theorem in a Banach space, we establish the existence of denumerably many positive solutions for dynamical iterative system of two-point boundary value problem with n singularities and involving IHPHO on time scales,

$$\left.\begin{aligned}\phi\left(\vartheta_j^{\Delta\nabla}(t)\right)+\zeta(t)f_j\left(\vartheta_{j+1}(t)\right)=0,\ 1\le j\le \ell,\ t\in[0,\mathfrak{T}]_{\mathbb{T}}\\ \vartheta_{\ell+1}(t)=\vartheta_1(t),\ t\in[0,\mathfrak{T}]_{\mathbb{T}},\end{aligned}\right\} \tag{1.1}$$

$$\left.\begin{aligned}\vartheta_j(0)=\int_0^{\mathfrak{T}}\kappa(s)\vartheta_j(s)\nabla s,\ 1\le j\le\ell,\\ \vartheta_j(\mathfrak{T})=\int_0^{\mathfrak{T}}\kappa(s)\vartheta_j(s)\nabla s,\ 1\le j\le\ell,\end{aligned}\right\} \tag{1.2}$$

where $\ell\in\mathbb{N}$, $\zeta(t)=\prod_{i=1}^{n}\zeta_i(t)$ and each $\zeta_i(t)\in L_{\nabla}^{p_i}([0,\mathfrak{T}]_{\mathbb{T}})(p_i\ge 1)$ has a singularity in the interval $\left(0,\frac{\mathfrak{T}}{2}\right)$ and $\phi:\mathbb{R}\to\mathbb{R}$ is an IHPHO with $\phi(0)=0$.

A projection $\phi:\mathbb{R}\to\mathbb{R}$ is called a IHPHO, if the following three conditions are fulfilled,

- (a) $\phi(\tau_1)\le\phi(\tau_2)$ whenever $\tau_1\le\tau_2$, for any real numbers τ_1,τ_2.
- (b) ϕ is a continuous bijection and its inverse ϕ^{-1} is continuous.
- (c) $\phi(\tau_1\tau_2)=\phi(\tau_1)\phi(\tau_2)$ for any real numbers τ_1,τ_2.

We assume the following conditions are true in the entire paper:

(H_1) $f_j:[0,+\infty)\to[0,+\infty)$ and $\kappa:[0,\mathfrak{T}]_{\mathbb{T}}\to[0,+\infty)$ are continuous,

(H_2) there exists a sequence $\{t_r\}_{r=1}^{\infty}$ such that $0<t_{r+1}<t_r<\frac{\mathfrak{T}}{2}$,

$$\lim_{r\to\infty}t_r=t^*<\frac{\mathfrak{T}}{2},\ \lim_{t\to t_r}\zeta_i(t)=+\infty,\ i,r\in\mathbb{N}$$

and each $\zeta_i(t)$ does not vanish identically on any subinterval of $[0,\mathfrak{T}]_{\mathbb{T}}$. Moreover, there exists $\delta_i>0$ such that

$$\delta_i<\phi^{-1}\left(\zeta_i(t)\right)<\infty\ a.e.\ \text{on}\ [0,\mathfrak{T}]_{\mathbb{T}},\ i=1,2,\cdots,n.$$

2 Preliminaries

In this section, we introduce some basic definitions and lemmas which are useful for our later discussions; for details, see [3–6, 11, 18, 19].

Definition 2.1. A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$. $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology. It follows that the jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$, and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined by $\sigma(t) = \inf\{\tau \in \mathbb{T} : \tau > t\}$, $\rho(t) = \sup\{\tau \in \mathbb{T} : \tau < t\}$, and $\mu(t) = \rho(t) - t$, respectively.

- The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively.
- If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$.
- If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$.
- A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of all rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$.
- A function $f : \mathbb{T} \to \mathbb{R}$ is called ld-continuous provided it is continuous at left-dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at right-dense points in $\mathbb{T}$. The set of all ld-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{ld} = C_{ld}(\mathbb{T}) = C_{ld}(\mathbb{T}, \mathbb{R})$.
- By an interval time scale, we mean the intersection of a real interval with a given time scale. i.e., $[a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T}$ other intervals can be defined similarly.

Definition 2.2. Let μ_Δ and μ_∇ be the Lebesgue $\Delta-$ measure and the Lebesgue $\nabla-$ measure on $\mathbb{T}$, respectively. If $A \subset \mathbb{T}$ satisfies $\mu_\Delta(A) = \mu_\nabla(A)$, then we call A is measurable on $\mathbb{T}$, denoted $\mu(A)$ and this value is called the Lebesgue measure of A. Let P denote a proposition with respect to $t \in \mathbb{T}$.

(i) If there exists $E_1 \subset A$ with $\mu_\Delta(E_1) = 0$ such that P holds on $A\backslash E_1$, then P is said to hold Δ–a.e. on A.

(ii) If there exists $E_2 \subset A$ with $\mu_\nabla(E_2) = 0$ such that P holds on $A\backslash E_2$, then P is said to hold ∇–a.e. on A.

Definition 2.3. Let $E \subset \mathbb{T}$ be a ∇–measurable set and $p \in \bar{\mathbb{R}} \equiv \mathbb{R} \cup \{-\infty, +\infty\}$ be such that $p \geq 1$ and let $f : E \to \bar{\mathbb{R}}$ be ∇–measurable function. We say that f belongs to $L^p_\nabla(E)$ provided that either

$$\int_E |f|^p(s)\nabla s < \infty \quad \text{if} \quad p \in \mathbb{R},$$

or there exists a constant $M \in \mathbb{R}$ such that

$$|f| \leq M, \ \ \nabla - a.e. \ on \ E \ \text{ if } \ p = +\infty.$$

Lemma 2.4. *Let $E \subset \mathbb{T}$ be a ∇–measurable set. If $f : \mathbb{T} \to \mathbb{R}$ is a ∇–integrable on E, then*

$$\int_E f(s)\nabla s = \int_E f(s)ds + \sum_{i \in I_E} \big(t_i - \rho(t_i)\big) f(t_i),$$

where $I_E := \{i \in I : t_i \in E\}$ and $\{t_i\}_{i \in I}, I \subset \mathbb{N}$, is the set of all left-scattered points of $\mathbb{T}$.

Lemma 2.5. *Suppose $0 < \eta < 1$, where $\eta = \int_0^{\mathfrak{T}} \kappa(\tau)\nabla\tau$. Then for any $\varrho(t) \in C([0, \mathfrak{T}]_{\mathbb{T}})$, boundary value problem,*

$$-\phi(\vartheta_1^{\Delta\nabla}(t)) = \varrho(t),\ t \in [0, \mathfrak{T}]_{\mathbb{T}}, \tag{2.1}$$

$$\vartheta_1(0) = \vartheta_1(\mathfrak{T}) = \int_0^{\mathfrak{T}} \kappa(\tau)\vartheta_1(\tau)\nabla\tau, \tag{2.2}$$

has a unique solution

$$\vartheta_1(t) = \int_0^{\mathfrak{T}} \aleph(t, \tau)\phi^{-1}(\varrho(\tau))\nabla\tau, \tag{2.3}$$

where

$$\aleph(t, \tau) = \aleph_0(t, \tau) + \frac{1}{1 - \eta}\int_0^{\mathfrak{T}} \aleph_0(\tau_1, \tau)\kappa(\tau_1)\nabla\tau_1, \tag{2.4}$$

in which

$$\aleph_0(t, \tau) = \frac{1}{\mathfrak{T}}\begin{cases} t(\mathfrak{T} - \tau), & t \le \tau, \\ \tau(\mathfrak{T} - t), & \tau \le t. \end{cases} \tag{2.5}$$

