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Period-Doubling and Naimark–Sacker Bifurcations of Certain Second Order Quadratic Fractional Difference Equations

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Abstract

We investigate the period-doubling and Naimark–Sacker bifurcations of the equilibrium of the difference equation

$$x_{n+1} = \frac{\gamma x_{n-1}^2 + \delta x_n}{C x_{n-1}^2 + x_n}$$

where the parameters γ, δ, C are positive numbers and the initial conditions x_{-1} and x_0 are arbitrary nonnegative numbers such that $x_{-1} + x_0 > 0$.

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1 Introduction and Preliminaries

In this paper we study the period-doubling and the Naimark–Sacker bifurcations of the equilibrium of the difference equation

$$x_{n+1} = \frac{\gamma x_{n-1}^2 + \delta x_n}{C x_{n-1}^2 + x_n}, \quad n = 0, 1, \dots \tag{1.1}$$

where the parameters γ, δ, C are positive numbers and the initial conditions x_{-1} and x_0 are arbitrary nonnegative numbers such that $x_{-1} + x_0 > 0$.

Equation (1.1) is a special case of a general second order quadratic fractional equation of the form

$$x_{n+1} = \frac{A x_n^2 + B x_n x_{n-1} + C x_{n-1}^2 + D x_n + E x_{n-1} + F}{a x_n^2 + b x_n x_{n-1} + c x_{n-1}^2 + d x_n + e x_{n-1} + f}, \quad n = 0, 1, \dots \tag{1.2}$$

with non-negative parameters and initial conditions such that $A + B + C > 0$, $a + b + c + d + e + f > 0$ and $a x_n^2 + b x_n x_{n-1} + c x_{n-1}^2 + d x_n + e x_{n-1} + f > 0, n = 0, 1, \dots$. Several global asymptotic results for some special cases of Equation (1.2) were obtained in [14, 17, 18, 34].

Equation (1.1) has some similarities in appearance with the difference equation

$$x_{n+1} = \frac{\gamma x_{n-1} + \delta x_n}{C x_{n-1} + x_n}, \quad n = 0, 1, \dots \tag{1.3}$$

where the parameters γ, δ, C are positive numbers and the initial conditions x_{-1} and x_0 are arbitrary nonnegative numbers such that $x_{-1} + x_0 > 0$, studied in detail in [20, 21]. The global dynamics of Equation (1.3) is simple and consists of a single global period-two bifurcation. Namely, the unique equilibrium is globally asymptotically stable for $\gamma \leq C\gamma + \delta + 3C\delta$ and the unique period-two solution emerges for $\gamma > C\gamma + \delta + 3C\delta$ which attracts all solutions except those that start on the global stable manifold of the equilibrium. As we will show the global dynamics of Equation (1.1) is quite different than global dynamics of Equation (1.3) exhibiting several global period-two bifurcations as well as Naimark–Sacker bifurcation, which is local and does not give information about the global behavior of solutions. In addition, Equation (1.1) can have up to three period-two solutions in the parametric region $C\delta < \gamma$. The mathematical techniques used to analyze Equation (1.3) were straightforward and the proofs of global results were based on the mixture of fixed point techniques from [20, 21, 23] and monotone maps techniques from [24–26]. In addition to those techniques, the mathematical tools that we will use to analyze Equation (1.1) involve complicated analysis of number of solutions of polynomial equations, use of resultants and use of Naimark–Sacker bifurcation result. These will involve the new method of proving local stability as demonstrated in the proofs of theorems in Section 3. Equation (1.3) plays an important role in biochemical networks [10].

The first systematic study of global dynamics of a special quadratic fractional case of Equation (1.2) where $A = C = D = a = c = d = 0$ was performed in [3, 4]. Dynamics of some related quadratic fractional difference equations was considered in the papers [14, 17, 18, 30, 34].

The following global dynamics result will be useful in our investigation of the global character of solutions of the difference equations, see [7].

Theorem 1.1. *Let I be a set of real numbers and let*

$$F : I \times I \to I$$

be a function $F(u, v)$ which decreases in u and increases in v. Then for every solution $\{x_n\}_{n=-1}^{\infty}$ of the equation

$$x_{n+1} = F(x_n, x_{n-1}), \quad n = 0, 1, ... \tag{1.4}$$

the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n+1}\}_{n=-1}^{\infty}$ of even and odd terms are eventually monotonic.

Consider a first order system of difference equations of the form

$$\left\{ \begin{array}{lcl} x_{n+1} & = & f(x_n, y_n) \\ y_{n+1} & = & g(x_n, y_n) \end{array} \right. , \quad n = 0, 1, \ldots, \quad (x_{-1}, x_0) \in \mathcal{I} \times \mathcal{I} \tag{1.5}$$

where $f, g : \mathcal{I} \times \mathcal{I} \to \mathcal{I}$ are continuous functions on an interval $\mathcal{I} \subset \mathbb{R}$, $f(x, y)$ is non-decreasing in x and non-increasing in y, and $g(x, y)$ is non-increasing in x and non-decreasing in y. Such system is called *competitive*. One may associate a competitive map T to a competitive system (1.5) by setting $T = (f, g)$ and considering T on $\mathcal{B} = \mathcal{I} \times \mathcal{I}$. Theory of competitive systems and maps in the plane have been extensively developed and main results are given in [24–27]. The general theory of monotone systems in ordered Banach spaces is given in [12] with many applications to different types of dynamical systems such as elliptic and parabolic differential equations, ordinary differential equations and delay differential equations. The advantage of the results in [24–27] is that they provide powerful tool for determining basins of attraction of equilibrium and periodic points which becomes the main objective in problems with several or infinite number of equilibrium or periodic points. See also [16, 28, 29, 31, 35] for different examples of planar competitive systems and their applications.

We now present some basic notions about competitive maps in the plane. Define a South-east partial order $\preceq_{se}$ on $\mathbb{R}^2$ so that the positive cone is the fourth quadrant, that is, $(x^1, y^1) \preceq_{se} (x^2, y^2)$ if and only if $x^1 \leq x^2$ and $y^1 \geq y^2$. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ the *order interval* $[\![\mathbf{x}, \mathbf{y}]\!]$ is the set of all $\mathbf{z}$ such that $\mathbf{x} \preceq \mathbf{z} \preceq \mathbf{y}$. A set $\mathcal{A}$ is said to be *linearly ordered* if $\preceq$ is a total order on $\mathcal{A}$. If a set $\mathcal{A} \subset \mathbb{R}^2$ is linearly ordered by $\preceq$, then the infimum $\mathbf{i} = \inf \mathcal{A}$ and supremum $\mathbf{s} = \sup \mathcal{A}$ of $\mathcal{A}$ exist in $\overline{\mathbb{R}}^2 = [-\infty, \infty] \times [-\infty, \infty]$. If both $\mathbf{i}$ and $\mathbf{s}$ belong to $\mathbb{R}^2$, then the linearly ordered set $\mathcal{A}$ is bounded, and conversely.

We note that the ordering $\preceq$ may be extended to the extended plane $\overline{\mathbb{R}}^2$ in a natural way. For example, $(0,\infty) \preceq (a,b)$ if $a \geq 0$ or $a = \infty$. If $\mathbf{x} \in \mathbb{R}^2$, we denote with $Q_\ell(\mathbf{x})$, $\ell \in \{1,2,3,4\}$, the four quadrants in $\overline{\mathbb{R}}^2$ relative to $\mathbf{x}$, i.e., $Q_1(x,y) = \{\ (u,v) \in \overline{\mathbb{R}}^2\ :\ u \geq x,\ v \geq y\ \}$, $Q_2(x,y) = \{\ (u,v) \in \overline{\mathbb{R}}^2\ :\ x \geq u,\ v \geq y\ \}$, and so on.

A *map* T on a set $\mathcal{B} \subset \mathbb{R}^2$ with non-empty interior is a continuous function $T : \mathcal{B} \to \mathcal{B}$. The map is *smooth* on $\mathcal{B}$ if T is continuously differentiable on the interior of $\mathcal{B}$. A set $\mathcal{A} \subset \mathcal{B}$ is *invariant* for the map T if $T(\mathcal{A}) \subset \mathcal{A}$. A point $\mathbf{x} \in \mathcal{B}$ is a *fixed point* of T if $T(\mathbf{x}) = \mathbf{x}$, and a *minimal period-two point* if $T^2(\mathbf{x}) = \mathbf{x}$ and $T(\mathbf{x}) \neq \mathbf{x}$. A *period-two point* is either a fixed point or a minimal period-two point. In a similar fashion one can define a minimal period p point. The *orbit of* $\mathbf{x} \in \mathcal{B}$ is the sequence $\{T^\ell(\mathbf{x})\}_{\ell=0}^\infty$. A *minimal period-two orbit* is an orbit $\{\mathbf{x}_\ell\}_{\ell=0}^\infty$ for which $\mathbf{x}_0 \neq \mathbf{x}_1$ and $\mathbf{x}_0 = \mathbf{x}_2$. The *basin of attraction* of a fixed point $\mathbf{x}$ is the set of all $\mathbf{y}$ such that $T^n(\mathbf{y}) \to \mathbf{x}$. A fixed point $\mathbf{x}$ is a *global attractor* on a set $\mathcal{A}$ if $\mathcal{A}$ is a subset of the basin of attraction of $\mathbf{x}$. A fixed point $\mathbf{x}$ is a *saddle point* if T is differentiable at $\mathbf{x}$, and the eigenvalues of the Jacobian matrix of T at $\mathbf{x}$ are such that one of them lies in the interior of the unit circle in $\mathbb{R}^2$, while the other eigenvalue lies in the exterior of the unit circle. If $T = (T_1, T_2)$ is a map on $\mathcal{R} \subset \mathbb{R}^2$, define the sets $\mathcal{R}_T(-,+) := \{(x,y) \in \mathcal{R} : T_1(x,y) \leq x,\ T_2(x,y) \geq y\ \}$ and $\mathcal{R}_T(+,-) := \{(x,y) \in \mathcal{R} : T_1(x,y) \geq x,\ T_2(x,y) \leq y\ \}$. For $\mathcal{A} \subset \mathbb{R}^2$ and $\mathrm{x} \in \mathbb{R}^2$, define the *distance from* x *to* $\mathcal{A}$ as $\mathrm{dist}(\mathrm{x}, \mathcal{A}) := \inf\{\|\mathrm{x} - \mathrm{y}\|\ :\ \mathrm{y} \in \mathcal{A}\}$.

A map T is *competitive* if $T(\mathbf{x}) \preceq_{se} T(\mathbf{y})$ whenever $\mathbf{x} \preceq_{se} \mathbf{y}$, and T is *strongly competitive* if $\mathbf{x} \preceq_{se} \mathbf{y}$ implies $T(\mathbf{x}) - T(\mathbf{y}) \in \{(u,v) : u > 0,\ v < 0\}$. If T is differentiable, a sufficient condition for T to be strongly competitive is that the Jacobian matrix of T at any $\mathbf{x} \in \mathcal{B}$ has the sign configuration

$$\begin{pmatrix} + & - \\ - & + \end{pmatrix}.$$

For additional definitions and results (e.g., repeller, hyperbolic fixed points, stability, asymptotic stability, stable and unstable manifolds) see [12, 13, 36] for competitive maps, and [22, 23] for difference equations. The following result is from [26] and it will be useful in our investigation of the global character of solutions of the difference equations.

Theorem 1.2. *Let T be a map on a set $\mathcal{R} \subset R^2$ with nonempty interior, such that T has an isolated fixed point $\bar{e} \in \mathcal{R}$. If T is differentiable and strongly monotonic on a neighborhood $U \subset \mathcal{R}$ of $\bar{e}$, then,*

i) The eigenvalues λ_1 and λ_2 of the Jacobian $J_T(\bar{e})$ of T at $\bar{e}$ are both real and satisfy $0 \leq |\lambda_2| < \lambda_1$. Corresponding associated eigenvectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ can be chosen so that the signs of their entries are $(+,-)$ and $(+,+)$ respectively.

ii) There exists $r_0 > 0$ such that both $B(\bar{e}; r_0)$, and $T(B(\bar{e}; r_0))$ are subsets of U, and such that if $0 < r < r_0$, *(a)* $\bar{e} + r\boldsymbol{v}_1 \preceq_{se} T(\bar{e} + r\boldsymbol{v}_1)$ *and* $T(\bar{e} - r\boldsymbol{v}_1) \preceq_{se} \bar{e} - r\boldsymbol{v}_1$ *if* $\lambda_1 > 1$.

(b) $T(\bar{e} + r\boldsymbol{v}_1) \preceq_{se} \bar{e} + r\boldsymbol{v}_1$ *and* $\bar{e} - rv_1 \preceq_{se} T(\bar{e} - r\boldsymbol{v}_1)$ *if* $\lambda_1 < 1$.

The following result is a direct consequence of the trichotomy theorem of Dancer and Hess, see [12] and [24], and is helpful for determining the basins of attraction of the equilibrium points.

