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Oscillation Criteria for Difference Equations with Several Arguments

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Abstract

Consider the first-order retarded difference equation

$$\Delta x(n) + \sum_{i=1}^{m} p_i(n) x(\tau_i(n)) = 0, \;\; n \geq 0$$

where $(p_i(n))$ $(i = 1, \ldots, m)$ are sequences of nonnegative real numbers, $(\tau_i(n))$ are sequences of positive real numbers such that $\tau_i(n) \leq n - 1$ for $n \geq 0$ and $\lim_{n\to\infty} \tau_i(n) = \infty$ $(i = 1, \ldots, m)$. Under the assumption that the deviating arguments are not necessarily monotone, some new oscillation criteria, involving $\liminf$, are established. An example illustrating the results is also given.

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1 Introduction

Consider the retarded difference equation

$$\Delta x(n) + \sum_{i=1}^{m} p_i(n) x(\tau_i(n)) = 0, \quad n = 0, 1, \ldots \tag{1.1}$$

where $(p_i(n))$ $(i = 1, \ldots, m)$ are sequences of nonnegative real numbers, $(\tau_i(n))$ $(i = 1, \ldots, m)$ are sequences of positive real numbers such that

$$\tau_i(n) \le n - 1 \text{ for } n \ge 0 \text{ and } \lim_{n\to\infty} \tau_i(n) = \infty, \ 1 \le i \le m. \tag{1.2}$$

Δ denotes the forward difference operator $\Delta x(n) = x(n+1) - x(n)$. Define

$$k = -\min_{\substack{n\ge 0\\ 1\le i\le m}} \tau_i(n). \quad \text{(Clearly, } k \text{ is a positive integer.)}$$

By a solution of the difference equation (1.1), we mean a sequence of real numbers $(x(n))$ which satisfies (1.1) for all $n \ge 0$. It is clear that, for each choice of real numbers $c_{-k}, c_{-k+1}, \ldots, c_{-1}, c_0$, there exists a unique solution $(x(n))$ of (1.1) which satisfies the initial conditions $x(-k) = c_{-k}, \ x(-k+1) = c_{-k+1}, \ldots, \ x(-1) = c_{-1}, \ x(0) = c_0$. A solution $(x(n))$ of the difference equation (1.1) is called oscillatory, if the terms $x(n)$ of the sequence are neither eventually positive nor eventually negative. Otherwise, the solution is said to be nonoscillatory. If $m = 1$, then Eq. (1.1) takes the form

$$\Delta x(n) + p(n) x(\tau(n)) = 0, \quad n \in \mathbb{N}_0, \tag{1.3}$$

where $(p(n))_{n\in\mathbb{N}_0}$ is a sequence of nonnegative real numbers and $(\tau(n))_{n\in\mathbb{N}_0}$ is a sequence of integers such that

$$\tau(n) \le n - 1 \ \text{ for } n \ge 0 \quad \text{and} \quad \lim_{n\to\infty} \tau(n) = \infty.$$

The problem of establishing sufficient conditions for the oscillation of all solutions of (1.1) and (1.3) has been the subject of many investigations. See [1–21] and the references cited therein. Most of these papers are concerning the special case of the retarded difference equation (1.3) with the retarded argument $(\tau(n))$ is not necessarily monotone, while a small number are dealing with the general case of the retarded difference equation (1.1), in which the retarded arguments $(\tau_i(n))$ $(i = 1, \ldots, m)$ are not necessarily monotone.

In 1998, Zhang and Tian [21] studied the equation (1.3) and proved that, if $(\tau(n))$ is not necessarily monotone and

$$\limsup_{n\to\infty} p(n) > 0 \quad \text{and} \quad \liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} p(j) > \frac{1}{e}, \tag{1.4}$$

then all solutions of (1.3) oscillate. In 2006, Chatzarakis, Koplatadze and Stavroulakis [4, 5] when $(\tau(n))$ is not necessarily monotone, studied the equation (1.3) and proved that, if one of the following conditions

$$\limsup_{n\to\infty} \sum_{j=h(n)}^{n} p(j) > 1, \quad \text{where } h(n) = \max_{0\le s\le n} \tau(s),\, n \ge 0 \tag{1.5}$$

or

$$\limsup_{n\to\infty} \sum_{j=\tau(n)}^{n-1} p(j) < \infty \quad \text{and} \quad \liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} p(j) > \frac{1}{e} \tag{1.6}$$

is satisfied, then all solutions of (1.3) oscillate.

