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On Hilfer-Type Nabla Fractional Differences

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Abstract

In this work, we introduce the nabla analogue of Hilfer fractional derivative and derive a few of its important properties such as composition and power rules. Further, we consider an initial value problem for a class of nonlinear Hilfer nabla fractional difference equations and obtain its equivalent Volterra summation equation, using these properties. We also determine expressions for general solutions of various classes of linear Hilfer nabla fractional difference equations by employing the discrete Laplace transform. We conclude this article with a discussion on the asymptotic behaviour of solutions of linear Hilfer nabla fractional difference equations.

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1 Introduction

Fractional calculus [19,21] has been of great interest over the last few decades due to the nonlocal nature of fractional derivatives. The contributions of several mathematicians resulted a robust theory of fractional calculus for functions of a real variable. Thus, numerous definitions of fractional integrals and fractional derivatives were proposed including the Hilfer's definition, which is the focus of this article. See for example [1, 10, 14, 16, 18, 19, 21, 22], and the references therein.

On the other hand, there has been a rise of interest in the development of nabla fractional calculus, within the past one decade. Gray and Zhang [12] introduced the notion

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of nabla fractional difference and obtained some of its elementary properties. Following their work, Miller & Ross [20], Atici & Eloe [5] and Anastassiou [4] defined Riemann–Liouville and Caputo nabla fractional differences and established several properties. The basic notions and properties of nabla fractional differences can be found in a recent monograph [11] and the references therein.

Hilfer [14] introduced the concept of generalized Riemann–Liouville fractional order derivative that contains Riemann–Liouville and Caputo fractional derivatives as particular cases.

Definition 1.1 (See [14]). The Riemann–Liouville fractional integral of order $\nu \in \mathbb{R}^+$ of a locally integrable function y is defined as

$$\big(I_a^\nu y\big)(t) = \frac{1}{\Gamma(\nu)} \int_a^t (t-s)^{\nu-1} y(s) ds, \quad t > a.$$

Let α, $\beta \in \mathbb{R}$, $n \in \mathbb{N}$ such that $n-1 < \alpha \leq n$ and $0 \leq \beta \leq 1$. The generalized Riemann–Liouville fractional derivative of order α and type β is defined as

$$\big(D_a^{\alpha,\beta} y\big)(t) = \left(I_a^{\beta(n-\alpha)} D^n I_a^{(1-\beta)(n-\alpha)} y\right)(t), \quad t > a. \tag{1.1}$$

Here D^n denotes the n^{th}-order differential operator.

The type β allows to interpolate continuously from the Riemann–Liouville case $D_a^{\alpha,0} \equiv D_a^\alpha$ to the Caputo case $D_a^{\alpha,1} \equiv D_{*a}^\alpha$. The type-parameter provides more types of stationary states and give an extra degree of freedom on the initial condition. Motivated by the definition of Hilfer fractional derivative, in this article, we propose its nabla analogue and demonstrate its properties. To the best of our knowledge, this notion was not yet reported.

The present article is structured as follows: Section 2 contains preliminaries on discrete fractional calculus. In section 3, we introduce the notion of Hilfer nabla fractional difference and discuss some of its important properties. We also obtain an equivalent Volterra summation equation for an initial value problem involving a particular class of nonlinear Hilfer nabla fractional difference equations. In section 4, we consider various classes of linear Hilfer nabla fractional difference equations and obtain their general solutions using the discrete Laplace transform. In section 5, we establish sufficient conditions on the asymptotic behaviour of solutions of linear Hilfer nabla fractional difference equations.

2 Preliminaries

Denote the set of all real numbers by $\mathbb{R}$. Define $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$, for any $a \in \mathbb{R}$. Assume empty sums and products are 0 and 1, respectively.

Definition 2.1 (See [13])**.** Let $\mu \in \mathbb{R} \setminus \{\dots,-2,-1\}$. The μ^{th}-order nabla fractional Taylor monomial is given by

$$H_\mu(t,a) = \frac{(t-a)^{\overline{\mu}}}{\Gamma(\mu+1)} = \frac{\Gamma(t-a+\mu)}{\Gamma(t-a)\Gamma(\mu+1)},$$

provided the right-hand side exists. Here $\Gamma(\cdot)$ denotes the Euler gamma function.

Let $u : \mathbb{N}_a \to \mathbb{R}$ and $N \in \mathbb{N}_1$. The first order backward (nabla) difference of u is defined by $\big(\nabla u\big)(t) = u(t) - u(t-1)$ for $t \in \mathbb{N}_{a+1}$, and the N^{th}-order nabla difference of u is defined recursively by $\big(\nabla^N u\big)(t) = \Big(\nabla\big(\nabla^{N-1}u\big)\Big)(t)$ for $t \in \mathbb{N}_{a+N}$.