Corollary 1.3. *If the nonnegative cone of* $\preceq$ *is a generalized quadrant in* $\mathbb{R}^n$*, and if* T *has no fixed points in* $[\![u_1, u_2]\!]$ *other than* u_1 *and* u_2*, then the interior of* $[\![u_1, u_2]\!]$ *is either a subset of the basin of attraction of* u_1 *or a subset of the basin of attraction of* u_2*.*

Remark 1.4. The connection between the theory of monotone maps and the asymptotic behavior of Equation (1.4) follows from the fact that if F is strongly decreasing in the first argument and strongly increasing in the second argument, then the second iterate of a map associated to Equation (1.4) is a strictly competitive map on $I \times I$, see [24–26].

This paper is organized as follows. The next section deals with the local stability of the equilibrium solution of Equation (1.1). In view of results for monotone maps in [24–27] and the way how these results are applied to second order difference equations in [6] the local dynamics of the equilibrium solutions and period-two solutions will determine the global dynamics in hyperbolic cases and some non-hyperbolic cases as well. The third section deals with the case $C\delta < \gamma$ where these results are applied to give global dynamics of Equation (1.1), which can be described as the sequence of transcritical (exchange of stability) bifurcations resulting in appearance of up to three period-two solutions. The fourth section deals with the case $C\delta > \gamma$ where we prove that in the parametric subregion of local asymptotic stability of the equilibrium one has also global asymptotic stability, while in the parametric region where the equilibrium solution is unstable the Naimark–Sacker bifurcation takes place and we provide the asymptotic formulas for parametric equations for invariant curve.

2 Linearized Stability Analysis of the Equilibrium Point

Let

$$F(u, v) = \frac{\gamma v^2 + \delta u}{Cv^2 + u}$$

be the map associated to Equation (1.1). Then

$$F'_u = \frac{v^2(C\delta - \gamma)}{(Cv^2 + u)^2} \quad \text{and} \quad F'_v = \frac{2uv(\gamma - C\delta)}{(Cv^2 + u)^2}.$$

Set $u_n = x_{n-1}$ and $v_n = x_n$ for $n = 0, 1, \ldots$ and write Equation (1.1) in the equivalent form:

$$u_{n+1} = v_n \tag{2.1}$$

$$v_{n+1} = \frac{\gamma u_n^2 + \delta v_n}{C u_n^2 + v_n}, \quad n = 0, 1, \ldots$$

Let T be the function defined by:

$$T(u, v) = (v, F(v, u)) = \left(v, \frac{\gamma u^2 + \delta v}{C u^2 + D v}\right). \tag{2.2}$$

then

$$(u_{n+1}, v_{n+1}) = T(u_n, v_n). \tag{2.3}$$

Now we have that

$$\begin{aligned} T^2(u, v) = T(T(u, v)) &= (G(u, v), H(u, v)) \\ &= \left(\frac{u^2\gamma + v\delta}{Cu^2 + v}, \frac{Cu^2v^2 + \gamma\delta u^2 + \gamma v^3 + \delta^2 v}{C^2u^2v^2 + Cv^3 + \gamma u^2 + \delta v}\right) \end{aligned}$$

from which it follows that

$$(u_{2n+2}, v_{2n+2}) = T^2(u_{2n}, v_{2n}) \tag{2.4}$$

which is equivalent to

$$(x_{2n+1}, x_{2n+2}) = T^2(x_{2n-1}, x_{2n}).$$

The Jacobian matrix of the map T has the form:

$$J_T(u, v) = \begin{pmatrix} 0 & 1 \\ \dfrac{2uv(\gamma - C\delta)}{(Cu^2 + v)^2} & \dfrac{u^2(C\delta - \gamma)}{(Cu^2 + v)^2} \end{pmatrix}. \tag{2.5}$$

The Jacobian matrix of the map T^2 has the form:

$$J_{T^2}(u, v) = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}, \tag{2.6}$$

where

$$\begin{aligned} J_{11} &= \frac{2uv(\gamma - C\delta)}{(Cu^2 + v)^2} \\ J_{12} &= \frac{u^2(C\delta - \gamma)}{(Cu^2 + v)^2} \\ J_{21} &= -\frac{2uv^3(\gamma - C\delta)^2}{(Cv^3 + C^2u^2v^2 + \delta v + u^2\gamma)^2} \\ J_{22} &= \frac{v(\gamma - C\delta)\left(v\left(3\gamma u^2 + 2v\delta\right) + C\left(2\gamma u^4 + v\delta u^2\right)\right)}{(Cv^3 + C^2u^2v^2 + \delta v + u^2\gamma)^2}. \end{aligned}$$

The determinant of (2.6) is given by

$$\det J_{T^2}(u,v) = \frac{4uv^2(\gamma - C\delta)^2\left(u^2\gamma + v\delta\right)}{\left(Cu^2+v\right)\left(C^2u^2v^2 + Cv^3 + u^2\gamma + v\delta\right)^2}. \tag{2.7}$$

The equilibrium points are positive solution of the equation $F(x,x) = x$, which is equivalent to $f(x) = 0$ where

$$f(x) = Cx^2 + x(1-\gamma) - \delta.$$

The following results follow from [30, Theorem 5].

Theorem 2.1. *Equation* (1.1) *has the unique equilibrium point*

$$\bar{x} = \frac{\sqrt{4\delta C + (1-\gamma)^2} - 1 + \gamma}{2C}$$

which is

i) locally asymptotically stable if and only if

$$-\frac{(1-3\gamma)(3-\gamma)}{16C} < \delta < \frac{(1+2\gamma)(2+\gamma)}{C};$$

ii) a repeller if and only if

$$\delta > \frac{(1+2\gamma)(2+\gamma)}{C};$$

iii) a saddle point if and only if

$$\delta < -\frac{(1-3\gamma)(3-\gamma)}{16C};$$

iv) a non-hyperbolic equilibrium if and only if

$$\delta = -\frac{(1-3\gamma)(3-\gamma)}{16C} \text{ or } \delta = \frac{(1+2\gamma)(2+\gamma)}{C}.$$

By direct inspection of (2.7) we obtain the following result.

Lemma 2.2. *If* $C\delta < \gamma$ *then the map* T^2 *is competitive on* $[0,\infty)^2 \setminus \{(0,0)\}$ *and strongly competitive on* $(0,\infty)^2$.

The following lemma which proof we skip is needed for global dynamics.

Lemma 2.3. *Assume that* $x_{-1}, x_0 \in [0,\infty)$ *and* $x_{-1} + x_0 > 0$. *The following holds:*

(a) If $C\delta < \gamma$, *then* $\delta < x_n < \dfrac{\gamma}{C}$ *for all* $n > 1$.

(b) If $C\delta > \gamma$, *then* $\dfrac{\gamma}{C} < x_n < \delta$ *for all* $n > 1$.

The proof of the following theorem follows directly from Theorem 1.1 and Lemmas 2.2 and 2.3.

Theorem 2.4. *Assume that* $C\delta < \gamma$. *Then every solution of Equation* (1.1) *converges to a period-two solution.*

3 The Case $0 < C\delta < \gamma$.

In this section we consider the case $0 < C\delta < \gamma$, when the transition function $f(x, y)$ decreases in x and increases in y, which in view of Theorem 1.1 implies that every solution of Equation (1.1) converges to either the unique equilibrium or period-two solution. Thus the global dynamics of Equation (1.1) is determined by the local stability of the unique equilibrium and the period-two solutions.

3.1 Existence of Period-Two Solutions

Assume that $\{\phi, \psi\}$ is a minimal period-two solution of Equation (1.1). Then

$$\phi = G(\psi, \phi), \psi = H(\phi, \psi) \quad \text{with } \psi, \phi \in [0, \infty), \phi \neq \psi$$

which is equivalent to

$$\phi = \frac{\gamma\phi^2 + \delta\psi}{C\phi^2 + \psi} \quad \text{and} \quad \psi = \frac{\gamma\psi^2 + \delta\phi}{C\psi^2 + \phi},$$

from which it immediately follows that

$$\phi\left(C\phi^2 + \psi\right) = \gamma\phi^2 + \delta\psi \tag{3.1}$$

and

$$\psi\left(C\psi^2 + \phi\right) = \gamma\psi^2 + \delta\phi. \tag{3.2}$$

By eliminating ψ and ϕ from (3.1) and (3.2) we get $g(\phi) = g(\psi) = 0,\ (\phi \neq \psi)$ where

$$\begin{aligned} g(u) := {} & C^3u^6 - C^2u^5(2\gamma + 1) + Cu^4\left(C\delta + \gamma^2 + 2\gamma + 1\right) \\ & - u^3(\gamma + 1)(2C\delta + \gamma) + u^2\delta\left(C\delta + \gamma^2 + 2\gamma + 1\right) - u(\gamma + 2)\delta^2 + \delta^3. \end{aligned}$$

Subtracting equations (3.1) and (3.2) we get

$$(\phi - \psi)\left(C\phi\psi + C\left(\phi^2 + \psi^2\right) - \gamma(\phi + \psi) + \delta\right) = 0. \tag{3.3}$$

Dividing Equation (3.1) by ϕ and Equation (3.2) by ψ and subtracting them we get

$$\frac{(\phi - \psi)(\phi\psi(C(\phi + \psi) - 1 - \gamma) + \delta(\phi + \psi))}{\phi\psi} = 0. \tag{3.4}$$

If we set

$$\phi + \psi = u, \quad \phi\psi = v$$

where $u, v > 0$, then ϕ and ψ are positive and different solutions of the quadratic equation

$$t^2 - ut + v = 0. \tag{3.5}$$

In addition to condition $u, v > 0$ it is necessary that $u^2 - 4v > 0$.

From (3.3) and (3.4) we get the system

$$\begin{cases} v(Cu - 1 - \gamma) + u\delta = 0 \\ -Cv + Cu^2 - u\gamma + \delta = 0. \end{cases} \tag{3.6}$$

Solving the second equation of system (3.6) by v we get

$$v = \frac{Cu^2 - u\gamma + \delta}{C}. \tag{3.7}$$

Substituting (3.7) in the first equation of system (3.6) we get that u satisfies the following equation

$$C^2u^3 - C(2\gamma + 1)u^2 + \left(2C\delta + \gamma^2 + \gamma\right)u - \gamma\delta - \delta = 0. \tag{3.8}$$

In a similar way one can show that v satisfies the following equation

$$C^3v^3 - C\left(\gamma - C\delta + 1\right)v^2 - \delta\left(C\delta - \gamma^2 - \gamma\right)v - \delta^3 = 0. \tag{3.9}$$

From this we obtain the following result.

Lemma 3.1. *i) Equation* (1.1) *has at most three distinct minimal period-two solutions.*

ii) $\{\Phi, \Psi\}$ *is a minimal period-two solution of Equation* (1.1) *if and only if there exists a positive solution* $\tilde{u}$ *of* (3.8) *such that* $\tilde{u}, \tilde{v} > 0$ *and* $\tilde{u}^2 - 4\tilde{v} > 0$, *where* $\tilde{v} = \dfrac{C\tilde{u}^2 - \tilde{u}\gamma + \delta}{C}$ *and*

$$\Phi = \frac{\tilde{u} - \sqrt{\tilde{u}^2 - 4\tilde{v}}}{2}, \quad \Psi = \frac{\tilde{u} + \sqrt{\tilde{u}^2 - 4\tilde{v}}}{2}.$$

3.2 Local Stability of Period-Two Solutions

Let $f_1(x) := a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a_0$ and $g_1(x) := b_mx^m + b_{m-1}x^{m-1} + ... + b_1x + b_0$ be two polynomials of degrees n and m, respectively. Their resultant (see [15, 19, 38]) $Res(f_1, g_1)$ is the determinant of the $(m+n) \times (m+n)$ Sylvester matrix given by

$$Syl(f_1, g_1) = \begin{pmatrix} a_n & a_{n-1} & \cdots & a_1 & a_0 & 0 & \cdots & 0 \\ 0 & a_n & a_{n-1} & \cdots & a_1 & a_0 & \cdots & 0 \\ \vdots & & & & & & & \\ 0 & \cdots & a_n & a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \\ b_m & b_{m-1} & \cdots & b_1 & b_0 & 0 & \cdots & 0 \\ 0 & b_m & b_{n-1} & \cdots & b_1 & b_0 & \cdots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & \cdots & b_m & b_{m-1} & b_{m-2} & \cdots & b_0 \end{pmatrix}.$$

The following theorem is from [15].

Theorem 3.2. *Assume that $f_1, g_1 \in \mathbb{R}[x]$ and $n, m \geq 1$. Then the following hold:*

(i) The discriminant of f_1 is given by

$$Dis(f_1) = (-1)^{n(n-1)/2} \frac{1}{a_n} Res(f_1, f_1')$$

and

$$Dis(f_1 \cdot g_1) = Dis(f_1) Dis(g_1) Res(f_1, g_1)^2.$$

(ii) $Dis(f_1) = 0 \Leftrightarrow f_1$ has a double root in $\mathbb{C}$. Equivalently, f_1 has n distinct roots in $\mathbb{C}$ if and only if $Dis(f_1) \neq 0$.

For two bivariate polynomials $f, g \in \mathbb{R}[x, y]$ the following holds (see [15, 38]).

Theorem 3.3. *Let $f(x, y), g(x, y) \in \mathbb{R}[x, y]$ where*

$$f(x, y) = \sum_{i=0}^{n} f_i(x) y^i \text{ and } g(x, y) = \sum_{i=0}^{m} g_i(x) y^i,$$

and let $r(x) = Res_y(f, g) \in \mathbb{R}[x]$ be the resultant of f and g with respect to the variable y. Then

(a) f and g have a nontrivial common factor if and only if r is identically zero.