Proof. Suppose ϑ_1 is a solution of (2.1), then

$$\begin{aligned}\vartheta_1(t) &= -\int_0^t \int_0^\tau \phi^{-1}(\varrho(\tau_1))\nabla\tau_1\Delta\tau + At + B \\ &= -\int_0^t (t - \tau)\phi^{-1}(\varrho(\tau))\nabla\tau + A_1 t + A_2,\end{aligned}$$

where $A_1 = \vartheta_1^{\Delta}(0)$ and $A_2 = \vartheta_1(0)$. By the conditions (2.2), we get

$$A_1 = \frac{1}{\mathfrak{T}}\int_0^{\mathfrak{T}} (\mathfrak{T} - \tau)\phi^{-1}(\varrho(\tau))\nabla\tau,$$

and

$$
\begin{aligned}
A_2 &= \int_0^{\mathfrak{T}} \kappa(\tau)\vartheta_1(\tau)\nabla\tau \\
&= \int_0^{\mathfrak{T}} \kappa(\tau)\Big[-\int_0^{\tau}(\tau-\tau_1)\phi^{-1}(\varrho(\tau_1))\nabla\tau_1 + A_1\tau + A_2\Big]\nabla\tau \\
&= \int_0^{\mathfrak{T}} \kappa(\tau)\Big[-\int_0^{\tau}(\tau-\tau_1)\phi^{-1}(\varrho(\tau_1))\nabla\tau_1 \\
&\qquad\qquad + \frac{\tau}{\mathfrak{T}}\int_0^{\mathfrak{T}}(\mathfrak{T}-\tau_1)\phi^{-1}(\varrho(\tau_1))\nabla\tau_1\Big]\nabla\tau + A_2\eta \\
&= \int_0^{\mathfrak{T}} \kappa(\tau)\Big[\int_0^{\tau}\frac{s}{\mathfrak{T}}(\mathfrak{T}-\tau_1)\phi^{-1}(\varrho(\tau_1))\nabla\tau_1 \\
&\qquad\qquad + \int_{\tau}^{\mathfrak{T}}\frac{\tau}{\mathfrak{T}}(\mathfrak{T}-\tau_1)\phi^{-1}(\varrho(\tau_1))\nabla\tau_1\Big]\nabla\tau + A_2\eta \\
&= \int_0^{\mathfrak{T}} \kappa(\tau)\Big[\int_0^{\mathfrak{T}}\aleph_0(\tau,s)\phi^{-1}(\varrho(\tau))\nabla\tau\Big]\nabla\tau + A_2\eta \\
&= \int_0^{\mathfrak{T}} \Big[\int_0^{\mathfrak{T}}\aleph_0(\tau,s)\kappa(\tau)\nabla\tau\Big]\phi^{-1}(\varrho(\tau))\nabla\tau + A_2\eta \\
&= \frac{1}{1-\eta}\int_0^{\mathfrak{T}} \Big[\int_0^{\mathfrak{T}}\aleph_0(\tau,s)\kappa(\tau)\nabla\tau\Big]\phi^{-1}(\varrho(\tau))\nabla\tau.
\end{aligned}
$$

So, we have

$$
\begin{aligned}
\vartheta_1(t) &= -\int_0^{t}(t-\tau)\phi^{-1}(\varrho(\tau))\nabla\tau + \int_0^{\mathfrak{T}}\frac{t}{\mathfrak{T}}(\mathfrak{T}-\tau)\phi^{-1}(\varrho(\tau))\nabla\tau \\
&\quad + \frac{1}{1-\eta}\int_0^{\mathfrak{T}}\Big[\int_0^{\mathfrak{T}}\aleph_0(\tau_1,\tau)\kappa(\tau_1)\nabla\tau_1\Big]\phi^{-1}(\varrho(\tau))\nabla\tau \\
&= \int_0^{t}\frac{\tau}{\mathfrak{T}}(\mathfrak{T}-t)\phi^{-1}(\varrho(\tau))\nabla\tau + \int_t^{\mathfrak{T}}\frac{t}{\mathfrak{T}}(\mathfrak{T}-\tau)\phi^{-1}(\varrho(\tau))\nabla\tau \\
&\quad + \frac{1}{1-\eta}\int_0^{\mathfrak{T}}\Big[\int_0^{\mathfrak{T}}\aleph_0(\tau_1,\tau)\kappa(\tau_1)\nabla\tau_1\Big]\phi^{-1}(\varrho(\tau))\nabla\tau \\
&= \int_0^{\mathfrak{T}}\aleph_0(t,\tau)\phi^{-1}(\varrho(\tau))\nabla\tau \\
&\quad + \frac{1}{1-\eta}\int_0^{\mathfrak{T}}\Big[\int_0^{\mathfrak{T}}\aleph_0(\tau_1,\tau)\kappa(\tau_1)\nabla\tau_1\Big]\phi^{-1}(\varrho(\tau))\nabla\tau \\
&= \int_0^{\mathfrak{T}}\Big[\aleph_0(t,\tau) + \frac{1}{1-\eta}\int_0^{\mathfrak{T}}\aleph_0(\tau_1,\tau)\kappa(\tau_1)\nabla\tau_1\Big]\phi^{-1}(\varrho(\tau))\nabla\tau \\
&= \int_0^{\mathfrak{T}}\aleph(t,\tau)\phi^{-1}(\varrho(\tau))\nabla\tau,
\end{aligned}
$$

This completes the proof. □

Lemma 2.6. *Assume that* (H_1) *holds and let* $\mathfrak{z} \in \left(0, \frac{\mathfrak{T}}{2}\right)_{\mathbb{T}}$ *and* $\eta_{\mathfrak{z}} = \int_{\mathfrak{z}}^{\mathfrak{T}-\mathfrak{z}} \kappa(t)\nabla t$. *Then* $\aleph_0(t,\tau)$ *and* $\aleph(t,\tau)$ *have the following properties:*

(i) $\aleph_0(t,\tau) > 0$ *and* $\aleph(t,\tau) > 0$ *for all* $t,\tau \in [0,\mathfrak{T}]_{\mathbb{T}}$,

(ii) $\aleph_0(t,\tau) \le \aleph_0(\tau,\tau),\ \aleph(t,\tau) \le \aleph(\tau,\tau) \le \frac{1}{1-\eta}\aleph_0(\tau,\tau)$ *for all* $t,\tau \in [0,\mathfrak{T}]_{\mathbb{T}}$,

(iii) $\aleph_0(t,\tau) \ge \frac{\mathfrak{z}}{\mathfrak{T}}\aleph_0(\tau,\tau)$ *for all* $t \in [\mathfrak{z},\mathfrak{T}-\mathfrak{z}]_{\mathbb{T}}$ *and* $\tau \in [0,\mathfrak{T}]_{\mathbb{T}}$

(iv) $\aleph(t,\tau) \ge \lambda_{\mathfrak{z}}\aleph_0(\tau,\tau)$ *where* $\lambda_{\mathfrak{z}} = \frac{\mathfrak{z}}{\mathfrak{T}}\left[1+\frac{\eta_{\mathfrak{z}}}{1-\eta}\right]$, *for all* $t \in [\mathfrak{z},\mathfrak{T}-\mathfrak{z}]_{\mathbb{T}}$ *and* $\tau \in [0,\mathfrak{T}]_{\mathbb{T}}$.

Proof. Inequalities (i) and (ii) are obvious. To prove (iii), let $t \in [\mathfrak{z},\mathfrak{T}-\mathfrak{z}]_{\mathbb{T}}$. Then, for $0 < t < \tau < \mathfrak{T}$,

$$\frac{\aleph_0(t,\tau)}{\aleph_0(\tau,\tau)} = \frac{t}{\tau} \ge \frac{\mathfrak{z}}{\mathfrak{T}},$$

and for $0 < \tau < t < \mathfrak{T}$,

$$\frac{\aleph_0(t,\tau)}{\aleph_0(\tau,\tau)} = \frac{\mathfrak{T}-t}{\mathfrak{T}-\tau} \ge \frac{\mathfrak{z}}{\mathfrak{T}}.$$

This proves (iii). Next, for $t \in [\mathfrak{z},\mathfrak{T}-\mathfrak{z}]_{\mathbb{T}}$, we have

$$\begin{aligned}
\aleph(t,\tau) &= \aleph_0(t,\tau) + \frac{1}{1-\eta}\int_0^{\mathfrak{T}} \aleph_0(\tau_1,\tau)\kappa(\tau_1)\nabla\tau_1 \\
&\ge \frac{\mathfrak{z}}{\mathfrak{T}}\aleph_0(\tau,\tau) + \frac{1}{1-\eta}\int_{\mathfrak{z}}^{\mathfrak{T}-\mathfrak{z}} \aleph_0(\tau_1,\tau)\kappa(\tau_1)\nabla\tau_1 \\
&\ge \frac{\mathfrak{z}}{\mathfrak{T}}\aleph_0(\tau,\tau) + \frac{1}{1-\eta}\int_{\mathfrak{z}}^{\mathfrak{T}-\mathfrak{z}} \frac{\mathfrak{z}}{\mathfrak{T}}\aleph_0(\tau,\tau)\kappa(\tau_1)\nabla\tau_1 \\
&\ge \frac{\mathfrak{z}}{\mathfrak{T}}\aleph_0(\tau,\tau) + \frac{\eta_{\mathfrak{z}}}{1-\eta}\frac{\mathfrak{z}}{\mathfrak{T}}\aleph_0(\tau,\tau) \\
&= \lambda_{\mathfrak{z}}\aleph_0(\tau,\tau).
\end{aligned}$$