Assume that the argument $(\tau(n))$ is not necessarily monotone. Set

$$h(n) := \max_{s\le n} \tau(s), \quad n \ge 0. \tag{1.7}$$

Obviously, h is nondecreasing and $\tau(n) \le h(n) \le n-1$ for all $n \ge 0$.

In 2011, Braverman and Karpuz [2] proved that, if $(\tau(n))$ is not necessarily monotone and

$$\limsup_{n\to\infty} \sum_{j=h(n)}^{n} p(j) \prod_{i=\tau(j)}^{h(n)-1} \frac{1}{1-p(i)} > 1, \tag{1.8}$$

then all solutions of (1.3) oscillate.

Very recently, Öcalan [17] proved that, if $(\tau(n))$ is not necessarily monotone and

$$\liminf_{n\to\infty} \sum_{j=h(n)}^{n-1} p(j) \prod_{i=\tau(j)}^{h(j)-1} \frac{1}{1-p(i)} > \frac{1}{e}, \tag{1.9}$$

then all solutions of (1.3) oscillate.

Now, we return to the equation (1.1). In 2006, Berezansky and Braverman [1] established the following result for Eq. (1.1). If $(\tau_i(n))$ $(i = 1, \ldots, m)$ are not necessarily monotone and

$$\limsup_{n\to\infty} \sum_{i=1}^{m} p_i(n) > 0 \text{ and } \liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} \sum_{i=1}^{m} p_i(j) > \frac{1}{e}, \tag{1.10}$$

where $\tau(n) = \max_{1\le i\le m} \tau_i(n)$, then all solutions of (1.1) oscillate.

In 2013, Chatzarakis et al. [6] studied (1.1) and proved that, if $(\tau_i(n))(i = 1, \ldots, m)$ are nondecreasing and

$$\limsup_{n\to\infty} \sum_{j=\tau(n)}^{n} \sum_{i=1}^{m} p_i(j) > 1, \tag{1.11}$$

where $\tau(n) = \max_{1\le i\le m} \tau_i(n)$, then all solutions of (1.1) oscillate. Set

$$h_i(n) := \max_{s\le n} \tau_i(s), \quad n \ge 0 \text{ and } h(n) = \max_{1\le i\le m} h_i(n). \tag{1.12}$$

Clearly, $(h_i(n))$ $(i = 1, \dots, m)$ are nondecreasing and $1 \le i \le m$, $\tau_i(n) \le h_i(n) \le h(n)$ for all $n \ge 0$.

We remark that, in [3], Braverman et al. obtained the following result; assume that there exists a subsequence $\theta(n)$, $n \in \mathbb{N}$ of positive integers such that

$$\sum_{i=1}^{m} p_i(\theta(n)) \ge 1.$$

Then all solutions of (1.1) oscillate. Therefore, throughout this paper, we assume that $\sum_{i=1}^{m} p_i(n) < 1$ and

$$\sum_{i=k}^{k-1} A(i) = 0 \quad \text{and} \quad \prod_{i=k}^{k-1} A(i) = 1.$$

In 2015, Braverman et al. [3] analyzed the equation (1.1) and proved that, if $(\tau_i(n))$ $(i = 1, \dots, m)$ are not necessarily monotone and

$$\limsup_{n\to\infty} \sum_{j=h(n)}^{n} \sum_{i=1}^{m} p_i(j) \prod_{l=\tau_i(j)}^{h(n)-1} \frac{1}{\left(1 - \sum_{k=1}^{m} p_k(l)\right)} > 1, \tag{1.13}$$

where $h(n)$ is defined by (1.12), then all solutions of (1.1) oscillate.

Clearly, the following result is obtained by (1.13) immediately: if $(\tau_i(n))$ $(i = 1, \dots, m)$ are not necessarily monotone and

$$\limsup_{n\to\infty} \sum_{j=h(n)}^{n} \sum_{i=1}^{m} p_i(j) > 1, \tag{1.14}$$

where $h(n)$ is defined by (1.12), then all solutions of (1.1) oscillate.