(b) If f and g are co-prime (do not have a common factor), the following conditions are equivalent:

- *$\alpha \in \mathbb{C}$ is a root of r.*
- *$f_n(\alpha) = g_m(\alpha) = 0$ or there is $\beta \in \mathbb{C}$ with $f(\alpha, \beta) = 0 = g(\alpha, \beta) = 0$.*

(c) For all $(\alpha, \beta) \in \mathbb{C} \times \mathbb{C}$: If $f(\alpha, \beta) = 0 = g(\alpha, \beta) = 0$ then $r(\alpha) = 0$.

Period-two curves, that is the curves which intersection is a period-two solution, are given by

$$\mathcal{C}_{\tilde{F}} := \{(x, y) : F(y, x) = x\} = \{(x, y) : G(x, y) = x\},$$
$$\mathcal{C}_{\tilde{G}} := \{(x, y) : F(x, y) = y\} = \{(x, y) : H(x, y) = y\}.$$

Let

$$\tilde{F}(x, y) := Cx^3 - x^2\gamma + xy - y\delta,$$
$$\tilde{G}(x, y) := Cy^3 + xy - x\delta - y^2\gamma.$$

We see that $F(y, x) = x$ and $F(x, y) = y$ if and only if $\tilde{F}(x, y) = \tilde{G}(x, y) = 0$. We can view $\tilde{F}$ and $\tilde{G}$ as polynomials in y with coefficients in $\mathbb{R}[x]$ and as polynomials in x with coefficients in $\mathbb{R}[y]$. Let $Res_x(\tilde{F}, \tilde{G})$ and $Res_y(\tilde{F}, \tilde{G})$ be resultants (see [15,38]) of the polynomials $\tilde{F}(x, y)$ and $\tilde{G}(x, y)$ with respect to the variables x and y, respectively.

The following lemma can be verified by the package *Mathematica*.

Lemma 3.4. *The following hold:*

(i) $\tilde{F}(x,y) = \tilde{G}(y,x)$.

(ii) *If* $s(y) = Res_x(\tilde{F},\tilde{G})$ *and* $r(x) = Res_y(\tilde{F},\tilde{G})$ *then* $s(y) = -r(y)$ *where* $s(y) = yf(y)g(y)$.

(ii) *Discriminants and resultant of* f *and* g *are given by*

$$\begin{aligned}
\textit{Dis}(f) &= 4C\delta + (\gamma-1)^2 > 0\\
\textit{Dis}(g) &= -C^6\delta^6(C\delta-\gamma)^2\left(16C\delta+3\gamma^2-10\gamma+3\right)\big(32C^3\delta^3-13C^2\gamma^2\delta^2\\
&\quad -22C^2\gamma\delta-13C^2\delta^2+4C\gamma^4\delta+6C\gamma^3\delta+4C\gamma^2\delta+6C\gamma\delta+4C\delta\\
&\quad -\gamma^4-2\gamma^3-\gamma^2\big)^2.\\
\textit{Res}(f,g) &= C^3\delta^3(C\delta-\gamma)^2\left(16C\delta+3\gamma^2-10\gamma+3\right);
\end{aligned}$$

The following lemma gives the necessary condition under which an isolated period-two solution is non-hyperbolic.

Lemma 3.5. *If Equation* (1.1) *has a non-hyperbolic minimal period-two solution then* $Dis(g) = 0$.

Proof. Suppose that (Φ,Ψ) is a non-hyperbolic minimal period-two solution. It follows that $G'_y(\Phi,\Psi) \neq 0$ and $H'_y(\Phi,\Psi) \neq 1$. This implies that $\tilde{F}(\Phi,\Psi) = 0$, $\tilde{G}(\Phi,\Psi) = 0$. By Theorem 3.3 we have that $f(\Phi)g(\Phi) = 0$. Since Φ is not an equilibrium point we obtain $f(\Phi) \neq 0$ and $g(\Phi) = 0$. By the Implicit Function Theorem there are neighborhoods $I \subset \mathbb{R}$ of the point Φ and $J \subset \mathbb{R}$ of the point Ψ such that sets $\mathcal{C}_{\tilde{F}} \cap (I \times J)$ and $\mathcal{C}_{\tilde{G}} \cap (I \times J)$ are the graphs of the monotonic functions $y_{\tilde{F}}(x)$ and $y_{\tilde{G}}(x)$ for $x \in I$. Furthermore, taking derivatives of $G(x,y) = x$ and $H(x,y) = y$ with respect to x for $x \in I$ we get

$$y'_{\tilde{F}}(x) = \frac{1-G'_x(x,y)}{G'_y(x,y)} \quad \text{and} \quad y'_{\tilde{G}}(x) = \frac{H'_x(x,y)}{1-H'_y(x,y)}.$$

One can see that

$$\begin{aligned}
y'_{\tilde{F}}(\Phi) - y'_{\tilde{G}}(\Phi) &= \frac{1-G'_x(\Phi,\Psi)}{G'_y(\Phi,\Psi)} - \frac{H'_x(\Phi,\Psi)}{1-H'_y(\Phi,\Psi)}\\
&= \frac{1-e_1}{f_1} - \frac{g_1}{1-h_1} = \frac{1-(e_1+h_1)+(e_1h_1-f_1g_1)}{f_1(1-h_1)}\\
&= \frac{p(1)}{f_1(1-h_1)} = \frac{(1-\mu_1)(1-\mu_2)}{f_1(1-h_1)},
\end{aligned}$$

where $p(\mu)$ is the characteristic equation of the matrix

$$J_{T^2}(\Phi, \Psi) = \begin{pmatrix} e_1 & f_1 \\ g_1 & h_1 \end{pmatrix}.$$

By (2.6) and (2.7) we have that $\mu_1, \mu_2 \geq 0$. Since (Φ, Ψ) is a non-hyperbolic minimal period-two solution, we have that $\mu_1 = 1$ or $\mu_2 = 1$, which implies $y'_{\tilde{F}}(\Phi) - y'_{\tilde{G}}(\Phi) = 0$. By Theorem 3.3 the curves $\mathcal{C}_{\tilde{F}}$ and $\mathcal{C}_{\tilde{G}}$ have no common components. In view of [5, Lemmas 6 and 7], the curves $\mathcal{C}_{\tilde{F}}$ and $\mathcal{C}_{\tilde{G}}$ intersect tangentially at (Φ, Ψ) (i.e. $y'_{\tilde{F}}(\Phi) - y'_{\tilde{G}}(\Phi) = 0$) if and only if Φ is zero of $f(y)g(y)$ of multiplicity greater than one. Since $f(\Phi) \neq 0$ we have that Φ is zero of $g(y)$ of multiplicity greater than one. By Theorem 3.2, $g(y)$ has zero of multiplicity greater than one if and only if discriminant $Dis(g) = 0$, from which the proof follows. □

Lemma 3.6. *Assume that $\bar{x}$ is locally asymptotically stable and $Dis(g) \neq 0$. The following statements hold:*

i) Set of period-two solutions of T is totally ordered under South-East order $\preceq_{se}$.

ii) If $\{\phi_1, \psi_1\}$ and $\{\phi_2, \psi_2\}$ are two distinct minimal period-two solutions of Equation (1.1) *such that T has no other period-two solutions in $[[(\phi_1, \psi_1), (\phi_2, \psi_2)]]$ then, if one of them is locally asymptotically stable, the other is a saddle point.*

iii) Equation (1.1) *has zero or two minimal period-two solutions. If $\{\phi_1, \psi_1\}$ and $\{\phi_2, \psi_2\}$ are two distinct minimal period-two solutions of Equation* (1.1) *such that $(\phi_1, \psi_1) \ll_{se} (\phi_2, \psi_2) \ll_{se} (\bar{x}, \bar{x})$ then $\{\phi_1, \psi_1\}$ is locally asymptotically stable and $\{\phi_2, \psi_2\}$ is a saddle point.*

Proof. i) The period-two solutions are positive roots of the system $\tilde{F}(x, y) = 0$, $\tilde{G}(x, y) = 0$. Solving $\tilde{F}(x, y) = 0$ for y and $\tilde{G}(x, y) = 0$ for x we obtain

$$x_{\tilde{G}}(y) = \frac{y^2\gamma - Cy^3}{y - \delta}, \quad y_{\tilde{F}}(x) = \frac{x^2\gamma - Cx^3}{x - \delta}$$

and

$$x'_{\tilde{G}}(y) = \frac{y(Cy(3\delta - 2y) + \gamma(y - 2\delta))}{(y - \delta)^2}, \; y'_{\tilde{F}}(x) = \frac{x(Cx(3\delta - 2x) + \gamma(x - 2\delta))}{(x - \delta)^2}.$$

Since $\bar{x}$ is locally asymptotically stable, by Theorem 2.1 and the fact that $C\delta < \gamma$ we obtain that

$$-\frac{(1 - 3\gamma)(3 - \gamma)}{16C} < \delta < \frac{\gamma}{C}.$$

Using this one can prove that either

$$x'_{\tilde{G}}(y) < 0 \text{ for } 0 < y < \bar{x} \text{ and } y'_{\tilde{F}}(x) < 0 \text{ for } \bar{x} < x < \infty$$

or

$$y'_{\tilde{F}}(x) < 0 \text{ for } 0 < x < \bar{x} \text{ and } x'_{\tilde{G}}(y) < 0 \text{ for } \bar{x} < y < \infty.$$

Since, $x_{\tilde{G}}(\bar{x}) = y_{\tilde{F}}(\bar{x}) = \bar{x}$ we obtain that the period-two solutions belong to the decreasing curve. This completes the proof of the statement.

ii) Assume that, for example,

$$x'_{\tilde{G}}(y) < 0 \text{ for } 0 < y < \bar{x} \text{ and } y'_{\tilde{F}}(x) < 0 \text{ for } \bar{x} < x < \infty.$$

Since $T(\phi_1, \psi_1) = (\psi_1, \phi_1)$ and $T(\phi_2, \psi_2) = (\psi_2, \phi_2)$ we can suppose that $(\phi_1, \psi_1) \preceq_{se} (\phi_2, \psi_2) \preceq_{se} (\bar{x}, \bar{x})$. Let $p(\mu) = (\lambda - \mu_1^{(i)})(\lambda - \mu_2^{(i)})$ be the characteristic equation of the matrix

$$J_{T^2}(\Phi_i, \Psi_i) = \begin{pmatrix} e_i & f_i \\ g_i & h_i \end{pmatrix}.$$

Since $x'_{\tilde{G}}(\Psi_1) < 0$, $x'_{\tilde{G}}(\Psi_2) < 0$, $g_i = H'_x(\Phi_i, \Psi_i) < 0$ we obtain

$$x'_{\tilde{G}}(\Psi_i) = \frac{1 - H'_y(\Phi_i, \Psi_i)}{H'_x(\Phi_i, \Psi_i)} < 0, \ i = 1, 2,$$

which implies $h_i = H'_y(\Phi_i, \Psi_i) < 1, \ i = 1, 2$. Let $\tilde{y}(x) = y_{\tilde{F}}(x) - y_{\tilde{G}}(x)$. Now we have that $\tilde{y}(x)$ is continuously differentiable function on $\left(\delta, \dfrac{\gamma}{C}\right)$. Suppose that (ϕ_1, ψ_1) is locally asymptotically stable. Since

$$\tilde{y}'(\Phi_i) = \frac{(1 - \mu_1^{(i)})(1 - \mu_2^{(i)})}{f_i(1 - h_i)}$$

we obtain that $\tilde{y}'(\Phi_1) < 0$. Using $\tilde{y}(\Phi_i) = 0, \ i = 1, 2$, and continuity of $\tilde{y}(x)$ on $\left(\delta, \dfrac{\gamma}{C}\right)$ we get $\tilde{y}'(\Phi_2) > 0$. Since $f_2 < 0$ and $h_2 < 1$ we obtain $(1 - \mu_1^{(i)})(1 - \mu_2^{(i)}) < 0$, which shows that (ϕ_2, ψ_2) is a saddle point.

iii) Assume that Equation (1.1) has at least one prime period-two solution and let (ϕ_1, ψ_1) be the smallest prime period-two solution with respect to the order $\preceq_{se}$. By Lemma 2.3 interval $[[(\delta, \frac{\gamma}{C}), (\frac{\gamma}{C}, \delta)]]$ is an attracting set and it is invariant under the map T, which implies that $[[(\delta, \frac{\gamma}{C}), (\phi_1, \psi_1)]]$ is invariant under T^2. In view of Lemma 3.5 we obtain that (ϕ_1, ψ_1) is hyperbolic. By Lemma 2.4 we have that $T^{2n}(x_0, y_0)$ converge to (ϕ_1, ψ_1) as $n \to \infty$ for all $(x_0, y_0) \in [[(\delta, \frac{\gamma}{C}), (\phi_1, \psi_1)]]$, which implies that (ϕ_1, ψ_1) is locally asymptotically stable. By Corollary 1.3 the set $[[(\phi_1, \psi_1), (\bar{x}, \bar{x})]]$ contains at least one fixed point of T^2. Let (ϕ_2, ψ_2) be fixed point of T^2 such that T^2 has no other fixed point in $[[(\phi_1, \psi_1), (\phi_2, \psi_2)]]$. From ii) it follows that (ϕ_2, ψ_2) is a saddle point. Since T has at most three prime period-two solutions, statement i) and Corollary 1.3 implies that T has exactly two prime period-two solutions.