This completes the proof. □

Notice that an $\ell-$tuple $(\vartheta_1(t),\vartheta_2(t),\vartheta_3(t),\cdots,\vartheta_\ell(t))$ is a solution of the iterative boundary value problem (1.1)–(1.2) if and only if

$$\vartheta_j(t) = \int_0^{\mathfrak{T}} \aleph(t,\tau)\phi^{-1}\big[\zeta(\tau)f_j(\vartheta_{j+1}(\tau))\big]\nabla\tau,\ t \in [0,\mathfrak{T}]_{\mathbb{T}},\ 1 \le j \le \ell,$$

$$\vartheta_{\ell+1}(t) = \vartheta_1(t),\ t \in [0,\mathfrak{T}]_{\mathbb{T}},$$

i.e.,

$$\vartheta_1(t)=\int_0^{\mathfrak{T}}\aleph(t,\tau_1)\phi^{-1}\bigg[\zeta(\tau_1)f_1\bigg(\int_0^{\mathfrak{T}}\aleph(\tau_1,\tau_2)\phi^{-1}\bigg[\zeta(\tau_2)f_2\bigg(\int_0^{\mathfrak{T}}\aleph(\tau_2,\tau_3)\times\phi^{-1}\bigg[\zeta(\tau_3)f_3\bigg(\int_0^{\mathfrak{T}}\aleph(\tau_3,\tau_4)\cdots\times f_{\ell-1}\bigg(\int_0^{\mathfrak{T}}\aleph(\tau_{\ell-1},\tau_\ell)\phi^{-1}\big[\zeta(\tau_\ell)f_\ell(\vartheta_1(\tau_\ell))\big]\nabla\tau_\ell\bigg)\cdots\nabla\tau_3\bigg]\nabla\tau_2\bigg]\nabla\tau_1.$$

Let $\mathtt{X}$ be the Banach space $C_{ld}([0,\mathfrak{T}]_{\mathbb{T}},\mathbb{R})$ with the norm $\|\vartheta\| = \max_{t\in[0,\mathfrak{T}]_{\mathbb{T}}}|\vartheta(t)|$. For $\mathfrak{z}\in\left(0,\frac{\mathfrak{T}}{2}\right)$, we define the cone $\mathtt{P}_{\mathfrak{z}}\subset\mathtt{X}$ as

$$\mathtt{P}_{\mathfrak{z}}=\Big\{\vartheta\in\mathtt{X}:\vartheta(t)\text{ is nonnegative and }\min_{t\in[\mathfrak{z},\,1-\mathfrak{z}]_{\mathbb{T}}}\vartheta(t)\geq\lambda_{\mathfrak{z}}(1-\eta)\|\vartheta(t)\|\Big\},$$

For any $\vartheta_1\in\mathtt{P}_{\mathfrak{z}}$, define an operator $\Omega:\mathtt{P}_{\mathfrak{z}}\to\mathtt{X}$ by

$$(\Omega\vartheta_1)(t)=\int_0^{\mathfrak{T}}\aleph(t,\tau_1)\phi^{-1}\bigg[\zeta(\tau_1)f_1\bigg(\int_0^{\mathfrak{T}}\aleph(\tau_1,\tau_2)\phi^{-1}\bigg[\zeta(\tau_2)f_2\bigg(\int_0^{\mathfrak{T}}\aleph(\tau_2,\tau_3)\times\phi^{-1}\bigg[\zeta(\tau_3)f_3\bigg(\int_0^{\mathfrak{T}}\aleph(\tau_3,\tau_4)\cdots\times f_{\ell-1}\bigg(\int_0^{\mathfrak{T}}\aleph(\tau_{\ell-1},\tau_\ell)\phi^{-1}\big[\zeta(\tau_\ell)f_\ell(\vartheta_1(\tau_\ell))\big]\nabla\tau_\ell\bigg)\cdots\nabla\tau_3\bigg]\nabla\tau_2\bigg]\nabla\tau_1.$$

Lemma 2.7. *Assume that* (H_1)–(H_2) *hold. Then for each* $\mathfrak{z}\in\left(0,\frac{\mathfrak{T}}{2}\right)$, $\Omega(\mathtt{P}_{\mathfrak{z}})\subset\mathtt{P}_{\mathfrak{z}}$ *and* $\Omega:\mathtt{P}_{\mathfrak{z}}\to\mathtt{P}_{\mathfrak{z}}$ *is completely continuous.*

Proof. From Lemma 2.6, $\aleph(t,\tau)\geq 0$ for all $t,\tau\in[0,\mathfrak{T}]_{\mathbb{T}}$. So, $(\Omega\vartheta_1)(t)\geq 0$. Also, for $\vartheta_1\in\mathtt{P}$, we have

$$(\Omega\vartheta_1)(t)\leq\frac{1}{1-\eta}\int_0^{\mathfrak{T}}\aleph_0(\tau_1,\tau_1)\phi^{-1}\bigg[\zeta(\tau_1)f_1\bigg(\int_0^{\mathfrak{T}}\aleph(\tau_1,\tau_2)\phi^{-1}\bigg[\zeta(\tau_2)\times f_2\bigg(\int_0^{\mathfrak{T}}\aleph(\tau_2,\tau_3)\phi^{-1}\bigg[\zeta(\tau_3)f_3\bigg(\int_0^{\mathfrak{T}}\aleph(\tau_3,\tau_4)\cdots\times f_{\ell-1}\bigg(\int_0^{\mathfrak{T}}\aleph(\tau_{\ell-1},\tau_\ell)\phi^{-1}\big[\zeta(\tau_\ell)f_\ell(\vartheta_1(\tau_\ell))\big]\nabla\tau_\ell\bigg)\cdots\nabla\tau_3\bigg]\nabla\tau_2\bigg]\nabla\tau_1.$$

So,

$$\begin{aligned}\|\Omega\vartheta_1\| \leq & \frac{1}{1-\eta}\int_0^{\mathfrak{T}} \aleph_0(\tau_1,\tau_1)\phi^{-1}\Bigg[\zeta(\tau_1) f_1\Bigg(\int_0^{\mathfrak{T}} \aleph(\tau_1,\tau_2)\phi^{-1}\Bigg[\zeta(\tau_2) \\ & \times f_2\Bigg(\int_0^{\mathfrak{T}} \aleph(\tau_2,\tau_3)\phi^{-1}\Bigg[\zeta(\tau_3) f_3\Bigg(\int_0^{\mathfrak{T}} \aleph(\tau_3,\tau_4)\cdots \\ & \times f_{\ell-1}\Bigg(\int_0^{\mathfrak{T}} \aleph(\tau_{\ell-1},\tau_\ell)\phi^{-1}\big[\zeta(\tau_\ell) f_\ell(\vartheta_1(\tau_\ell))\big]\nabla\tau_\ell\Bigg)\cdots\nabla\tau_3\Bigg]\nabla\tau_2\Bigg]\nabla\tau_1.\end{aligned}$$