2 Main Results

In this section, we present a new sufficient condition for the oscillation of all solutions of (1.1), under the assumption that the arguments $(\tau_i(n))$ $(i = 1, \dots, m)$ are not necessarily monotone. The following result was given in [9].

Lemma 2.1 (See [9])**.** *Assume that* (1.1) *holds and* $\alpha > 0$. *Then, we have*

$$\alpha = \liminf_{n\to\infty} \sum_{j=h(n)}^{n-1} \sum_{i=1}^{m} p_i(j) = \liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} \sum_{i=1}^{m} p_i(j), \tag{2.1}$$

where $h(n)$ *is defined by* (1.12) *and* $\tau(n) = \max_{1\le i\le m} \tau_i(n)$.

The following result will be useful for the proof of the main theorem. Also, this result is easily obtained by [2].

Lemma 2.2 (Discrete Grönwall Inequality, see [2] for $m = 1$)**.** *Let* $p_i(n) > 0$ $(i = 1,\dots,m)$ *and suppose that* $(x(n))$ *is a positive solution of the equation*

$$\Delta x(n) + \left(\sum_{i=1}^{m} p_i(n)\right) x(n) \le 0, \quad n \ge s. \tag{2.2}$$

Then

$$x(s) \ge \prod_{j=s}^{n-1} \frac{1}{\left(1 - \sum\limits_{k=1}^{m} p_k(j)\right)} x(n), \quad n \ge s. \tag{2.3}$$

Theorem 2.3. *Assume that* (1.2) *holds. If* $(\tau_i(n))$ *(*$i = 1,\dots,m$*) are not necessarily monotone and*

$$\liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} \sum_{i=1}^{m} p_i(j) \prod_{l=\tau_i(j)}^{h(j)-1} \frac{1}{\left(1 - \sum\limits_{k=1}^{m} p_k(l)\right)} > \frac{1}{e}, \tag{2.4}$$

where $h(n)$ *is defined by* (1.12) *and* $\tau(n) = \max_{1\le i\le m} \tau_i(n)$*, then all solutions of* (1.1) *oscillate.*

Proof. Assume, for the sake of contradiction, that there exists a nonoscillatory solution $x(n)$ of (1.1). Since $-x(n)$ is also a solution of (1.1), we can confine our discussion only to the case where the solution $x(n)$ is eventually positive. Then, there exists $n_1 > n_0$ such that $x(n),\ x(\tau(n)),\ x(h(n)) > 0$, for all $n \ge n_1$. Thus, from (1.1) we have

$$\Delta x(n) = -\sum_{i=1}^{m} p_i(n) x(\tau_i(n)) \le 0 \ \text{ for all } n \ge n_1,$$

which means that $(x(n))$ is an eventually nonincreasing sequence of positive numbers. In view of this and taking into account that $\tau_i(n) \le h_i(n) \le h(n)$ for all $n \ge 0$ and $1 \le i \le m$, Eq. (1.2) gives

$$\Delta x(n) + \left(\sum_{i=1}^{m} p_i(n)\right) x(h(n)) \le 0 \ \text{ for } n \ge n_1 \tag{2.5}$$

and

$$\Delta x(n)+\left(\sum_{i=1}^{m}p_i(n)\right)x\left(n\right)\leq 0 \text{ for } n\geq n_1. \tag{2.6}$$

Therefore, from (2.6) we have Lemma 2.2. On the other hand, we know from Lemma 2.1 that

$$\liminf_{n\to\infty}\sum_{j=\tau(n)}^{n-1}\sum_{i=1}^{m}p_i(j)\prod_{l=\tau_i(j)}^{h(j)-1}\frac{1}{\left(1-\sum\limits_{k=1}^{m}p_k(l)\right)}$$
$$=\liminf_{n\to\infty}\sum_{j=h(n)}^{n-1}\sum_{i=1}^{m}p_i(j)\prod_{l=\tau_i(j)}^{h(j)-1}\frac{1}{\left(1-\sum\limits_{k=1}^{m}p_k(l)\right)}.$$