□

Lemma 3.7. *Assume that $\bar{x}$ is a saddle point and $Dis(g) \neq 0$. Then Equation* (1.1) *has one or three minimal period-two solutions. If Equation* (1.1) *has only one minimal period-two solution then it is locally asymptotically stable. If Equation* (1.1) *has three minimal period-two solutions $\{\Phi_1, \Psi_1\}$, $\{\Phi_2, \Psi_2\}$ and $\{\Phi_3, \Psi_3\}$ and if $\Phi_1 < \Phi_2 < \Phi_3 < \bar{x}$, then $\{\Phi_1, \Psi_1\}$ and $\{\Phi_3, \Psi_3\}$ are locally asymptotically stable and $\{\Phi_2, \Psi_2\}$ is a saddle point.*

Proof. In view of Theorem 1.2, since $(\bar{x}, \bar{x})$ is a saddle point, there exists $r_0 > 0$ such that $T^2((\bar{x}, \bar{x}) - r_0\mathbf{v}_1) \preceq_{se} (\bar{x}, \bar{x}) - r_0\mathbf{v}_1 \ll_{se} (\bar{x}, \bar{x})$. By monotonicity of T^2, we obtain $T^{2n+2}((\bar{x}, \bar{x}) - r_0\mathbf{v}_1) \preceq_{se} T^{2n}((\bar{x}, \bar{x}) - r_0\mathbf{v}_1) \ll_{se} (\bar{x}, \bar{x})$. In view of Lemma 2.3 we have that $T\left([0,\infty)^2 \setminus \{(0,0)\}\right) \subset (\delta, \gamma/C)^2$. Since $\{T^{2n}((\bar{x}, \bar{x}) - r_0\mathbf{v}_1)\}$ is bounded and monotonic, then $\{T^{2n}((\bar{x}, \bar{x}) - r_0\mathbf{v}_1)\}$ is convergent. Let $T^{2n}((\bar{x}, \bar{x}) - r_0\mathbf{v}_1) \to (\Phi, \Psi)$ as $n \to \infty$. From Lemma 3.5 we have that all minimal period two solutions are hyperbolic. For the competitive map, the stable set of a saddle point is contained in the union of first and third quadrants relative to the saddle point [26, 27]. Since $(\Phi, \Psi) \preceq_{se} T^{2n}((\bar{x}, \bar{x}) - r_0\mathbf{v}_1) \ll_{se} (\bar{x}, \bar{x})$, we have that $\{(\Phi, \Psi), (\Psi, \Phi)\}$ is locally asymptotically stable. Therefore, there exists at least one minimal period-two solution which is locally asymptotically stable. Assume that there exist two minimal period-two solutions $\{\Phi_1, \Psi_1\}$ and $\{\Phi_2, \Psi_2\}$. Then $(\Phi_1, \Psi_1) \ll_{se} (\Phi_2, \Psi_2)$ or $(\Phi_2, \Psi_2) \ll_{se} (\Phi_1, \Psi_1)$. Indeed if, for example, $(\Phi_1, \Psi_1) \ll_{ne} (\Phi_2, \Psi_2)$ then $[[(\delta, \gamma/C), (\Phi_1, \Psi_2)]]$ is invariant under T^2 which implies that there exist third minimal period-two solution contained in $[[(\delta, \gamma/C), (\Phi_1, \Psi_2)]]$, which is contradiction, since there exist only two minimal period-two solutions. Assume that, for example, $(\Phi_1, \Psi_1) \ll_{se} (\Phi_2, \Psi_2) \ll_s (\bar{x}, \bar{x})$. Since $[[(\delta, \frac{\gamma}{C}), (\Phi_1, \Psi_1)]]$ is invariant under map T^2, by Theorem 2.4 we have $T^{2n}(x_0, y_0) \to (\Phi_1, \Psi_1)$ for all $(x_0, y_0) \in [[(\delta, \frac{\gamma}{C}), (\Phi_1, \Psi_1)]]$ from which it follows that $\{\Phi_1, \Psi_1\}$ is locally asymptotically stable. By Corollary 1.3 we have that $[[(\Phi_1, \Psi_1), (\Phi_2, \Psi_2)]] \subseteq \mathcal{B}((\Phi_1, \Psi_1))$ and $[[(\Phi_2, \Psi_2), (\bar{x}, \bar{x})]] \subseteq \mathcal{B}((\Phi_2, \Psi_2))$, from which it follows that $\{\Phi_2, \Psi_2\}$ is non-hyperbolic which is contradiction with the fact that all minimal period-two solutions are hyperbolic. Hence, Equation (1.1) has one or three minimal period-two solutions. Assume that Equation (1.1) has three minimal period-two solutions $\{\Phi_1, \Psi_1\}$, $\{\Phi_2, \Psi_2\}$ and $\{\Phi_3, \Psi_3\}$ and $\Phi_1 < \Phi_2 < \Phi_3 < \bar{x}$. As we have seen, we have $(\Phi_1, \Psi_1) \ll_{se} (\Phi_2, \Psi_2)$ and $\{\Phi_1, \Psi_1\}$ is locally asymptotically stable. Now, we prove that $\{\Phi_2, \Psi_2\}$ is a saddle point. Let $\mathcal{B}((\Phi_1, \Psi_1))$ be the basin of attraction of (Φ_1, Ψ_1) with respect to the map T^2. By using Theorem 1.2 one can prove that $int(Q_2(\Phi_2, \Psi_2)) \subset \mathcal{B}(\Phi_1, \Psi_1)$. Let $\mathcal{S}_1$ denote the boundary of $\mathcal{B}(\Phi_1, \Psi_1)$ considered as a subset of $Q_1(\Phi_2, \Psi_2)$ and $\mathcal{S}_2$ denote the boundary of $\mathcal{B}(\Phi_1, \Psi_1)$ considered as a subset of $Q_3(\Phi_2, \Psi_2)$. One can see that $(\Phi_2, \Psi_2) \in \mathcal{S}_1$, $(\Phi_2, \Psi_2) \in \mathcal{S}_2$. Similarly as in [11] one can prove that $(\mathcal{S}_1 \cup \mathcal{S}_2, \ll_{ne})$ is a totally ordered set which is invariant under T^2. If $(x_0, y_0) \in \mathcal{S}_1 \cup \mathcal{S}_2$ then $\{T^{(2n)}(x_0, y_0)\}$ is eventually componentwise monotone. Then $T^{(2n)}(x_0, y_0) \to (\Phi_2, \Psi_2)$ as $n \to \infty$ for all $(x_0, y_0) \in \mathcal{S}_1 \cup \mathcal{S}_2$. From all of this it follows that $\{\Phi_2, \Psi_2\}$ is a saddle point. By Corollary 1.3 and [25, Theorem 4] applied to $(\bar{x}, \bar{x})$ it can be proved that $(\Phi_2, \Psi_2) \ll_{se} (\Phi_3, \Psi_3) \ll_s (\bar{x}, \bar{x})$ and $\{\Phi_3, \Psi_3\}$ is locally asymptotically stable □

Lemma 3.8. *Assume that $\bar{x}$ is a non-hyperbolic equilibrium. Then the following holds:*

i) If $11\gamma^2 - 26\gamma + 11 \leq 0$ then $\bar{x}$ is global attractor and $\mathcal{B}((\bar{x}, \bar{x}) = [0, \infty)^2 \setminus \{(0, 0)\}$.

ii) If $11\gamma^2 - 26\gamma + 11 > 0$ then there exist one minimal period-two solution which is locally asymptotically stable.

Proof. By Theorem 2.1 we have that

$$\delta = \frac{(3\gamma - 1)(3 - \gamma)}{16C}.$$

Substituting this into $g(u)$ we obtain that

$$g(u) = \frac{(4Cu - 3\gamma + 1)^2 \tilde{g}(u)}{4096C^3},$$

where

$$\tilde{g}(u) = 256C^4u^4 - 128C^3u^3(\gamma + 3) - 128C^2u^2(\gamma - 3)(\gamma + 1)$$
$$-8Cu(\gamma - 3)^2(3\gamma + 1) - (\gamma - 3)^3(3\gamma - 1).$$

The minimal period-two solutions of Equation (1.1) are solutions of equation $g(u) = 0$. Since $u = \dfrac{3\gamma - 1}{4C}$ is an equilibrium point, we have that the minimal period-two solutions satisfy equation $\tilde{g}(u) = 0$. The following matrix, called the discrimination matrix of $\tilde{g}(y)$ and $\tilde{g}'(u)$ in [37], is actually the Sylvester matrix of $\tilde{g}(u)$ and $\tilde{g}'(u)$ with some permuted rows, is given by

$$Discr(\tilde{f}) = \begin{pmatrix} a_4 & a_3 & a_2 & a_1 & a_0 & 0 & 0 & 0 \\ 0 & 4a_4 & 3a_3 & 2a_2 & a_1 & 0 & 0 & 0 \\ 0 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 & 0 \\ 0 & 0 & 4a_4 & 3a_3 & 2a_2 & a_1 & 0 & 0 \\ 0 & 0 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & 0 & 4a_4 & 3a_3 & 2a_2 & a_1 & 0 \\ 0 & 0 & 0 & a_4 & a_3 & a_2 & a_1 & a_0 \\ 0 & 0 & 0 & 0 & 4a_4 & 3a_3 & 2a_2 & a_1 \end{pmatrix},$$

where a_0, a_1, a_2, a_3, a_4 are coefficients of $g(u)$. Let D_k denote the determinant of the submatrix of $Discr(\tilde{u})$, formed by the first $2k$ row and the first $2k$ columns, for $k =$

$1, 2, 3, 4$. By straightforward calculation one can see that

$$\begin{aligned} D_1 =& 262144C^8, \\ D_2 =& 1073741824C^{14}\left(19\gamma^2 - 14\gamma - 21\right), \\ D_3 =& 2199023255552C^{18}(\gamma-3)^2\left(3\gamma^2+2\gamma+3\right)\left(13\gamma^2+2\gamma-47\right), \\ D_4 =& -633318697598976C^{20}(\gamma-3)^6\left(3\gamma^2+2\gamma+3\right)^2\left(11\gamma^2-26\gamma+11\right). \end{aligned}$$

Assume that $11\gamma^2 - 26\gamma + 11 < 0$. The sign list of the sequence $\{D_1, D_2, D_3, D_4\}$ is given by

$$[1, \text{sign}(D_2), -1, 1], \tag{3.10}$$

from which it follows that the number of sign changes of the revised sign list of the list (3.10) is at least two. In view of [37, Theorem 1] $\tilde{g}(y)$ has two pairs of conjugate imaginary roots from which the statement a) follows. Assume that $11\gamma^2-26\gamma+11 > 0$. In this case the sign list is given by

$$[1, \text{sign}(D_2), \text{sign}(D_3), -1]. \tag{3.11}$$

One can see that the number of sign changes of the revised sign list of the list (3.11) is one. By [37, Theorem 1] $\tilde{g}(y)$ has two distinct real roots. Since all coefficients of $\tilde{g}(-y)$ are nonnegative numbers, the Descartes' Rule of Signs implies that all real roots are positive numbers, from which the statement b) follows. □

3.3 Global Behavior

Theorem 3.9. *Assume that $\bar{x}$ is locally asymptotically stable and $Dis(g) \neq 0$.*

If Equation (1.1) *has no minimal period-two solutions then the equilibrium $\bar{x}$ is globally asymptotically stable.*