Again from Lemma 2.6, we get

$$\begin{aligned}\min_{t\in[\mathfrak{z},\mathfrak{T}-\mathfrak{z}]_{\mathbb{T}}}\{(\Omega\vartheta_1)(t)\} \geq \lambda_{\mathfrak{z}} & \int_0^{\mathfrak{T}} \aleph_0(\tau_1,\tau_1)\phi^{-1}\Bigg[\zeta(\tau_1) f_1\Bigg(\int_0^{\mathfrak{T}} \aleph(\tau_1,\tau_2)\phi^{-1}\Bigg[\zeta(\tau_2) \\ & \times f_2\Bigg(\int_0^{\mathfrak{T}} \aleph(\tau_2,\tau_3)\phi^{-1}\Bigg[\zeta(\tau_3) f_3\Bigg(\int_0^{\mathfrak{T}} \aleph(\tau_3,\tau_4)\cdots \\ & \times f_{\ell-1}\Bigg(\int_0^{\mathfrak{T}} \aleph(\tau_{\ell-1},\tau_\ell)\phi^{-1}\big[\zeta(\tau_\ell) f_\ell(\vartheta_1(\tau_\ell))\big]\nabla\tau_\ell\Bigg)\cdots\nabla\tau_3\Bigg]\nabla\tau_2\Bigg]\nabla\tau_1.\end{aligned}$$

It follows from the above two inequalities that

$$\min_{t\in[\mathfrak{z},\mathfrak{T}-\mathfrak{z}]_{\mathbb{T}}}\{(\Omega\vartheta_1)(t)\} \geq \lambda_{\mathfrak{z}}(1-\eta)\|\Omega\vartheta_1\|.$$

So, $\Omega\vartheta_1 \in \mathtt{P}_{\mathfrak{z}}$ and thus $\Omega(\mathtt{P}_{\mathfrak{z}}) \subset \mathtt{P}_{\mathfrak{z}}$. Next, by standard methods and Arzela–Ascoli theorem, it can be proves easily that the operator Ω is completely continuous. The proof is complete. □

3 Denumerably Infinitely Many Positive Solutions

For the existence of denumerably many positive solutions for iterative system of boundary value problem (1.1). We apply following theorems.

Theorem 3.1. *[10] Let $\mathcal{E}$ be a cone in a Banach space $\mathcal{X}$ and $\Lambda_1,\ \Lambda_2$ are open sets with $0 \in \Lambda_1, \overline{\Lambda}_1 \subset \Lambda_2$. Let $\mathcal{A} : \mathcal{E} \cap (\overline{\Lambda}_2 \backslash \Lambda_1) \to \mathcal{E}$ be a completely continuous operator such that*

(a) $\|\mathcal{A}z\| \leq \|z\|,\ z \in \mathcal{E} \cap \partial\Lambda_1$, and $\|\mathcal{A}z\| \geq \|z\|,\ z \in \mathcal{E} \cap \partial\Lambda_2$, or

(b) $\|\mathcal{A}z\| \geq \|z\|,\ z \in \mathcal{E} \cap \partial\Lambda_1$, and $\|\mathcal{A}z\| \leq \|z\|,\ z \in \mathcal{E} \cap \partial\Lambda_2$.

Then $\mathcal{A}$ has a fixed point in $\mathcal{E} \cap (\overline{\Lambda}_2 \backslash \Lambda_1)$.

Theorem 3.2 (See [7, 17]). *Let $f \in L^p_\nabla(J)$ with $p > 1, g \in L^q_\nabla(J)$ with $q > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$. Then $fg \in L^1_\nabla(J)$ and $\|fg\|_{L^1_\nabla} \le \|f\|_{L^p_\nabla}\|g\|_{L^q_\nabla}$.*
where

$$\|f\|_{L^p_\nabla} := \begin{cases} \Big[\displaystyle\int_J |f|^p(s)\nabla s\Big]^{\frac{1}{p}}, & p \in \mathbb{R}, \\ \inf\Big\{M \in \mathbb{R} \,/\, |f| \le M \ \nabla - a.e., \ on\ J\Big\}, & p = \infty, \end{cases}$$

and $J = (a, b]_{\mathbb{T}}$.

Theorem 3.3 (Hölder). *Let $f \in L^{p_i}_\nabla(J)$ with $p_i > 1$, for $i = 1, 2, \cdots, n$ and $\displaystyle\sum_{i=1}^{n} \frac{1}{p_i} = 1$. Then $\displaystyle\prod_{i=1}^{n} f_i \in L^1_\nabla(J)$ and*

$$\left\|\prod_{i=1}^{n} f_i\right\|_1 \le \prod_{i=1}^{n} \|f_i\|_{p_i}.$$

Further, if $f \in L^1_\nabla(J)$ and $g \in L^\infty_\nabla(J)$. Then $fg \in L^1_\nabla(J)$ and

$$\|fg\|_1 \le \|f\|_1\|g\|_\infty.$$

Consider the following three possible cases for $\zeta_i \in L^{p_i}_\nabla([0, \mathfrak{T}]_{\mathbb{T}})$:
$(i) \displaystyle\sum_{i=1}^{n} \frac{1}{p_i} < 1,\ (ii) \sum_{i=1}^{n} \frac{1}{p_i} = 1,\ (iii) \sum_{i=1}^{n} \frac{1}{p_i} > 1.$

Firstly, we seek denumerably many positive solutions for the case $\displaystyle\sum_{i=1}^{n} \frac{1}{p_i} < 1$.

Theorem 3.4. *Suppose $(H_1)-(H_2)$ hold, let $\{\mathfrak{z}_r\}_{r=1}^{\infty}$ be a sequence with $\mathfrak{z}_r \in (t_{r+1}, t_r)$. Let $\{E_r\}_{r=1}^{\infty}$ and $\{O_r\}_{r=1}^{\infty}$ be such that*

$$E_{r+1} < \frac{\mathfrak{z}_r}{\mathfrak{T}} O_r < O_r < \mathfrak{Z} O_r < E_r,\ r \in \mathbb{N},$$

where

$$\mathfrak{Z} = \max\left\{ \left[\lambda_{\mathfrak{z}_1} \prod_{i=1}^{n} \delta_i \int_{\mathfrak{z}_1}^{\mathfrak{T}-\mathfrak{z}_1} \aleph_0(\tau, \tau)\nabla\tau\right]^{-1},\ 1\right\}.$$

Assume that f satisfies

(C_1) $f_j(\vartheta) \le \phi(\mathfrak{M}_1 E_r)\ \forall\, t \in [0, \mathfrak{T}]_{\mathbb{T}},\ 0 \le \vartheta \le E_r,$
where

$$\mathfrak{M}_1 < \left[\frac{1}{1-\eta}\|\aleph_0\|_{L^q_\nabla} \prod_{i=1}^{n} \left\|\phi^{-1}(\zeta_i)\right\|_{L^{p_i}_\nabla}\right]^{-1},$$

(C_2) $f_j(\vartheta) \geq \phi(\mathfrak{Z}O_r)\ \forall\, t \in [\mathfrak{z}_r, 1-\mathfrak{z}_r]_{\mathbb{T}},\ \frac{\mathfrak{z}_r}{\mathfrak{T}}O_r \leq \vartheta \leq O_r.$

Then the iterative boundary value problem (1.1)–(1.2) *has denumerably many solutions* $\{(\vartheta_1^{[r]}, \vartheta_2^{[r]}, \cdots, \vartheta_\ell^{[r]})\}_{r=1}^{\infty}$ *such that* $\vartheta_j^{[r]}(t) \geq 0$ *on* $[0, \mathfrak{T}]_{\mathbb{T}}$, $j = 1, 2, \cdots, \ell$ *and* $r \in \mathbb{N}$.

Proof. Let

$$\Lambda_{1,r} = \{\vartheta \in \mathtt{X} : \|\vartheta\| < E_r\}, \ \ \Lambda_{2,r} = \{\vartheta \in \mathtt{X} : \|\vartheta\| < O_r\}$$

be open subsets of $\mathtt{X}$. Let $\{\mathfrak{z}_r\}_{r=1}^{\infty}$ be given in the hypothesis and we note that

$$t^* < t_{r+1} < \mathfrak{z}_r < t_r < \frac{\mathfrak{T}}{2},$$

for all $r \in \mathbb{N}$.