It follows that there exists a constant $c>0$ such that

$$\sum_{j=h(n)}^{n-1}\sum_{i=1}^{m}p_i(j)\prod_{l=\tau_i(j)}^{h(j)-1}\frac{1}{\left(1-\sum\limits_{k=1}^{m}p_k(l)\right)}\geq c>\frac{1}{e},\quad n\geq n_2>n_1. \tag{2.7}$$

Now, dividing (1.1) by $x(n)$, then we obtain the following equation:

$$\frac{\Delta x(n)}{x(n)}+\sum_{i=1}^{m}p_i(n)\frac{x\left(\tau_i(n)\right)}{x(n)}=0. \tag{2.8}$$

Summing up (2.8) from $h(n)$ to $n-1$, using by Lemma 2.2 and the fact that $(x(n))$ is nonincreasing and $(h(n))$ is nondecreasing, we obtain

$$\sum_{j=h(n)}^{n-1}\frac{\Delta x(j)}{x(j)}+\sum_{j=h(n)}^{n-1}\sum_{i=1}^{m}p_i(j)\frac{x\left(h(j)\right)}{x(j)}\prod_{l=\tau_i(j)}^{h(j)-1}\frac{1}{\left(1-\sum\limits_{k=1}^{m}p_k(l)\right)}\leq 0. \tag{2.9}$$

Since $\sum\limits_{j=h(n)}^{n-1}\dfrac{\Delta x(j)}{x(j)}\geq\ln\dfrac{x(n)}{x\left(h(n)\right)}$ and $\dfrac{x\left(h(n)\right)}{x(n)}>1$, from (2.9) we have

$$\ln\frac{x(n)}{x\left(h(n)\right)}+\sum_{j=h(n)}^{n-1}\sum_{i=1}^{m}p_i(j)\prod_{l=\tau_i(j)}^{h(j)-1}\frac{1}{\left(1-\sum\limits_{k=1}^{m}p_k(l)\right)}\leq 0$$

and

$$\frac{x\left(h(n)\right)}{x(n)}\geq\exp\left\{\sum_{j=h(n)}^{n-1}\sum_{i=1}^{m}p_i(j)\prod_{l=\tau_i(j)}^{h(j)-1}\frac{1}{\left(1-\sum\limits_{k=1}^{m}p_k(l)\right)}\right\}. \tag{2.10}$$

So, from (2.7) and (2.10), we have

$$\frac{x\left(h(n)\right)}{x(n)} \geq e^{c} \geq ec.$$

Substituting the above inequality into (2.5), we have

$$\Delta x(n) + \left(\sum_{i=1}^{m} p_i(n)\right)(ec)x\left(n\right) \leq 0.$$

Therefore, repeating the above procedure, it follows by induction that for any positive integer k

$$\frac{x\left(h(n)\right)}{x(n)} \geq (ec)^{k} \text{ for sufficiently large } n. \tag{2.11}$$

On the other hand, from (2.7) there exists an integer $n^{*} \in (h(n), n]$ for all $n \geq n_1$ such that

$$\sum_{j=h(n)}^{n^{*}-1} \sum_{i=1}^{m} p_i(j) \prod_{l=\tau_i(j)}^{h(j)-1} \frac{1}{\left(1 - \sum\limits_{k=1}^{m} p_k(l)\right)} \geq \frac{c}{2} \tag{2.12}$$

and

$$\sum_{j=n^{*}}^{n} \sum_{i=1}^{m} p_i(j) \prod_{l=\tau_i(j)}^{h(j)-1} \frac{1}{\left(1 - \sum\limits_{k=1}^{m} p_k(l)\right)} \geq \frac{c}{2}. \tag{2.13}$$