If Equation (1.1) *has two minimal period-two solutions: $\{(\Phi_1, \Psi_1), (\Psi_1, \Phi_1)\}$ and $\{(\Phi_2, \Psi_2), (\Psi_2, \Phi_2)\}$ with $(\Phi_1, \Psi_1) \preceq_{se} (\Phi_2, \Psi_2) \ll_{se} (\bar{x}, \bar{x})$, then $\{(\Phi_1, \Psi_1), (\Psi_1, \Phi_1)\}$ is locally asymptotically stable and $\{(\Phi_2, \Psi_2), (\Psi_2, \Phi_2)\}$ is a saddle point. The global stable manifold of the period-two solution $\{(\Phi_2, \Psi_2), (\Psi_2, \Phi_2)\}$ is given by $\mathcal{W}^s(\{(\Phi_2, \Psi_2), (\Psi_2, \Phi_2)\}) = \mathcal{W}^s((\Phi_2, \Psi_2)) \cup \mathcal{W}^s((\Psi_2, \Phi_2))$ where $\mathcal{W}^s((\Phi_2, \Psi_2))$ and $\mathcal{W}^s((\Psi_2, \Phi_2))$ are continuous increasing curves, invariant under the map T^2 and $T(\mathcal{W}^s((\Phi_2, \Psi_2))) = \mathcal{W}^s((\Psi_2, \Phi_2))$, and divide the first quadrant into two connected components, namely*

$$\begin{aligned} \mathcal{W}_1^- : &= \{x \in \mathcal{R} \setminus \mathcal{W}^s((\Phi_2, \Psi_2)) : \exists y \in \mathcal{W}^s((\Phi_2, \Psi_2)) \text{ with } y \preceq_{se} x\} \\ \mathcal{W}_1^+ : &= \{x \in \mathcal{R} \setminus \mathcal{W}^s((\Phi_2, \Psi_2)) : \exists y \in \mathcal{W}^s((\Phi_2, \Psi_2)) \text{ with } x \preceq_{se} y\} \\ \mathcal{W}_2^- : &= \{x \in \mathcal{R} \setminus \mathcal{W}^s((\Psi_2, \Phi_2)) : \exists y \in \mathcal{W}^s((\Psi_2, \Phi_2)) \text{ with } y \preceq_{se} x\} \\ \mathcal{W}_2^+ : &= \{x \in \mathcal{R} \setminus \mathcal{W}^s((\Psi_2, \Phi_2)) : \exists y \in \mathcal{W}^s((\Psi_2, \Phi_2)) \text{ with } x \preceq_{se} y\} \end{aligned}$$

respectively. In addition, $\mathcal{W}^s((\Phi_2, \Psi_2))$ passing through the point (Φ_2, Ψ_2) and $\mathcal{W}^s((\Psi_2, \Phi_2))$ passing through the point (Ψ_2, Φ_2) and the following holds:

i) If $(u_0, v_0) \in \mathcal{W}^s((\Phi_2, \Psi_2))$ *then the subsequence of even-indexed terms* $\{(u_{2n}, v_{2n})\}$ *is attracted to* (Φ_2, Ψ_2), *and the subsequence of odd-indexed terms* $\{(u_{2n+1}, v_{2n+1})\}$ *is attracted to* (Ψ_2, Φ_2).

ii) If $(u_0, v_0) \in \mathcal{W}^s((\Psi_2, \Phi_2))$ *then the subsequence of even-indexed terms* $\{(u_{2n}, v_{2n})\}$ *is attracted to* (Ψ_2, Φ_2), *and the subsequence of odd-indexed terms* $\{(u_{2n+1}, v_{2n+1})\}$ *is attracted to* (Φ_2, Ψ_2).

iii) If $(u_0, v_0) \in \mathcal{W}_1^+$ *(the region above* $\mathcal{W}^s((\Phi_2, \Psi_2))$*) then the subsequence of even-indexed terms* $\{(u_{2n}, v_{2n})\}$ *is attracted to* (Φ_1, Ψ_1), *and the subsequence of odd-indexed terms* $\{(u_{2n+1}, v_{2n+1})\}$ *is attracted to* (Ψ_1, Φ_1).

iv) If $(u_0, v_0) \in \mathcal{W}_2^-$ *(the region below* $\mathcal{W}^s((\Psi_2, \Phi_2))$*) then the subsequence of even-indexed terms* $\{(u_{2n}, v_{2n})\}$ *is attracted to* (Ψ_1, Φ_1), *and the subsequence of odd-indexed terms* $\{(u_{2n+1}, v_{2n+1})\}$ *is attracted to* (Φ_1, Ψ_1).

v) If $(u_0, v_0) \in \mathcal{W}_1^- \cap \mathcal{W}_2^+$ *(the region between* $\mathcal{W}_1^-$ *and* $\mathcal{W}_2^+$*) then the sequence* $\{(u_n, v_n)\}$ *is attracted to* $(\bar{x}, \bar{x})$.

Proof. See Figure 3.1 for visual illustration of Theorem 3.9.

If Equation (1.1) has no minimal period-two solutions then by Theorem 2.4 the equilibrium $\bar{x}$ is globally asymptotically stable.

By Lemma 3.6 we have that $\{(\Phi_1, \Psi_1), (\Psi_1, \Phi_1)\}$ is locally asymptotically stable and $\{(\Phi_2, \Psi_2), (\Psi_2, \Phi_2)\}$ is a saddle point. In view of Lemma 2.2 the map $T^2(u, v) = T(T(u, v))$ is competitive on $\mathcal{R} = \mathbb{R}_+^2 \setminus \{(0, 0)\}$ and strongly competitive on $int(\mathcal{R})$. It follows from the Perron–Frobenius theorem and a change of variables [36] that at each point, the Jacobian matrix of a strongly competitive map has two real and distinct eigenvalues, the larger one in absolute value being positive, and that corresponding eigenvectors may be chosen to point in the direction of the second and first quadrant, respectively. Also, one can show that if the map is strongly competitive then no eigenvector is aligned with a coordinate axis.

In view of Theorem 2.4 we have that all solutions converge to an equilibrium or period-two solution. Hence, all conditions of [25, Theorem 4] are satisfies, which yields the existence of the global stable manifolds $\mathcal{W}^s((\Phi_2, \Psi_2))$ and $\mathcal{W}^s((\Psi_2, \Phi_2))$ which are the graphs of strictly increasing functions of the first coordinate on an interval.

By [25, Theorem 4] we have that if $(u_0, v_0) \in \mathcal{W}^s((\Phi_2, \Psi_2))$ then $(u_{2n}, v_{2n}) = T^{2n}(u_0, v_0) \to (\Phi_2, \Psi_2)$ as $n \to \infty$ which implies that $(u_{2n+1}, v_{2n+1}) = T(T^{2n}(u_0, v_0)) \to T((\Phi_2, \Psi_2)) = (\Psi_2, \Phi_2)$ as $n \to \infty$ which implies the statement i). The proof of the statement ii) is similar.

Take $(u_0, v_0) \in \mathcal{W}_1^+ \cap \mathcal{R}$. By [25, Theorem 4] we have that there exists $n_0 > 0$ such that, $T^{2n}(u_0, v_0) \in int(Q_2((\Phi_2, \Psi_2)) \cap \mathcal{R}), \quad n > n_0$. In view of Theorem 1.2, since (Φ_2, Ψ_2) is a saddle point, we obtain that for all $(u_0, v_0) \in int(Q_2((\Phi_2, \Psi_2)) \cap \mathcal{R})$, there exists $r_0 > 0$ such that $(u_0, v_0) \preceq_{se} (\Phi_2, \Psi_2) - r_0 \mathbf{v}_1$ and $T^2((\Phi_2, \Psi_2) - r_0 \mathbf{v}_1) \preceq_{se}$

$(\Phi_2,\Psi_2)-r_0\mathbf{v}_1$. By monotonicity $T^{2n+2}((\Phi_2,\Psi_2)-r_0\mathbf{v}_1)\preceq_{se} T^{2n}((\Phi_2,\Psi_2)-r_0\mathbf{v}_1)\ll(\Phi_2,\Psi_2)$. In view of Lemma 2.3 we have that $T\left([0,\infty)^2\setminus\{(0,0)\}\right)\subset(\delta,\gamma/C)^2$. From this and the fact that $(\Phi_1,\Psi_1)\ll(\Phi_2,\Psi_2)\ll(\bar{x},\bar{x})\ll(\Psi_2,\Phi_2)\ll(\Psi_1,\Phi_1)$ we have that $T^{2n}((\Phi_2,\Psi_2)-r_0\mathbf{v}_1)\to(\Phi_1,\Psi_1)$ as $n\to\infty$. By monotonicity we have that $(\Phi_1,\Psi_1)\preceq_{se} T^{2n}(u_0,v_0)\preceq_{se} T^{2n}((\Phi_2,\Psi_2)-r_0\mathbf{v}_1)\ll(\Phi_2,\Psi_2)$ which implies that $T^{2n}(u_0,v_0)\to(\Phi_1,\Psi_1)$ and $T^{2n+1}(u_0,v_0)=T(T^{2n}(u_0,v_0))\to T((\Phi_1,\Psi_1))=(\Psi_1,\Phi_1)$ as $n\to\infty$ which proves the statement iii).

Take $(u_0,v_0)\in\mathcal{W}_2^-\cap\mathcal{R}$. By [25, Theorem 4] we have that there exists $n_1>0$ such that, $T^{2n}(u_0,v_0)\in int(Q_4((\Psi_2,\Phi_2))\cap\mathcal{R}),\quad n>n_1$. In view of Theorem 1.2, since (Ψ_2,Φ_2) is a saddle point, we obtain that for all $(u_0,v_0)\in int(Q_4((\Psi_2,\Phi_2))\cap\mathcal{R})$, there exists $r_1>0$ such that $(\Psi_2,\Phi_2)+r_1\mathbf{v}_1\preceq_{se}(u_0,v_0)$ and $(\Psi_2,\Phi_2)+r_1\mathbf{v}_1\preceq_{se} T^2((\Psi_2,\Phi_2)+r_1\mathbf{v}_1)$. The rest of the proof of the statement iv) is similar to the proof of the statement iii) and will be omitted.

Now, we show that each orbit starting in the region $\mathcal{W}_1^-\cap\mathcal{W}_2^+$ converges to $E(\bar{x},\bar{x})$. Take $(u_0,v_0)\in\mathcal{W}_1^-\cap\mathcal{W}_2^+$. By [25, Theorem 4] we have that there exists $n_2>0$ such that, $T^{2n}(u_0,v_0)\in int(Q_4((\Phi_2,\Psi_2))\cap Q_2(P_4)\cap\mathcal{R})=[[(\Phi_2,\Psi_2),(\Psi_2,\Phi_2)]]$, for $n>n_2$. Since (Φ_2,Ψ_2) and (Ψ_2,Φ_2) are the saddle points and E is locally asymptotically stable, in view of Corollary 1.3 we have that $T^{2n}(u',v')\to E$ and $T^{2n+1}(u',v')=T(T^{2n}(u',v'))\to T((\bar{x},\bar{x}))=(\bar{x},\bar{x})$ as $n\to\infty$ for all $(u',v')\in[[(\Phi_2,\Psi_2),(\bar{x},\bar{x})]]$ and that $T^{2n}(u'',v'')\to(\bar{x},\bar{x})$ and $T^{2n+1}(u'',v'')=T(T^{2n}(u'',v''))\to T((\bar{x},\bar{x}))=(\bar{x},\bar{x})$ as $n\to\infty$ for all $(u'',v'')\in[[E,(\Psi_2,\Phi_2)]]$. Then there exist the points $(u_0',v_0')\in[[(\Phi_2,\Psi_2),(\bar{x},\bar{x})]]$ and $(u_0'',v_0'')\in[[(\bar{x},\bar{x}),(\Psi_2,\Phi_2)]]$ such that

$$(u_0',v_0')\preceq_{se} T^{2n_2+2}(u_0,v_0)\preceq_{se}(u_0'',v_0'').$$

By monotonicity of the map T^2 we have that $T^{2n}(u_0,v_0)\to(\bar{x},\bar{x})$ and $T^{2n+1}(u_0,v_0)=T(T^{2n}(u_0,v_0))\to T((\bar{x},\bar{x}))=(\bar{x},\bar{x})$ as $n\to\infty$ for all $(u_0,v_0)\in\mathcal{W}_1^-\cap\mathcal{W}_2^+$. This completes the proof of statement v) of the Theorem. □

Theorem 3.10. *Assume that the unique equilibrium point $(\bar{x},\bar{x})$ is a saddle point and there exist one minimal period-two solution $\{(\Phi_1,\Psi_1),(\Psi_1,\Phi_1)\}$ such that $(\Phi_1,\Psi_1)\ll_{se}(\bar{x},\bar{x})$. Then $\{(\Phi_1,\Psi_1),(\Psi_1,\Phi_1)\}$ is locally asymptotically stable. Further, the global stable manifold $\mathcal{W}^s((\bar{x},\bar{x}))$, which is continuous increasing curve, divides the first quadrant into two connected components*

$$\begin{aligned}\mathcal{W}^-((\bar{x},\bar{x})):&=\{x\in\mathcal{R}\setminus\mathcal{W}^s((\bar{x},\bar{x})):\exists y\in\mathcal{W}^s((\bar{x},\bar{x}))\ \textit{with}\ y\preceq_{se}x\}\\ \mathcal{W}^+((\bar{x},\bar{x})):&=\{x\in\mathcal{R}\setminus\mathcal{W}^s((\bar{x},\bar{x})):\exists y\in\mathcal{W}^s((\bar{x},\bar{x}))\ \textit{with}\ x\preceq_{se}y\}\end{aligned}$$

such that

$$\mathbb{R}_+^2=\mathcal{W}^-((\bar{x},\bar{x}))\cup\mathcal{W}^+((\bar{x},\bar{x}))\cup\mathcal{W}^s((\bar{x},\bar{x})).$$

In addition, $\mathcal{W}^s((\bar{x},\bar{x}))$ passing through the point $(\bar{x},\bar{x})$ and the following holds:

i) Every initial point (u_0,v_0) in $\mathcal{W}^s((\bar{x},\bar{x}))$ is attracted to $(\bar{x},\bar{x})$.