For each $r \in \mathbb{N}$, we define the cone $\mathtt{P}_{\mathfrak{z}_r}$ by

$$\mathtt{P}_{\mathfrak{z}_r} = \Big\{\vartheta \in \mathtt{X} : \vartheta(t) \geq 0, \min_{t \in [\mathfrak{z}_r, \mathfrak{T}-\mathfrak{z}_r]_{\mathbb{T}}} \vartheta(t) \geq \frac{\mathfrak{z}_r}{\mathfrak{T}}\|\vartheta(t)\|\Big\}.$$

Let $\vartheta_1 \in \mathtt{P}_{\mathfrak{z}_r} \cap \partial\Lambda_{1,r}$. Then, $\vartheta_1(\tau) \leq E_r = \|\vartheta_1\|$ for all $\tau \in [0, \mathfrak{T}]_{\mathbb{T}}$. By (C_1) and for $\tau_{\ell-1} \in [0, \mathfrak{T}]_{\mathbb{T}}$, we have

$$\begin{aligned}
&\int_0^{\mathfrak{T}} \aleph(\tau_{\ell-1}, \tau_\ell)\phi^{-1}\big[\zeta(\tau_\ell) f_\ell(\vartheta_1(\tau_\ell))\big]\nabla\tau_\ell \\
&\leq \frac{1}{1-\eta}\int_0^{\mathfrak{T}} \aleph_0(\tau_\ell, \tau_\ell)\phi^{-1}\big[\zeta(\tau_\ell) f_\ell(\vartheta_1(\tau_\ell))\big]\nabla\tau_\ell \\
&\leq \frac{\mathfrak{M}_1 E_r}{1-\eta}\int_0^{\mathfrak{T}} \aleph_0(\tau_\ell, \tau_\ell)\phi^{-1}\big[\zeta(\tau_\ell)\big]\nabla\tau_\ell \\
&\leq \frac{\mathfrak{M}_1 E_r}{1-\eta}\int_0^{\mathfrak{T}} \aleph_0(\tau_\ell, \tau_\ell)\phi^{-1}\Big[\prod_{i=1}^{n}\zeta_i(\tau_\ell)\Big]\nabla\tau_\ell \\
&\leq \frac{\mathfrak{M}_1 E_r}{1-\eta}\int_0^{\mathfrak{T}} \aleph_0(\tau_\ell, \tau_\ell)\prod_{i=1}^{n}\phi^{-1}(\zeta_i(\tau_\ell))\nabla\tau_\ell.
\end{aligned}$$

There exists a $q > 1$ such that $\frac{1}{q} + \sum_{i=1}^{n}\frac{1}{p_i} = 1$. So,

$$\begin{aligned}
\int_0^{\mathfrak{T}} \aleph(\tau_{\ell-1}, \tau_\ell)\phi^{-1}\big[\zeta(\tau_\ell) f_\ell(\vartheta_1(\tau_\ell))\big]\nabla\tau_\ell &\leq \frac{\mathfrak{M}_1 E_r}{1-\eta}\big\|\aleph_0\big\|_{L^q_\nabla}\Big\|\prod_{i=1}^{n}\phi^{-1}(\zeta_i)\Big\|_{L^{p_i}_\nabla} \\
&\leq \frac{\mathfrak{M}_1 E_r}{1-\eta}\|\aleph_0\|_{L^q_\nabla}\prod_{i=1}^{n}\big\|\phi^{-1}(\zeta_i)\big\|_{L^{p_i}_\nabla} \\
&\leq E_r.
\end{aligned}$$

It follows in similar manner (for $\tau_{\ell-2}\in[0,\mathfrak{T}]_{\mathbb{T}},$) that

$$\begin{aligned}\int_0^{\mathfrak{T}}\aleph(\tau_{\ell-2},\tau_{\ell-1})\phi^{-1}\Bigg[\zeta(\tau_{\ell-1})f_{\ell-1}\bigg(\int_0^{\mathfrak{T}}\aleph(\tau_{\ell-1},\tau_\ell)\phi^{-1}\big[\zeta(\tau_\ell)f_\ell(\vartheta_1(\tau_\ell))\big]\nabla\tau_\ell\bigg)\Bigg]\nabla\tau_{\ell-1}\\
&\le\int_0^{\mathfrak{T}}\aleph(\tau_{\ell-2},\tau_{\ell-1})\phi^{-1}\big[\zeta(\tau_{\ell-1})f_{\ell-1}(E_r)\big]\nabla\tau_{\ell-1}\\
&\le\frac{1}{1-\eta}\int_0^{\mathfrak{T}}\aleph_0(\tau_{\ell-1},\tau_{\ell-1})\phi^{-1}\big[\zeta(\tau_{\ell-1})f_{\ell-1}(E_r)\big]\nabla\tau_{\ell-1}\\
&\le\frac{\mathfrak{M}_1E_r}{1-\eta}\int_0^{\mathfrak{T}}\aleph_0(\tau_{\ell-1},\tau_{\ell-1})\phi^{-1}\big[\zeta(\tau_{\ell-1})\big]\nabla\tau_{\ell-1}\\
&\le\frac{\mathfrak{M}_1E_r}{1-\eta}\int_0^{\mathfrak{T}}\aleph_0(\tau_{\ell-1},\tau_{\ell-1})\phi^{-1}\bigg[\prod_{i=1}^n\zeta_i(\tau_{\ell-1})\bigg]\nabla\tau_{\ell-1}\\
&\le\frac{\mathfrak{M}_1E_r}{1-\eta}\int_0^{\mathfrak{T}}\aleph_0(\tau_{\ell-1},\tau_{\ell-1})\prod_{i=1}^n\phi^{-1}(\zeta_i(\tau_{\ell-1}))\nabla\tau_{\ell-1}\\
&\le\frac{\mathfrak{M}_1E_r}{1-\eta}\|\aleph_0\|_{L^q_\nabla}\prod_{i=1}^n\big\|\phi^{-1}(\zeta_i)\big\|_{L^{p_i}_\nabla}\\
&\le E_r.\end{aligned}$$

Continuing with this bootstrapping argument, we get

$$\begin{aligned}(\Omega\vartheta_1)(t)&=\int_0^{\mathfrak{T}}\aleph(t,\tau_1)\phi^{-1}\Bigg[\zeta(\tau_1)f_1\bigg(\int_0^{\mathfrak{T}}\aleph(\tau_1,\tau_2)\phi^{-1}\Bigg[\zeta(\tau_2)f_2\bigg(\int_0^{\mathfrak{T}}\aleph(\tau_2,\tau_3)\\
&\quad\times\phi^{-1}\Bigg[\zeta(\tau_3)f_3\bigg(\int_0^{\mathfrak{T}}\aleph(\tau_3,\tau_4)\cdots\\
&\quad\times f_{\ell-1}\bigg(\int_0^{\mathfrak{T}}\aleph(\tau_{\ell-1},\tau_\ell)\phi^{-1}\big[\zeta(\tau_\ell)f_\ell(\vartheta_1(\tau_\ell))\big]\nabla\tau_\ell\bigg)\cdots\nabla\tau_3\Bigg]\nabla\tau_2\Bigg]\nabla\tau_1\\
&\le E_r.\end{aligned}$$

Since $E_r=\|\vartheta_1\|$ for $\vartheta_1\in\mathtt{P}_{\mathfrak{z}_r}\cap\partial\Lambda_{1,r}$, we get

$$\|\Omega\vartheta_1\|\le\|\vartheta_1\|. \tag{3.1}$$

Let $t\in[\mathfrak{z}_r,\mathfrak{T}-\mathfrak{z}_r]_{\mathbb{T}}$. Then,

$$O_r=\|\vartheta_1\|\ge\vartheta_1(t)\ge\min_{t\in[\mathfrak{z}_r,\mathfrak{T}-\mathfrak{z}_r]_{\mathbb{T}}}\vartheta_1(t)\ge\frac{\mathfrak{z}_r}{\mathfrak{T}}\|\vartheta_1\|\ge\frac{\mathfrak{z}_r}{\mathfrak{T}}O_r.$$