Summing (1.1) from $h(n)$ to $n^{*}-1$ and using Lemma 2.2, we obtain

$$x(n^{*}) - x\left(h(n)\right) + \sum_{j=h(n)}^{n^{*}-1} \sum_{i=1}^{m} p_i(j) x\left(\tau_i(j)\right) = 0$$

or

$$x(n+1) - x\left(h(n)\right) + \sum_{j=h(n)}^{n^{*}-1} \sum_{i=1}^{m} p_i(j) x\left(h(j)\right) \prod_{l=\tau_i(j)}^{h(j)-1} \frac{1}{\left(1 - \sum\limits_{k=1}^{m} p_k(l)\right)} \leq 0$$

or

$$-x\left(h(n)\right) + x\left(h(n^{*})\right) \sum_{j=h(n)}^{n^{*}-1} \sum_{i=1}^{m} p_i(j) \prod_{l=\tau_i(j)}^{h(j)-1} \frac{1}{\left(1 - \sum\limits_{k=1}^{m} p_k(l)\right)} \leq 0. \tag{2.14}$$

Thus, from (2.12) and (2.14), we have

$$x\left(h(n^{*})\right)\frac{c}{2} \leq x\left(h(n)\right). \tag{2.15}$$

Summing (1.1) from n^* to n and using the same arguments, we get

$$x(n+1) - x(n^*) + x(h(n)) \sum_{j=n^*}^{n} \sum_{i=1}^{m} p_i(j) \prod_{l=\tau_i(j)}^{h(j)-1} \frac{1}{\left(1 - \sum_{k=1}^{m} p_k(l)\right)} \leq 0$$

and so we obtain

$$x(h(n)) \frac{c}{2} \leq x(n^*). \tag{2.16}$$

Combining the inequalities (2.15) and (2.16), we obtain

$$x(n^*) \geq x(h(n)) \frac{c}{2} \geq x(h(n^*)) \left(\frac{c}{2}\right)^2,$$

or

$$\frac{x(h(n^*))}{x(n^*)} \leq \left(\frac{2}{c}\right)^2 < +\infty$$

i.e., $\liminf\limits_{n\to\infty} \dfrac{x(h(n))}{x(n)}$ exists. This contradicts with (2.11). The proof of the theorem is completed. □

We note that, since for $i = 1, 2, \ldots, m$,

$$\prod_{l=\tau_i(j)}^{h(j)-1} \frac{1}{\left(1 - \sum_{k=1}^{m} p_k(l)\right)} > 1,$$

the following result is obtained by (2.4) immediately.

Corollary 2.4. *Assume that* (1.2) *holds. If* $(\tau_i(n))$ *(*$i = 1, \ldots, m$*) are not necessarily monotone and*

$$\liminf_{n\to\infty} \sum_{j=h(n)}^{n-1} \sum_{i=1}^{m} p_i(j) = \liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} \sum_{i=1}^{m} p_i(j) > \frac{1}{e}, \tag{2.17}$$

where $h(n)$ *is defined by* (1.12) *and* $\tau(n) = \max\limits_{1\leq i\leq m} \tau_i(n)$*, then all solutions of* (1.1) *oscillate.*

Example 2.5. Consider the delay difference equation

$$\Delta x(n) + (0,0167)\, x(n-10) + (0,020)\, x(n-11) = 0, \quad n \in \mathbb{N}. \tag{2.18}$$

Then, we have

$$\limsup_{n\to\infty} \sum_{j=n-10}^{n} \sum_{i=1}^{2} p_i(j) \cong 0,4037 < 1.$$

So, the condition (1.14) does not hold. Also, we observe that

$$\liminf_{n\to\infty} \sum_{j=n-10}^{n-1} \sum_{i=1}^{2} p_i(j) \cong 0,367 < \frac{1}{e},$$

this means that the condition (2.17) is not applicable for this equation.

On the other hand, since

$$\limsup_{n\to\infty} \sum_{j=n-10}^{n} \sum_{i=1}^{2} p_i(j) \prod_{l=\tau_i(j)}^{n-11} \frac{1}{\left(1-\sum\limits_{k=1}^{2} p_k(l)\right)} \cong 0,5 < 1,$$

which means that the condition (1.13) is not applicable for this equation.

However, it is easy to see that

$$\liminf_{n\to\infty} \sum_{j=n-10}^{n-1} \sum_{i=1}^{2} p_i(j) \prod_{l=\tau_i(j)}^{j-11} \frac{1}{\left(1-\sum\limits_{k=1}^{2} p_k(l)\right)} \cong 0,374 > \frac{1}{e}.$$

Thus, all the conditions of Theorem 2.3 are satisfied and therefore all solutions of Eq. (2.18) oscillate.

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