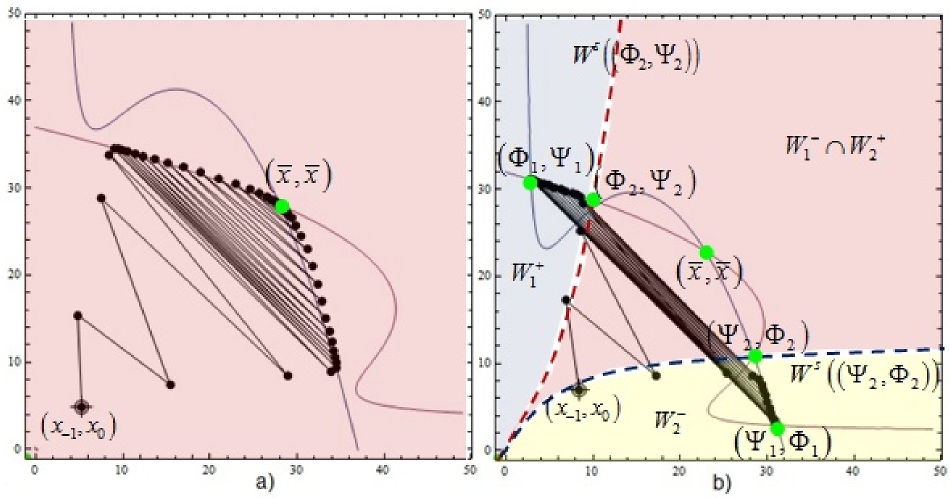

Figure 3.1: Visual illustration of Theorem 3.9. a) No period-two solutions b) Two period-two solutions.

ii) If $(u_0, v_0) \in \mathcal{W}^+((\bar{x}, \bar{x}))$ *(the region below* $\mathcal{W}^s((\bar{x}, \bar{x}))$*) then the subsequence of even-indexed terms* $\{(u_{2n}, v_{2n})\}$ *is attracted to* (Ψ_1, Φ_1)*, and the subsequence of odd-indexed terms* $\{(u_{2n+1}, v_{2n+1})\}$ *is attracted to* (Φ_1, Ψ_1)*.*

iii) If $(u_0, v_0) \in \mathcal{W}^-((\bar{x}, \bar{x}))$ *(the region above* $\mathcal{W}^s((\bar{x}, \bar{x}))$*) then the subsequence of even-indexed terms* $\{(x_{2n}, v_{2n})\}$ *is attracted to* (Φ_1, Ψ_1)*, and the subsequence of odd-indexed terms* $\{(u_{2n+1}, v_{2n+1})\}$ *is attracted to* (Ψ_1, Φ_1)*.*

Proof. See Figure 3.2 cases a) and b) for visual illustrations of Theorem 3.10.

By Lemma 3.7 we have that $\{(\Phi_1, \Psi_1), (\Psi_1, \Phi_1)\}$ is locally asymptotically stable. Similarly as in the proof of Theorem 3.9 all conditions of [25, Theorem 4] are satisfied, which yields the existence of the global stable manifold $\mathcal{W}^s((\bar{x}, \bar{x}))$ which is the graph of strictly increasing function of the first coordinate and the basin of attraction of $E(\bar{x}, \bar{x})$.

Take $(u_0, v_0) \in \mathcal{W}^+ \cap \mathcal{R}$. By [25, Theorem 4] we have that there exists $n_0 > 0$ such that, $T^{2n}(u_0, v_0) \in int(Q_2((\bar{x}, \bar{x})) \cap \mathcal{R}), \quad n > n_0$. In view of Theorem 1.2, since $(\bar{x}, \bar{x})$ is a saddle point, we obtain that for all $(u_0, v_0) \in int(Q_2((\bar{x}, \bar{x})) \cap \mathcal{R})$, there exists $r_0 > 0$ such that $(u_0, v_0) \preceq_{se} (\bar{x}, \bar{x}) - r_0\mathbf{v}_1 \preceq_{se} (\bar{x}, \bar{x})$ and $T^2((\bar{x}, \bar{x}) - r_0\mathbf{v}_1) \preceq_{se} (\bar{x}, \bar{x}) - r_0\mathbf{v}_1$. By monotonicity $T^{2n+2}((\bar{x}, \bar{x}) - r_0\mathbf{v}_1) \preceq_{se} T^{2n}((\bar{x}, \bar{x}) - r_0\mathbf{v}_1) \ll (\bar{x}, \bar{x})$. In view of Lemma 2.3 we have that $T^n(u, v) \in [0, \gamma/C)^2 \setminus \{(0, 0)\}$. From this and the fact that $(\Phi_1, \Psi_1) \ll (\bar{x}, \bar{x}) \ll (\Psi_1, \Phi_1)$ we have that $T^{2n}((\bar{x}, \bar{x}) - r_0\mathbf{v}_1) \to (\Phi_1, \Psi_1)$ as $n \to \infty$. By monotonicity, $P_1 \preceq_{se} T^{2n}(u_0, v_0) \preceq_{se} T^{2n}((\bar{x}, \bar{x}) - r_0\mathbf{v}_1) \ll (\bar{x}, \bar{x})$ which implies that $T^{2n}(u_0, v_0) \to (\bar{x}, \bar{x})$ and $T^{2n+1}(u_0, v_0) = T(T^{2n}(u_0, v_0)) \to T((\bar{x}, \bar{x})) = (\bar{x}, \bar{x})$ as $n \to \infty$ which proves the statement ii).

The proof of the statement iii) is similar and we skip it here. □

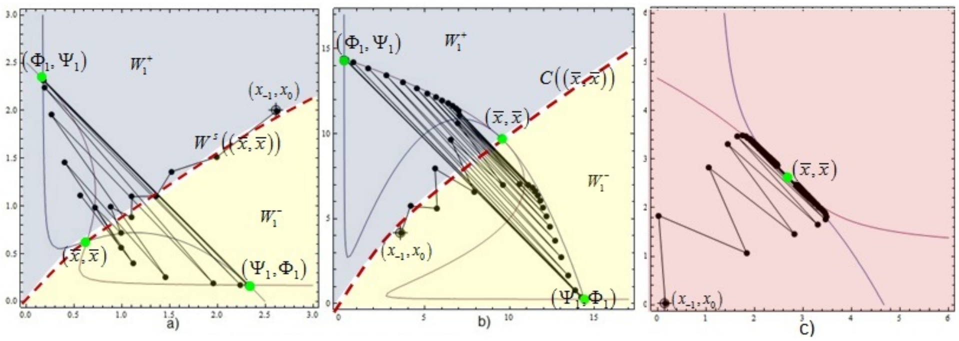

Figure 3.2: Visual illustration of Theorem 3.10 Cases a) and b). Visual illustration of Theorem 3.12 Cases a), b) and c).

Remark 3.11. If the unique equilibrium point $E(\bar{x}, \bar{x})$ is a saddle point and there exist three minimal period-two solutions $\{(\Phi_i, \Psi_i), (\Psi_i, \Phi_i)\}, i = 1, 2, 3$ and if $\Phi_1 < \Phi_2 < \Phi_3 < \bar{x}$, then $\{\Phi_1, \Psi_1\}$ and $\{\Phi_3, \Psi_3\}$ are locally asymptotically stable and $\{\Phi_2, \Psi_2\}$ is a saddle point. Global behavior in this case is replication of the global behavior given in Theorem 3.10 and can be described in a less formal way as the union of the basins of attraction of the unique equilibrium and the three period-two solutions where the basins of attraction of saddle points are their global stable manifolds and the basins of attractions of two locally asymptotically stable period-two points are their complements.

Theorem 3.12. *Assume that the unique equilibrium point $(\bar{x}, \bar{x})$ is non-hyperbolic. If $11\gamma^2 - 26\gamma + 11 \leq 0$ then there are no minimal period two solutions and $(\bar{x}, \bar{x})$ is global attractor. If $11\gamma^2 - 26\gamma + 11 > 0$ then there exists one minimal period-two solution $\{(\Phi_1, \Psi_1), (\Psi_1, \Phi_1)\}$ such that $(\Phi_1, \Psi_1) \ll_{se} (\bar{x}, \bar{x})$ which is locally asymptotically stable. Further, there exists a continuous increasing curve $\mathcal{C}_{(\bar{x},\bar{x})}$ which is the basin of attraction of $(\bar{x}, \bar{x})$ and it divides the first quadrant into two connected invariant components*

$$\begin{aligned} \mathcal{W}^-((\bar{x}, \bar{x})) : &= \{x \in \mathcal{R} \setminus \mathcal{C}_{(\bar{x},\bar{x})} : \exists y \in \mathcal{C}_{(\bar{x},\bar{x})} \text{ with } y \preceq_{se} x\} \\ \mathcal{W}^+((\bar{x}, \bar{x})) : &= \{x \in \mathcal{R} \setminus \mathcal{C}_{(\bar{x},\bar{x})} : \exists y \in \mathcal{C}_{(\bar{x},\bar{x})} \text{ with } x \preceq_{se} y\} \end{aligned}$$

such that the following holds true:

i) Every initial point (u_0, v_0) in $\mathcal{C}_{(\bar{x},\bar{x})}$ is attracted to $(\bar{x}, \bar{x})$.

ii) If $(u_0, v_0) \in \mathcal{W}^+(\bar{x}, \bar{x}))$ (the region above $\mathcal{C}_{(\bar{x},\bar{x})}$) then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ is attracted to (Φ_1, Ψ_1), and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ is attracted to (Ψ_1, Φ_1).

iii) If $(u_0, v_0) \in \mathcal{W}^-(\bar{x}, \bar{x}))$ (the region below $\mathcal{C}_{(\bar{x},\bar{x})}$) then the subsequence of even-indexed terms $\{(x_{2n}, v_{2n})\}$ is attracted to (Ψ_1, Φ_1), and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ is attracted to (Φ_1, Ψ_1).

Proof. See Figure 3.2 cases a), b) and c) for visual illustrations of Theorem 3.12.

If $11\gamma^2-26\gamma+11 \leq 0$ then by Lemma 3.8 there are no minimal period two solutions, and by Theorem 2.4 $E(\bar{x},\bar{x})$ is global attractor.

So, we suppose $11\gamma^2-26\gamma+11 > 0$. In view of Lemma 3.8 there exists one minimal period-two solution which is locally asymptotically stable. Theorem 2.1 implies $\delta = -\frac{(\gamma-3)(3\gamma-1)}{16C}$. In this case we have that $\bar{x} = \frac{3\gamma-1}{4C}$ and

$$J_T(\bar{x},\bar{x}) = \left(\begin{array}{cc} 0 & 1 \\ \frac{2}{3} & -\frac{1}{3} \end{array} \right).$$

An immediate calculation shows that eigenvalues of $J_T(\bar{x},\bar{x})$ are $\mu_1 = -1$ and $\mu_2 = \frac{2}{3}$ with corresponding eigenvectors $\nu_1 = (-1,1)^T$ and $\nu_1 = (3,2)^T$. Similarly as in the proof of Theorem 3.10 all conditions of [25, Theorem 4] are satisfied for map T^2, which yields the existence a continuous increasing curve $\mathcal{C}_{(\bar{x},\bar{x})}$ which is a subset of the basin of attraction of $(\bar{x},\bar{x})$ and for every $\mathrm{x} \in \mathcal{W}_+$ there exists $n_0 \in \mathbb{N}$ such that $T^{2n}(\mathrm{x}) \in \operatorname{int} \mathcal{Q}_2((\bar{x},\bar{x}))$ for $n \geq n_0$ and for every $\mathrm{x} \in \mathcal{W}_-$ there exists $n_1 \in \mathbb{N}$ such that $T^{2n}(\mathrm{x}) \in \operatorname{int} \mathcal{Q}_4((\bar{x},\bar{x}))$ for $n \geq n_1$. By Corollary 1.3 we have that $[[(\Phi_1,\Psi_1),(\bar{x},\bar{x})]] \subset B((\Phi_1,\Psi_1))$ and $[[(\bar{x},\bar{x}),(\Psi_1,\Phi_1)]] \subset B((\Psi_1,\Phi_1))$.

If $(x_0,y_0) \in int\mathcal{Q}_2(E)$ then there exists (u_0,v_0) such that $(x_0,y_0) \preceq_{se} (u_0,v_0) \ll_{se} (\bar{x},\bar{x})$. By monotonicity of the map T^2 we obtain that $(\delta,\gamma/C) \preceq_{se} T^{2n}((\delta,\gamma/C)) \preceq_{se} T^{2n}(x_0,y_0) \preceq_{se} T^{2n}(u_0,y_0) \ll_{se} (\bar{x},\bar{x})$ which implies that $T^{2n}(x_0,y_0) \to (\Phi_1,\Psi_1)$ and $T^{2n+1}(x_0,y_0) \to T((\Phi_1,\Psi_1)) = (\Psi_1,\Phi_1)$ as $n \to \infty$ which proves the statement ii).

Similarly one can prove the statement iii). □

4 The Case $C\delta > \gamma > 0$.

In this case dynamics is more complicated and there is a strong visual and numerical evidence for existence of chaotic dynamics in a substantial part of parametric region. As we have seen in Theorem 2.1 there is a subregion of the considered parametric region where the unique equilibrium solution is locally asymptotically stable and we will show that in its subregion equilibrium solution is also globally asymptotically stable. In the rest of the region we will prove the existence of the Naimark–Sacker bifurcation.

The following result is from [20].

Theorem 4.1. *Let $[a,b]$ be interval of real numbers and let $F : [a,b] \to [a,b]$ be continuous function that satisfies the following properties (a) and (b):*

(a) $F(x,y)$ *is non-decreasing in x and non-increasing in y;*

(b) $\forall (m,M) \in [a,b]^2 \Rightarrow (F(M,m) = M \wedge F(m,M) = m) \Rightarrow m = M.$

Then the difference equation $x_{n+1} = F(x_n, x_{n-1})$ has a unique equilibrium in $[a, b]$, and every solution with initial values in $[a, b]$ converges to the equilibrium.