By (C_2) and for $\tau_{\ell-1} \in [\mathfrak{z}_r, \mathfrak{T} - \mathfrak{z}_r]_{\mathbb{T}}$, we have

$$\begin{aligned}
&\int_0^{\mathfrak{T}} \aleph(\tau_{\ell-1}, \tau_\ell)\phi^{-1}\big[\zeta(\tau_\ell) f_\ell(\vartheta_1(\tau_\ell))\big]\nabla\tau_\ell \\
&\geq \lambda_{\mathfrak{z}_r} \int_{\mathfrak{z}_r}^{\mathfrak{T}-\mathfrak{z}_r} \aleph_0(\tau_\ell, \tau_\ell)\phi^{-1}\big[\zeta(\tau_\ell) f_\ell(\vartheta_1(\tau_\ell))\big]\nabla\tau_\ell \\
&\geq \lambda_{\mathfrak{z}_r} \mathfrak{Z} O_r \int_{\mathfrak{z}_r}^{\mathfrak{T}-\mathfrak{z}_r} \aleph_0(\tau_\ell, \tau_\ell)\phi^{-1}(\zeta(\tau_\ell))\nabla\tau_\ell \\
&\geq \lambda_{\mathfrak{z}_r} \mathfrak{Z} O_r \int_{\mathfrak{z}_r}^{\mathfrak{T}-\mathfrak{z}_r} \aleph_0(\tau_\ell, \tau_\ell) \prod_{i=1}^{n} \phi^{-1}(\zeta_i(\tau_\ell))\nabla\tau_\ell \\
&\geq \lambda_{\mathfrak{z}_1} \mathfrak{Z} O_r \prod_{i=1}^{n} \delta_i \int_{\mathfrak{z}_1}^{\mathfrak{T}-\mathfrak{z}_1} \aleph_0(\tau_\ell, \tau_\ell)\nabla\tau_\ell \\
&\geq O_r.
\end{aligned}$$

Continuing with bootstrapping argument, we get

$$\begin{aligned}
(\Omega\vartheta_1)(t) &= \int_0^{\mathfrak{T}} \aleph(t, \tau_1)\phi^{-1}\Bigg[\zeta(\tau_1) f_1\Bigg(\int_0^{\mathfrak{T}} \aleph(\tau_1, \tau_2)\phi^{-1}\Bigg[\zeta(\tau_2) f_2\Bigg(\int_0^{\mathfrak{T}} \aleph(\tau_2, \tau_3) \\
&\quad \times \phi^{-1}\Bigg[\zeta(\tau_3) f_3\Bigg(\int_0^{\mathfrak{T}} \aleph(\tau_3, \tau_4) \cdots \\
&\quad \times f_{\ell-1}\Bigg(\int_0^{\mathfrak{T}} \aleph(\tau_{\ell-1}, \tau_\ell)\phi^{-1}\big[\zeta(\tau_\ell) f_\ell(\vartheta_1(\tau_\ell))\big]\nabla\tau_\ell\Bigg) \cdots \nabla\tau_3\Bigg]\nabla\tau_2\Bigg]\nabla\tau_1 \\
&\geq O_r.
\end{aligned}$$

Thus, if $\vartheta_1 \in \mathtt{P}_{\mathfrak{z}_r} \cap \partial\Lambda_{2,r}$, then

$$\|\Omega\vartheta_1\| \geq \|\vartheta_1\|. \tag{3.2}$$

It is evident that $0 \in \Lambda_{2,k} \subset \overline{\Lambda}_{2,k} \subset \Lambda_{1,k}$. From (3.1),(3.2), it follows from Theorem 3.1 that the operator Ω has a fixed point $\vartheta_1^{[r]} \in \mathtt{P}_{\mathfrak{z}_r} \cap \left(\overline{\Lambda}_{1,r} \backslash \Lambda_{2,r}\right)$ such that $\vartheta_1^{[r]}(t) \geq 0$ on $[0, \mathfrak{T}]_{\mathbb{T}}$, and $r \in \mathbb{N}$. Next setting $\vartheta_{\ell+1} = \vartheta_1$, we obtain denumerably many positive solutions $\{(\vartheta_1^{[r]}, \vartheta_2^{[r]}, \cdots, \vartheta_\ell^{[r]})\}_{r=1}^{\infty}$ of (1.1)–(1.2) given iteratively by

$$\vartheta_j(t) = \int_0^{\mathfrak{T}} \aleph(t, \tau)\phi^{-1}\big[\zeta(\tau) f_j(\vartheta_{j+1}(\tau))\big]\nabla\tau,\ t \in [0, \mathfrak{T}]_{\mathbb{T}},\ j = \ell, \ell-1, \cdots, 1.$$

The proof is completed. □

For $\displaystyle\sum_{i=1}^{n} \frac{1}{p_i} = 1$, we have the following theorem.

Theorem 3.5. *Suppose $(H_1)-(H_2)$ hold, let $\{\mathfrak{z}_r\}_{r=1}^{\infty}$ be a sequence with $\mathfrak{z}_r \in (t_{r+1}, t_r)$. Let $\{E_r\}_{r=1}^{\infty}$ and $\{O_r\}_{r=1}^{\infty}$ be such that*

$$E_{r+1} < \frac{\mathfrak{z}_r}{\mathfrak{T}} O_r < O_r < \mathfrak{Z} O_r < E_r,\ r \in \mathbb{N},$$

where

$$\mathfrak{Z} = \max\left\{ \left[\lambda_{\mathfrak{z}_1} \prod_{i=1}^{n} \delta_i \int_{\mathfrak{z}_1}^{\mathfrak{T}-\mathfrak{z}_1} \aleph_0(\tau,\tau)\nabla\tau\right]^{-1},\ 1\right\}.$$

Assume that f satisfies

(C_3) $f_j(\vartheta) \le \phi(\mathfrak{M}_2 E_r)\ \forall\ t \in [0,\mathfrak{T}]_{\mathbb{T}},\ 0 \le \vartheta \le E_r,$
where

$$\mathfrak{M}_2 < \min\left\{ \left[\frac{1}{1-\eta}\|\aleph_0\|_{L^{\infty}_{\nabla}} \prod_{i=1}^{n} \left\|\phi^{-1}(\zeta_i)\right\|_{L^{p_i}_{\nabla}}\right]^{-1}, \mathfrak{Z}\right\},$$

(C_4) $f_j(\vartheta) \ge \phi(\mathfrak{Z} O_r)\ \forall\ t \in [\mathfrak{z}_r, 1-\mathfrak{z}_r]_{\mathbb{T}},\ \frac{\mathfrak{z}_r}{\mathfrak{T}} O_r \le \vartheta \le O_r.$

Then the iterative boundary value problem (1.1)–(1.2) *has denumerably many solutions $\{(\vartheta_1^{[r]}, \vartheta_2^{[r]}, \cdots, \vartheta_\ell^{[r]})\}_{r=1}^{\infty}$ such that $\vartheta_j^{[r]}(t) \ge 0$ on $[0,\mathfrak{T}]_{\mathbb{T}}$, $j = 1, 2, \cdots, \ell$ and $r \in \mathbb{N}$.*

Proof. For a fixed r, let $\Lambda_{1,r}$ be as in the proof of Theorem 3.4 and let $\vartheta_1 \in \mathsf{P}_{\mathfrak{z}_r} \cap \partial\Lambda_{2,r}$. Again

$$\vartheta_1(\tau) \le E_r = \|\vartheta_1\|,$$

for all $\tau \in [0,\mathfrak{T}]_{\mathbb{T}}$. By $(C3)$ and for $\tau_{\ell-1} \in [0,\mathfrak{T}]_{\mathbb{T}}$, we have

$$\begin{aligned}
&\int_0^{\mathfrak{T}} \aleph(\tau_{\ell-1},\tau_\ell)\phi^{-1}\big[\zeta(\tau_\ell) f_\ell(\vartheta_1(\tau_\ell))\big]\nabla\tau_\ell \\
&\le \frac{1}{1-\eta}\int_0^{\mathfrak{T}} \aleph_0(\tau_\ell,\tau_\ell)\phi^{-1}\big[\zeta(\tau_\ell) f_\ell(\vartheta_1(\tau_\ell))\big]\nabla\tau_\ell \\
&\le \frac{\mathfrak{M}_2 E_r}{1-\eta}\int_0^{\mathfrak{T}} \aleph_0(\tau_\ell,\tau_\ell)\phi^{-1}\big[\zeta(\tau_\ell)\big]\nabla\tau_\ell \\
&\le \frac{\mathfrak{M}_2 E_r}{1-\eta}\int_0^{\mathfrak{T}} \aleph_0(\tau_\ell,\tau_\ell)\phi^{-1}\left[\prod_{i=1}^{n}\zeta_i(\tau_\ell)\right]\nabla\tau_\ell \\
&\le \frac{\mathfrak{M}_2 E_r}{1-\eta}\int_0^{\mathfrak{T}} \aleph_0(\tau_\ell,\tau_\ell)\prod_{i=1}^{n}\phi^{-1}(\zeta_i(\tau_\ell))\nabla\tau_\ell \\
&\le \frac{\mathfrak{M}_2 E_r}{1-\eta}\|\aleph_0\|_{L^{\infty}_{\nabla}}\prod_{i=1}^{n}\left\|\phi^{-1}(\zeta_i)\right\|_{L^{p_i}_{\nabla}} \\
&\le E_r.
\end{aligned}$$

It follows in similar manner (for $\tau_{\ell-2} \in [0, \mathfrak{T}]_{\mathbb{T}}$,) that

$$\begin{aligned}
\int_0^{\mathfrak{T}} \aleph(\tau_{\ell-2}, \tau_{\ell-1}) \phi^{-1}\Bigg[\zeta(\tau_{\ell-1}) f_{\ell-1}\Bigg(\int_0^{\mathfrak{T}} \aleph(\tau_{\ell-1}, \tau_\ell)\phi^{-1}\big[\zeta(\tau_\ell) f_\ell(\vartheta_1(\tau_\ell))\big]\nabla\tau_\ell\Bigg)\Bigg]\nabla\tau_{\ell-1} \\
&\leq \int_0^{\mathfrak{T}} \aleph(\tau_{\ell-2}, \tau_{\ell-1}) \phi^{-1}\big[\zeta(\tau_{\ell-1}) f_{\ell-1}(E_r)\big]\nabla\tau_{\ell-1} \\
&\leq \frac{1}{1-\eta}\int_0^{\mathfrak{T}} \aleph_0(\tau_{\ell-1}, \tau_{\ell-1}) \phi^{-1}\big[\zeta(\tau_{\ell-1}) f_{\ell-1}(E_r)\big]\nabla\tau_{\ell-1} \\
&\leq \frac{\mathfrak{M}_2 E_r}{1-\eta}\int_0^{\mathfrak{T}} \aleph_0(\tau_{\ell-1}, \tau_{\ell-1}) \phi^{-1}\big[\zeta(\tau_{\ell-1})\big]\nabla\tau_{\ell-1} \\
&\leq \frac{\mathfrak{M}_2 E_r}{1-\eta}\int_0^{\mathfrak{T}} \aleph_0(\tau_{\ell-1}, \tau_{\ell-1}) \phi^{-1}\Bigg[\prod_{i=1}^{n}\zeta_i(\tau_{\ell-1})\Bigg]\nabla\tau_{\ell-1} \\
&\leq \frac{\mathfrak{M}_2 E_r}{1-\eta}\int_0^{\mathfrak{T}} \aleph_0(\tau_{\ell-1}, \tau_{\ell-1}) \prod_{i=1}^{n}\phi^{-1}(\zeta_i(\tau_{\ell-1}))\nabla\tau_{\ell-1} \\
&\leq \frac{\mathfrak{M}_2 E_r}{1-\eta}\|\aleph_0\|_{L^\infty_\nabla}\prod_{i=1}^{n}\big\|\phi^{-1}(\zeta_i)\big\|_{L^{p_i}_\nabla} \\
&\leq E_r.
\end{aligned}$$

Continuing with this bootstrapping argument, we get

$$\begin{aligned}
(\Omega\vartheta_1)(t) &= \int_0^{\mathfrak{T}} \aleph(t, \tau_1)\phi^{-1}\Bigg[\zeta(\tau_1) f_1\Bigg(\int_0^{\mathfrak{T}} \aleph(\tau_1, \tau_2)\phi^{-1}\Bigg[\zeta(\tau_2) f_2\Bigg(\int_0^{\mathfrak{T}} \aleph(\tau_2, \tau_3) \\
&\quad \times \phi^{-1}\Bigg[\zeta(\tau_3) f_3\Bigg(\int_0^{\mathfrak{T}} \aleph(\tau_3, \tau_4)\cdots \\
&\quad \times f_{\ell-1}\Bigg(\int_0^{\mathfrak{T}} \aleph(\tau_{\ell-1}, \tau_\ell)\phi^{-1}\big[\zeta(\tau_\ell) f_\ell(\vartheta_1(\tau_\ell))\big]\nabla\tau_\ell\Bigg)\cdots\nabla\tau_3\Bigg]\nabla\tau_2\Bigg]\nabla\tau_1 \\
&\leq E_r.
\end{aligned}$$

Since $E_r = \|\vartheta_1\|$ for $\vartheta_1 \in \mathtt{P}_{\mathfrak{z}_r} \cap \partial\Lambda_{1,r}$, we get

$$\|\Omega\vartheta_1\| \leq \|\vartheta_1\|. \tag{3.3}$$

Now define $\Lambda_{2,r} = \{\vartheta_1 \in \mathtt{X} : \|\vartheta_1\| < O_r\}$. Let $\vartheta \in \mathtt{P}_{\mathfrak{z}_r} \cap \partial\Lambda_{2,r}$ and let $\tau \in [\mathfrak{z}_r, \mathfrak{T} - \mathfrak{z}_r]_{\mathbb{T}}$. Then, the argument leading to (3.2) can be done to the present case. Hence, the theorem. □

Lastly, the case $\displaystyle\sum_{i=1}^{n}\frac{1}{p_i} > 1$.

Theorem 3.6. *Suppose* $(H_1)-(H_2)$ *hold, let* $\{\mathfrak{z}_r\}_{r=1}^{\infty}$ *be a sequence with* $\mathfrak{z}_r \in (t_{r+1}, t_r)$. *Let* $\{E_r\}_{r=1}^{\infty}$ *and* $\{O_r\}_{r=1}^{\infty}$ *be such that*

$$E_{r+1} < \frac{\mathfrak{z}_r}{\mathfrak{T}} O_r < O_r < \mathfrak{Z} O_r < E_r,\ r \in \mathbb{N},$$

where

$$\mathfrak{Z} = \max\left\{ \left[\lambda_{\mathfrak{z}_1} \prod_{i=1}^{n} \delta_i \int_{\mathfrak{z}_1}^{\mathfrak{T}-\mathfrak{z}_1} \aleph_0(\tau,\tau)\nabla\tau \right]^{-1},\ 1 \right\}.$$

Assume that f *satisfies*

(C_5) $f_j(\vartheta) \le \phi(\mathfrak{M}_3 E_r)\ \forall\, t \in [0, \mathfrak{T}]_{\mathbb{T}},\ 0 \le \vartheta \le E_r,$
where

$$\mathfrak{M}_3 < \min\left\{ \left[\frac{1}{1-\eta} \|\aleph_0\|_{L^{\infty}_{\nabla}} \prod_{i=1}^{n} \left\|\phi^{-1}(\zeta_i)\right\|_{L^1_{\nabla}} \right]^{-1}, \mathfrak{Z} \right\},$$

(C_6) $f_j(\vartheta) \ge \phi(\mathfrak{Z} O_r)\ \forall\, t \in [\mathfrak{z}_r, 1-\mathfrak{z}_r]_{\mathbb{T}},\ \frac{\mathfrak{z}_r}{\mathfrak{T}} O_r \le \vartheta \le O_r.$

Then the iterative boundary value problem (1.1)–(1.2) *has denumerably many solutions* $\{(\vartheta_1^{[r]}, \vartheta_2^{[r]}, \cdots, \vartheta_\ell^{[r]})\}_{r=1}^{\infty}$ *such that* $\vartheta_j^{[r]}(t) \ge 0$ *on* $[0, \mathfrak{T}]_{\mathbb{T}}$, $j = 1, 2, \cdots, \ell$ *and* $r \in \mathbb{N}$.

Proof. The proof is similar to the proof of Theorem 3.1. Therefore, we omit the details here. □

4 Examples

In this section, we present an example to check validity of our main results.