Theorem 4.2. *Suppose that*

$$\frac{\gamma}{C} < \delta < \frac{(\gamma+3)(3\gamma+1)}{4C}.$$

Then the unique equilibrium point $\bar{x}$ is globally asymptotically stable

Proof. From Lemma (2.3) we obtain $F(\mathbb{R}^+\backslash\{(0,0)\}) \subseteq [\frac{\gamma}{C}, \delta]$. Assume that $(m, M) \in [\frac{\gamma}{C}, \delta]$ Then the system

$$\begin{aligned} M &= F(M, m) \\ m &= F(m, M) \end{aligned}$$

is equivalent to

$$\begin{aligned} Cm^2M - m^2\gamma + M^2 - M\delta &= 0 \\ CmM^2 + m^2 - m\delta - M^2\gamma &= 0. \end{aligned}$$

Assume that $m \neq M$. By subtracting these two relations we obtain that

$$(m - M)(m(CM - \gamma - 1) - M(\gamma + 1) + \delta) = 0$$

from which it follows

$$m = \frac{\delta - M(\gamma+1)}{1 - CM + \gamma}.$$

Substituting this into the second equation of the previous system we have $\tilde{f}(M) = 0$ where

$$\tilde{f}(x) = C^2x^4 + Cx^3\left(-C\delta + \gamma^2 - 1\right) - x^2(\gamma-1)(\gamma+1)^2 + x\delta\left(C\delta + \gamma^2 - 1\right) - \gamma\delta^2 = 0.$$

Because of symmetry, we obtain $\tilde{f}(m) = 0$. The discriminant of a polynomial $\tilde{f}(x)$ is

$$\begin{aligned} Dis(\tilde{f}) = C^2\delta^2\left(C\delta - \gamma^2 - 2\gamma - 1\right)^2\left(4C\delta - 3\gamma^2 - 10\gamma - 3\right) \\ \times\left(C^2\delta^2 - 2C\gamma^2\delta + 4C\gamma\delta - 2C\delta + \gamma^4 - 2\gamma^2 + 1\right)^2. \end{aligned}$$

Since

$$\delta < \frac{(\gamma+3)(3\gamma+1)}{4C}$$

one can see that $Dis(\tilde{f}) < 0$. This implies that $\tilde{f}(x)$ has one pair of complex conjugate root and two real roots, all distinct. From Vieta's formulas the product of the roots of the polynomial $\tilde{f}(x)$ is $-\gamma\delta^2/C^2 < 0$. This implies that $mM < 0$, which contradict to $(m, M) \in [\frac{\gamma}{C}, \delta]$. Hence, $m = M$. The rest of the proof follows from Theorem 4.1. □

Now we consider bifurcation of a fixed point of map associated to Equation (1.1) in the case where the eigenvalues are complex conjugates and of unit module. We will use Naimark–Sacker bifurcation theorem, known also as Poincaré–Andronov–Hopf bifurcation theorem for maps, see [9, Theorem 4.5, p. 254].

Theorem 4.3 (Poincaré–Andronov–Hopf Bifurcation for Maps)**.** *Let*

$$F : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2; \quad (\lambda, x) \to F(\lambda, x)$$

be a C^4 map depending on real parameter λ satisfying the following conditions:

(i) $F(\lambda, 0) = 0$ *for λ near some fixed λ_0;*

(ii) $DF(\lambda, 0)$ *has two complex conjugate eigenvalues $\mu(\lambda)$ and $\bar{\mu}(\lambda)$ for λ near λ_0 with $|\mu(\lambda_0)| = 1$;*

(iii) $\dfrac{d}{d\lambda}|\mu(\lambda)| = d(\lambda_0) > 0$ *at $\lambda = \lambda_0$;*

(iv) $\mu^k(\lambda_0) \neq 1$ *for* $k = 1, 2, 3, 4$.

Then there is a smooth λ-dependent change of coordinate bringing f into the form

$$F(\lambda, x) = \mathcal{F}(\lambda, x) + O(\| x \|^5)$$

and there are smooth function $a(\lambda)$, $b(\lambda)$, and $\omega(\lambda)$ so that in polar coordinates the function $\mathcal{F}(\lambda, x)$ is given by

$$\mathcal{F}\left(\begin{pmatrix} r \\ \theta \end{pmatrix}\right) = \begin{pmatrix} |\mu(\lambda)|r + a(\lambda)r^3 \\ \theta + \omega(\lambda) + b(\lambda)r^2 \end{pmatrix}. \tag{4.1}$$

If $a(\lambda_0) < 0$, then there is a neighborhood U of the origin and a $\delta > 0$ such that for $|\lambda - \lambda_0| < \delta$ and $x_0 \in U$, then ω-limit set of x_0 is the origin if $\lambda < \lambda_0$ and belongs to a closed invariant C^1 curve $\Gamma(\lambda)$ encircling the origin if $\lambda > \lambda_0$. Furthermore, $\Gamma(\lambda_0) = 0$.

Consider a general map $F(\lambda, x)$ that has a fixed point at the origin with complex eigenvalues $\mu(\lambda) = \alpha(\lambda) + i\beta(\lambda)$ and $\bar{\mu}(\lambda) = \alpha(\lambda) - i\beta(\lambda)$ satisfying $\alpha(\lambda)^2 + \beta(\lambda)^2 = 1$ and $\beta(\lambda) \neq 0$. By putting the linear part of such a map into Jordan Canonical form, we may assume F to have the following form near the origin

$$F(\lambda, x) = A(\lambda)x + G(\lambda, x),$$

where

$$A(\lambda) = \begin{pmatrix} \alpha(\lambda) & -\beta(\lambda) \\ \beta(\lambda) & \alpha(\lambda) \end{pmatrix}, G(\lambda, x) = \begin{pmatrix} g_1(\lambda, x_1, x_2) \\ g_2(\lambda, x_1, x_2) \end{pmatrix}. \tag{4.2}$$

Let $\mathbf{p}$ and $\mathbf{q}$ be the eigenvectors of A associated with μ satisfying

$$A\mathbf{q} = \mu\mathbf{q},\ \mathbf{p}A = \mu\mathbf{p}, \mathbf{p}\mathbf{q} = 1$$

and

$$\Phi = (\mathbf{q}, \overline{\mathbf{q}}).$$

Assume that

$$G\left(\Phi\begin{pmatrix} z \\ \overline{z}\end{pmatrix}\right) = \frac{1}{2}(g_{20}z^2 + 2g_{11}z\overline{z} + g_{02}\overline{z}^2) + O(|z|^3)$$

and

$$\begin{array}{rcl} K_{20} & = & (\mu^2 I - A)^{-1}g_{20} \\ K_{11} & = & (I - A)^{-1}g_{11} \\ K_{02} & = & (\overline{\mu}^2 I - A)^{-1}g_{02}, \end{array} \tag{4.3}$$

where $A = A(\lambda_0)$. Let

$$G\left(\Phi\begin{pmatrix} z \\ \overline{z}\end{pmatrix} + \frac{1}{2}(K_{20}\xi^2 + 2K_{11}\xi\overline{\xi} + K_{02}\overline{\xi}^2)\right)$$

$$= \frac{1}{2}(g_{20}\xi^2 + 2g_{11}\xi\overline{\xi} + g_{02}\overline{\xi}^2) + \frac{1}{6}(g_{30}\xi^3 + 3g_{21}\xi^2\overline{\xi} + 3g_{12}\xi\overline{\xi}^2 + g_{03}\overline{\xi}^3) + O(|\xi|^4),$$

then

$$a(\lambda_0) = \frac{1}{2}Re(\mathbf{p}q_{21}\overline{\mu}).$$

The invariant curve from Theorem 4.3 can be approximated by using the following corollary [32].

Corollary 4.4. *Assume $a(\lambda_0) \neq 0$ and $\lambda = \lambda_0 + \eta$ where η is sufficient small parameter. If $\overline{x}$ is fixed point of F then invariant curve $\Gamma(\lambda)$ from Theorem 4.3 can be approximated by*

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \approx \overline{x} + 2\rho_0 Re(qe^{i\theta}) + \rho^2(Re(K_{20}e^{2i\theta}) + K_{11}),$$

where

$$d = \frac{d}{d\eta}|\mu(\lambda)|\bigg|_{\lambda=\lambda_0},\ \rho_0 = \sqrt{-\frac{d}{a}\eta},\ \theta \in \mathbb{R}.$$

Now we can state our main result.

Theorem 4.5. *Let*

$$\delta_0 = \frac{(2+\gamma)(1+2\gamma)}{C}, \quad \bar{x} = \frac{\sqrt{4C\delta + (1-\gamma)^2} - 1 + \gamma}{2C}.$$

Assume that $C\delta > \gamma$. Then there is a neighborhood U of the equilibrium point $\bar{x}$ and a $\rho > 0$ such that for $|\delta - \delta_0| < \rho$ and $x_0, x_{-1} \in U$, the ω-limit set of solution of Eq(1.1)*, with initial condition x_0, x_{-1} is equilibrium point $\bar{x}$ if $\delta < \delta_0$ and belongs to a closed invariant C^1 curve $\Gamma(\delta)$ encircling the $(\bar{x}, \bar{x})$ if $\delta > \delta_0$. Furthermore, $\Gamma(\delta_0) = 0$ and invariant curve can be approximated by*

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \approx \begin{pmatrix} x_1(\theta) \\ x_2(\theta) \end{pmatrix},$$

where

$$\begin{aligned}
x_1(\theta) \approx\ & \frac{\sqrt{\frac{(2\gamma+1)(-\gamma(2\gamma+5)+C\delta-2)}{(\gamma+2)C^2}}\left(\sqrt{15}\sin(\theta)+\cos(\theta)\right)}{2\sqrt{2}} \\
& - \frac{(\gamma(2\gamma+5)-C\delta+2)\left(9\sqrt{15}(\gamma+1)\sin(2\theta)-(95\gamma+31)\cos(2\theta)+24(5\gamma+1)\right)}{144(\gamma+1)(\gamma+2)C} \\
& + \frac{\gamma+\sqrt{(\gamma-1)^2+4C\delta}-1}{2C}; \\
x_2(\theta) \approx\ & \sqrt{2}\cos(\theta)\sqrt{\frac{(2\gamma+1)(-\gamma(2\gamma+5)+C\delta-2)}{(\gamma+2)C^2}} \\
& + \frac{(-\gamma(2\gamma+5)+C\delta-2)\left(\sqrt{15}(\gamma-1)\sin(2\theta)+(25\gamma+11)\cos(2\theta)+30\gamma+6\right)}{36(\gamma+1)(\gamma+2)C} \\
& + \frac{\gamma+\sqrt{(\gamma-1)^2+4C\delta}-1}{2C}.
\end{aligned}$$

Proof. See Figures 4.1 and 4.2 for visual illustration of Theorem 4.5 and bifurcation diagram for behavior in larger parametric region.

In this case the unique equilibrium $\bar{x}$ is non-hyperbolic. In order to apply Theorem 4.3 we make a change of variable $y_n = x_n - \bar{x}$. Then, the transformed equation is given by

$$y_{n+1} = \frac{\gamma\left(\bar{x}+y_{n-1}\right)^2+\delta\left(\bar{x}+y_n\right)}{C\left(\bar{x}+y_{n-1}\right)^2+\left(\bar{x}+y_n\right)} - \bar{x}. \tag{4.4}$$

By using the substitution $u_n = y_{n-1}$ and $v_n = y_n$ for $n = 0, 1, ...$ we write Equation (4.4) in the equivalent form

$$\begin{aligned}
u_{n+1} &= v_n \\
v_{n+1} &= \frac{\gamma\left(\bar{x}+u_n\right)^2+\delta\left(\bar{x}+v_n\right)}{C\left(\bar{x}+u_n\right)^2+\left(\bar{x}+v_n\right)} - \bar{x}.
\end{aligned} \tag{4.5}$$

Let F be the function defined by

$$F\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ \dfrac{\gamma\left(\bar{x}+u\right)^2+\delta\left(\bar{x}+v\right)}{C\left(\bar{x}+u\right)^2+\left(\bar{x}+v\right)} - \bar{x} \end{pmatrix}, \tag{4.6}$$

then

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = F \begin{pmatrix} u_n \\ v_n \end{pmatrix}.$$

Furthermore, F has the unique fixed point $(0,0)$ and maps $(-\bar{x},\infty)^2$ into $(-\bar{x},\infty)^2$. The Jacobian matrix of F is given by

$$Jac_F(u,v) = \left(\begin{array}{cc} 0 & 1 \\ \dfrac{2(\gamma - C\delta)\,(u+\bar{x})\,(v+\bar{x})}{\left(C\,(u+\bar{x})^2 + (v+\bar{x})\right)^2} & \dfrac{(C\delta - \gamma)\,(u+\bar{x})^2}{\left(C\,(u+\bar{x})^2 + 1\,(v+\bar{x})\right)^2} \end{array} \right).$$