Example 4.1. Consider the following boundary value problem on $\mathbb{T} = [0, 1]$.

$$\left.\begin{array}{c} \phi(\vartheta_j''(t)) + \zeta(t) f_j(\vartheta_{j+1}(t)) = 0, j = 1, 2, \\ \vartheta_3(t) = \vartheta_1(t), \\ \vartheta_j(0) = \vartheta_j(1) = \displaystyle\int_0^1 \frac{1}{2}\vartheta_j(\tau) d\tau \end{array}\right\} \tag{4.1}$$

where

$$\phi(\vartheta) = \begin{cases} \dfrac{\vartheta^3}{1+\vartheta^2}, & \vartheta \le 0, \\ \vartheta^2, & \vartheta > 0, \end{cases}$$

$$\zeta(t) = \zeta_1(t)\zeta_2(t)$$

in which

$$\zeta_1(t) = \frac{1}{|t-\frac{1}{4}|^{\frac{1}{2}}} \quad \text{and} \quad \zeta_2(t) = \frac{1}{|t-\frac{1}{3}|^{\frac{1}{2}}},$$

$$f_1(\vartheta) = f_2(\vartheta) = \begin{cases} \frac{1}{50}\times 10^{-16}, & \vartheta \in (10^{-16}, +\infty), \\ \frac{149125\times 10^{-(16r+8)} - \frac{1}{50}\times 10^{-16r}}{10^{-(16r+8)} - 10^{-16r}}(\vartheta - 10^{-16r}) + \frac{1}{50}\times 10^{-16r}, & \vartheta \in \left[10^{-(16r+8)}, 10^{-16r}\right], \\ 149125\times 10^{-(16r+8)}, & \vartheta \in \left(\frac{1}{5}\times 10^{-(16r+8)}, 10^{-(16r+8)}\right), \\ \frac{149125\times 10^{-(16r+8)} - \frac{1}{50}\times 10^{-(16r+16)}}{\frac{1}{5}\times 10^{-(16r+8)} - 10^{-(16r+16)}}(\vartheta - 10^{-(16r+16)}) + \frac{1}{50}\times 10^{-(16r+16)}, & \vartheta \in \left(10^{-(16r+16)}, \frac{1}{5}\times 10^{-(16r+8)}\right]. \end{cases}$$

Let

$$t_r = \frac{31}{64} - \sum_{k=1}^{r} \frac{1}{4(k+1)^4},\ \mathfrak{z}_r = \frac{1}{2}(t_r + t_{r+1}),\ r = 1,2,3,\cdots,$$

then

$$\mathfrak{z}_1 = \frac{15}{32} - \frac{1}{648} < \frac{15}{32}$$

and

$$t_{r+1} < \mathfrak{z}_r < t_r,\ \mathfrak{z}_r > \frac{1}{5}.$$

Therefore,

$$\frac{\mathfrak{z}_r}{\mathfrak{T}} = \mathfrak{z}_r > \frac{1}{5},\ j = 1,2,3,\cdots.$$

It is clear that

$$t_1 = \frac{15}{32} < \frac{1}{2},\ t_r - t_{r+1} = \frac{1}{4(r+2)^4},\ r = 1,2,3,\cdots.$$

Since $\sum_{j=1}^{\infty} \frac{1}{j^4} = \frac{\pi^4}{90}$ and $\sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6}$, it follows that

$$t^* = \lim_{r\to\infty} t_r = \frac{31}{64} - \sum_{k=1}^{\infty} \frac{1}{4(r+1)^4} = \frac{47}{64} - \frac{\pi^4}{360} > \frac{1}{5},$$

$$\zeta_1, \zeta_2 \in L^p[0,1] \quad \text{for all} \quad 0 < p < 2, \quad \text{so} \quad \delta_1 = \delta_2 = \frac{1}{\sqrt{3}}.$$

$$\eta = \int_0^{\mathfrak{T}} \kappa(\tau)\nabla\tau = \int_0^1 \frac{1}{2} d\tau = \frac{1}{2}, \eta_{\mathfrak{z}_1} = \int_{\mathfrak{z}_1}^{\mathfrak{T}-\mathfrak{z}_1} \kappa(t)\nabla t = \frac{1}{2}(1-2\mathfrak{z}_1) = 0.03279320988,$$

$$\lambda_{\mathfrak{z}_1} = \frac{\mathfrak{z}_1}{\mathfrak{T}}\left[1 + \frac{\eta_{\mathfrak{z}_1}}{1-\eta}\right] = \mathfrak{z}_1(1 + 2\eta_{\mathfrak{z}_1}) = 0.4978492109,$$

$$\int_{\mathfrak{z}_1}^{\mathfrak{T}-\mathfrak{z}_1} \aleph_0(\tau,\tau)\nabla\tau = \int_{\frac{15}{32}-\frac{1}{648}}^{1-\frac{15}{32}+\frac{1}{648}} \tau(1-\tau)d\tau = 0.01637309451.$$

So, we get

$$\begin{aligned} \mathfrak{Z} &= \max\left\{ \left[\lambda_{\mathfrak{z}_1} \prod_{i=1}^{n} \delta_i \int_{\mathfrak{z}_1}^{\mathfrak{T}-\mathfrak{z}_1} \aleph_0(\tau,\tau)\nabla\tau\right]^{-1}, \ 1\right\} \\ &= \max\left\{386.1654402, 1\right\} \\ &= 386.1654402. \end{aligned}$$

$$\|\aleph_0\|_{L^q_\nabla} = \left[\int_0^1 |\aleph_0(\tau,\tau)|^q d\tau\right]^{\frac{1}{q}} < 1 \ \text{ for any } \ 0 < q < 2.$$

Next, let $0 < \mathfrak{a} < 1$ be fixed. Then $\zeta_1, \zeta_2 \in L^{1+\mathfrak{a}}[0,1]$. It follows that

$$\|\phi^{-1}(\zeta_1)\|_{1+\mathfrak{a}} = \left[\frac{1}{3-\mathfrak{a}}\left(3^{\frac{3-\mathfrak{a}}{4}}+1\right)2^{\frac{1+\mathfrak{a}}{2}}\right]^{\frac{1}{1+\mathfrak{a}}}$$

$$\|\phi^{-1}(\zeta_2)\|_{1+\mathfrak{a}} = \left[\frac{4}{3-\mathfrak{a}}\left(2^{\frac{3-\mathfrak{a}}{4}}+1\right)(1/3)^{\frac{3-\mathfrak{a}}{4}}\right]^{\frac{1}{1+\mathfrak{a}}}.$$

So, for $0 < \mathfrak{a} < 1$, we have

$$0.1811770116 \le \left[\frac{1}{1-\eta}\|\aleph_0\|_{L^q_\nabla} \prod_{i=1}^{n} \left\|\phi^{-1}(\zeta_i)\right\|_{L^{p_i}_\nabla}\right]^{-1} \le 185.5612032.$$

Taking $\mathfrak{M}_1 = 0.17$. In addition if we take

$$E_r = 10^{-8r},\ O_r = 10^{-(8r+4)},$$

then

$$\begin{aligned} E_{r+1} = 10^{-(8r+8)} &< \frac{1}{5} \times 10^{-(8r+4)} < \frac{\mathfrak{z}_r}{\mathfrak{T}} O_r \\ &< O_r = 10^{-(8r+4)} < E_r = 10^{-8r}, \end{aligned}$$

$\mathfrak{Z}O_r = 386.1654402 \times 10^{-(8r+4)} < 0.17 \times 10^{-8r} = \mathfrak{M}_1 E_r$, $r = 1, 2, 3, \cdots$, and f_1, f_2 satisfies the following growth conditions:

$$f_1(\vartheta) = f_2(\vartheta) \leq \phi(\mathfrak{M}_1 E_r) = \mathfrak{M}_1^2 E_r^2 = 0.0289 \times 10^{-16r}, \ \ \vartheta \in \left[0, 10^{-16r}\right]$$

$$\begin{aligned} f_1(\vartheta) = f_2(\vartheta) &\geq \phi(\mathfrak{Z}O_r) = \mathfrak{Z}^2 O_r^2 \\ &= 149123.7162 \times 10^{-(16r+8)}, \ \ \vartheta \in \left[\frac{1}{5} \times 10^{-(16r+8)}, 10^{-(16r+8)}\right]. \end{aligned}$$

Then all the conditions of Theorem 3.4 are satisfied. Therefore, by Theorem 3.4, the iterative boundary value problem (1.1) has denumerably many solutions $\{(\vartheta_1^{[r]}, \vartheta_2^{[r]})\}_{r=1}^{\infty}$ such that $\vartheta_j^{[r]}(t) \geq 0$ on $[0, 1]$, $j = 1, 2$ and $r \in \mathbb{N}$.

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