At $(0,0)$, $Jac_F(u,v)$ has the form

$$J_0 = Jac_F(0,0) = \left(\begin{array}{cc} 0 & 1 \\ \dfrac{2(\gamma - C\delta)}{(1+C\bar{x})^2} & \dfrac{C\delta - \gamma}{(1+C\bar{x})^2} \end{array} \right). \tag{4.7}$$

Assume that $C\delta > \gamma$. Then the eigenvalues of (4.7) are $\mu(\delta)$ and $\bar{\mu}(\delta)$ where

$$\mu(\delta) = \frac{-\gamma + C\delta + i\sqrt{(C\delta - \gamma)\left(8\,(C\bar{x}+1)^2 + \gamma - C\delta\right)}}{2\,(C\bar{x}+1)^2}, \tag{4.8}$$

since

$$8\,(C\bar{x}+1)^2 - C\delta + \gamma = \left(4\sqrt{4C\delta + (1-\gamma)^2} + \gamma\right) \\ + 4\gamma\left(\sqrt{4C\delta + (1-\gamma)^2} + \gamma\right) + 7C\delta + 4 > 0.$$

Thus we have that

$$F\begin{pmatrix} u \\ v \end{pmatrix} = \left(\begin{array}{cc} 0 & 1 \\ \dfrac{2(\gamma - C\delta)}{(1+C\bar{x})^2} & \dfrac{C\delta - \gamma}{(1+C\bar{x})^2} \end{array} \right) \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f_1(\delta,u,v) \\ f_2(\delta,u,v) \end{pmatrix}, \tag{4.9}$$

and

$$f_1(\delta,u,v) = 0,$$

$$f_2(\delta,u,v) = \frac{\gamma\,(\bar{x}+u)^2 + \delta\,(\bar{x}+v)}{C\,(\bar{x}+u)^2 + (\bar{x}+v)} - \frac{2u(\gamma - C\delta)}{(C\bar{x}+1)^2} + \frac{v(\gamma - C\delta)}{(C\bar{x}+1)^2} - \bar{x}.$$

Let $\delta_0 = \dfrac{(2+\gamma)(1+2\gamma)}{C}$. For $\delta = \delta_0$ we obtain

$$\bar{x} = \frac{1+2\gamma}{C} \text{ and } J_0 = \begin{pmatrix} 0 & 1 \\ -1 & \dfrac{1}{2} \end{pmatrix}.$$

The eigenvalues of J_0 are $\mu(\delta_0)$ and $\overline{\mu(\delta_0)}$, where $\mu(\delta_0) = \frac{1}{4}\left(1 + i\sqrt{15}\right)$. The eigenvectors corresponding to $\mu(\delta_0)$ and $\overline{\mu(\delta_0)}$ are $v(\delta_0)$ and $\overline{v(\delta_0)}$, where

$$v(\delta_0) = \left(\frac{1}{4}\left(1 - i\sqrt{15}\right), 1\right).$$

We have that

$$|\mu(\delta_0)| = 1,$$

$$\mu^2(\delta_0) = -\frac{7}{8} + \frac{i\sqrt{15}}{8}, \quad \mu^3(\delta_0) = -\frac{11}{16} - \frac{3i\sqrt{15}}{16}, \quad \mu^4(\delta_0) = \frac{17}{32} - \frac{7i\sqrt{15}}{32}.$$

Substituting $\delta = \delta_0$ and $\bar{x}$ into (4.9) we get

$$F\begin{pmatrix} u \\ v \end{pmatrix} = J_0\begin{pmatrix} u \\ v \end{pmatrix} + G\begin{pmatrix} u \\ v \end{pmatrix}, \tag{4.10}$$

where

$$G\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{C\left(u(Cu(2u-v) + 6u\gamma - 4v\gamma) + (2u^2 - v^2)\right)}{2\left(C(2u+v) + 6\gamma\right) + (Cu + 2\gamma)^2 + 2)} \end{pmatrix}.$$

Hence, for $\delta = \delta_0$ system (4.5) is equivalent to

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = J_0\begin{pmatrix} u_n \\ v_n \end{pmatrix} + G\begin{pmatrix} u_n \\ v_n \end{pmatrix}. \tag{4.11}$$

Define the basis of $\mathbb{R}^2$ by $\Phi = (\mathbf{q}, \overline{\mathbf{q}})$ where $\mathbf{q} = \mathbf{q}(\delta_0)$. Then we can represent (u, v) as

$$\begin{pmatrix} u \\ v \end{pmatrix} = \Phi\begin{pmatrix} z \\ \overline{z} \end{pmatrix} = (\mathbf{q}z + \overline{\mathbf{q}z}) = \begin{pmatrix} \frac{1}{4}(1 + i\sqrt{15})z + \frac{1}{4}(1 - \sqrt{15})\overline{z} \\ z + \overline{z} \end{pmatrix}.$$

Now we have

$$G\left(\Phi\begin{pmatrix} z \\ \overline{z} \end{pmatrix}\right) = \begin{pmatrix} 0 \\ \dfrac{C\Gamma_1}{\Gamma_2} \end{pmatrix}, \tag{4.12}$$

where

$$\begin{aligned}
\Gamma_1 = {} & -C\left(\left(\sqrt{15} + i\right)z - \left(\sqrt{15} - i\right)\overline{z}\right)\left(4z^2 - 7z\overline{z} + 4\overline{z}^2\right) \\
& - 2\left(\left(\sqrt{15} + 25i\right)\gamma - \sqrt{15} + 11i\right)z^2 + 16iz(5\gamma\overline{z} + \overline{z}) \\
& + 2\left(\left(\sqrt{15} - 25i\right)\gamma - \sqrt{15} - 11i\right)\overline{z}^2; \\
\Gamma_2 = {} & 2C^2\left(\left(\sqrt{15} - 7i\right)z^2 + 16iz\overline{z} - \left(\sqrt{15} + 7i\right)\overline{z}^2\right) \\
& + 16\gamma C\left(\left(\sqrt{15} + i\right)z - \left(\sqrt{15} - i\right)\overline{z}\right) + 32i(\gamma + 1)(2\gamma + 1) \\
& + 8\left(\sqrt{15} + 3i\right)Cz - 8\left(\sqrt{15} - 3i\right)C\overline{z}.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
g_{20} &= \frac{\partial^2}{\partial z^2} G\left(\Phi\begin{pmatrix} z \\ \overline{z}\end{pmatrix}\right)\Bigg|_{z=0} = \begin{pmatrix} 0 \\ \dfrac{-25\gamma C - 11C}{8(\gamma+1)(2\gamma+1)} + i\dfrac{\sqrt{15}\gamma C - \sqrt{15}C}{8(\gamma+1)(2\gamma+1)}\end{pmatrix}, \\
g_{11} &= \frac{\partial^2}{\partial z \partial \overline{z}} G\left(\Phi\begin{pmatrix} z \\ \overline{z}\end{pmatrix}\right)\Bigg|_{z=0} = \begin{pmatrix} 0 \\ \dfrac{(5\gamma+1)C}{2(\gamma+1)(2\gamma+1)}\end{pmatrix}, \qquad (4.13) \\
g_{02} &= \frac{\partial^2}{\partial \overline{z} \partial \overline{z}} G\left(\Phi\begin{pmatrix} z \\ \overline{z}\end{pmatrix}\right)\Bigg|_{z=0} = \begin{pmatrix} 0 \\ \dfrac{-25\gamma C - 11C}{8(\gamma+1)(2\gamma+1)} + i\dfrac{\sqrt{15}C - \sqrt{15}\gamma C}{8(\gamma+1)(2\gamma+1)}\end{pmatrix}.
\end{aligned}$$

So

$$\begin{aligned}
K_{20} &= (\mu^2 I - A)^{-1} g_{20} = \begin{pmatrix} -\dfrac{(95\gamma+31)C}{72(\gamma+1)(2\gamma+1)} - \dfrac{3i\sqrt{15}C}{24(2\gamma+1)} \\ \dfrac{C(25\gamma+11)}{18(\gamma+1)(2\gamma+1)} + \dfrac{i\sqrt{15}C(1-\gamma)}{18(\gamma+1)(2\gamma+1)}\end{pmatrix}, \\
K_{11} &= (I - A)^{-1} g_{11} = \begin{pmatrix} \dfrac{(5\gamma+1)C}{3(\gamma+1)(2\gamma+1)} \\ \dfrac{(5\gamma+1)C}{3(\gamma+1)(2\gamma+1)}\end{pmatrix}, \qquad (4.14) \\
K_{02} &= (\overline{\mu}^2 I - A)^{-1} g_{02} = \overline{K_{20}}.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
g_{21} &= \frac{\partial^3}{\partial z^2 \partial \overline{z}} G\left(\Phi\begin{pmatrix} z \\ \overline{z}\end{pmatrix} + \frac{1}{2}K_{20}z^2 + K_{11}z\overline{z} + \frac{1}{2}K_{02}\overline{z}^2\right)\Bigg|_{z=0} \\
&= \begin{pmatrix} 0 \\ \dfrac{-2\gamma^2C^2 - 23\gamma C^2 - 2C^2}{18(\gamma+1)^2(2\gamma+1)^2} + \dfrac{i\left(-\sqrt{15}\gamma C^2 - 2\sqrt{15}C^2\right)}{6(\gamma+1)^2(2\gamma+1)}\end{pmatrix}. \qquad (4.15)
\end{aligned}$$

We have that $\mathbf{p}A = \mu\mathbf{p}$ and $\mathbf{pq} = 1$ where $\mathbf{p} = \left(\dfrac{2i}{\sqrt{15}},\ \dfrac{1}{2}\left(1 - \dfrac{i}{\sqrt{15}}\right)\right)$.

One can see that $a(\delta_0) = \dfrac{1}{2}Re(\mathbf{p}g_21\overline{\mu}) = -\dfrac{(\gamma+2)C^2}{6(\gamma+1)^2(2\gamma+1)} < 0$. Now we have

$$d = \frac{d}{d\eta}|\mu(\delta)|\Bigg|_{\delta=\delta_0} = \frac{C}{12(\gamma+1)^2}, \quad \rho_0 = \sqrt{-\frac{d}{a}\eta} = \sqrt{\frac{(2\gamma+1)(-\gamma(2\gamma+5)+C\delta-2)}{2(\gamma+2)C^2}},$$

which yields the asymptotic formula for the invariant curve $\begin{pmatrix} x_1 \\ x_2\end{pmatrix} \approx \begin{pmatrix} x_1(\theta) \\ x_2(\theta)\end{pmatrix}$. □

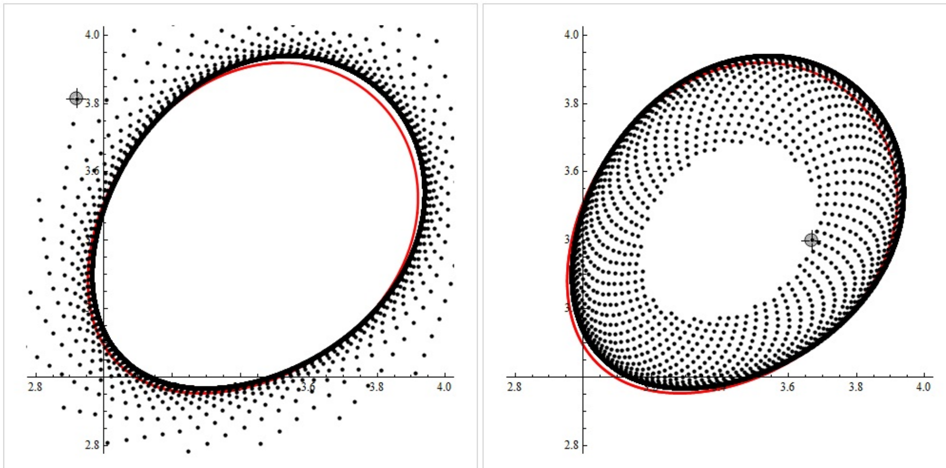

Figure 4.1: Visual illustration of the invariant curve in Theorem 4.5. Trajectories and invariant curve (red) for $\gamma = 1.2$, $C = 1$, $\delta_0 = 10.88$, $\delta = 10.99$.

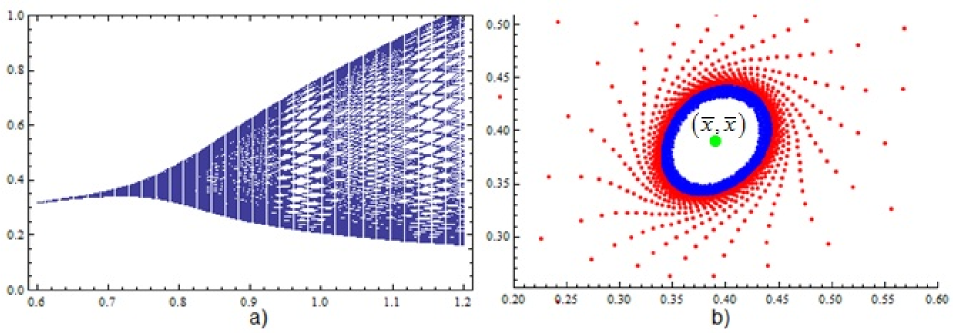

Figure 4.2: Bifurcation diagram in $(\delta - x)$ plane for $\gamma = 1.2; C = 1$, which indicates the chaotic behavior in large regions of parametric space.

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