Definition 2.2 (See [5])**.** Let $u : \mathbb{N}_a \to \mathbb{R}$ and $\nu > 0$. The ν^{th}-order nabla sum of u is given by

$$\big(\nabla_a^{-\nu}u\big)(t) = \sum_{s=a}^{t} H_{\nu-1}(t,\rho(s))u(s), \quad t \in \mathbb{N}_a,$$

where $\rho(s) = s-1$.

Definition 2.3 (See [5])**.** Let $u : \mathbb{N}_a \to \mathbb{R}$, $\nu > 0$ and choose $N \in \mathbb{N}_1$ such that $N-1 < \nu \le N$. The ν^{th}-order Riemann–Liouville nabla difference of u is given by

$$\big(\nabla_a^{\nu}u\big)(t) = \Big(\nabla^N\big(\nabla_a^{-(N-\nu)}u\big)\Big)(t), \quad t \in \mathbb{N}_{a+N}.$$

Remark 2.4. It is clear from Definition 2.2 and Definition 2.3 that if $u : \mathbb{N}_a \to \mathbb{R}$, then $\big(\nabla_a^{-\nu}u\big) : \mathbb{N}_a \to \mathbb{R}$ and $\big(\nabla_a^{\nu}u\big) : \mathbb{N}_{a+N} \to \mathbb{R}$.

Definition 2.5 (See [4])**.** Let $\nu > 0$, choose $N \in \mathbb{N}_1$ such that $N-1 < \nu \le N$ and $u : \mathbb{N}_{a-N} \to \mathbb{R}$. The ν^{th}-order Caputo nabla difference of u is given by

$$\big(\nabla_{*a}^{\nu}u\big)(t) = \Big(\nabla_a^{-(N-\nu)}\big(\nabla^N u\big)\Big)(t), \quad t \in \mathbb{N}_a.$$

We present a few useful properties of nabla fractional sum and differences in the following series of lemmas.

Lemma 2.6 (See [5,6])**.** *Assume* $u : \mathbb{N}_a \to \mathbb{R}$, *and* ν, $\mu > 0$. *Then,*

1. $\big(\nabla_a^{-\nu}\nabla_a^{-\mu}u\big)(t) = \big(\nabla_a^{-\nu-\mu}u\big)(t)$, $t \in \mathbb{N}_a$.
2. $\big(\nabla_{a+1}^{-\nu}\nabla u\big)(t) = \big(\nabla\nabla_a^{-\nu}u\big)(t) - H_{\nu-1}(t,\rho(a))u(a)$, $t \in \mathbb{N}_{a+1}$.

Lemma 2.7 (See [6, 11])**.** *The following fractional nabla Taylor monomials are well defined.*

1. *Let* $\nu > 0$ *and* $\mu \in \mathbb{R}$. *Then,* $\nabla_a^{-\nu}H_\mu(t,\rho(a)) = H_{\mu+\nu}(t,\rho(a))$, $t \in \mathbb{N}_a$.

2. *Let* ν, $\mu \in \mathbb{R}$ *and* $n \in \mathbb{N}_1$ *such that* $N-1 < \nu \leq N$. *Then,* $\nabla_a^\nu H_\mu(t, \rho(a)) = H_{\mu-\nu}(t, \rho(a))$, $t \in \mathbb{N}_{a+N}$.

Finally, we present the definitions of Mittag–Leffler function and discrete Laplace transform in nabla sense and state their useful properties.

Definition 2.8 (See [11]). For $|p| < 1$, $\alpha > 0$, $\beta \in \mathbb{R}$, the nabla Mittag–Leffler function is defined by

$$E_{p,\alpha,\beta}(t, a) = \sum_{k=0}^{\infty} p^k H_{\alpha k+\beta}(t, a), \quad t \in \mathbb{N}_a.$$

Obviously,

$$E_{p,\alpha,\beta}(t, \rho(t)) = \frac{1}{1-p}, \quad t \in \mathbb{N}_a.$$

Lemma 2.9 (See [7,9,11]). *Assume* $|p| < 1$, $\alpha > 0$, $\beta \in \mathbb{R}$.

1. *For* $\nu > 0$, $\nabla_a^{-\nu} E_{p,\alpha,\beta}(t, \rho(a)) = E_{p,\alpha,\beta+\nu}(t, \rho(a))$, $t \in \mathbb{N}_a$.
2. *For* $0 < \nu \leq 1$, $\nabla_a^\nu E_{p,\nu,\nu-1}(t, \rho(a)) = pE_{p,\nu,\nu-1}(t, \rho(a))$, $t \in \mathbb{N}_{a+1}$.
3. *For* $0 < \nu \leq 1$, $E_{p,\nu,-1}(t, \rho(a)) = pE_{p,\nu,\nu-1}(t, \rho(a))$, $t \in \mathbb{N}_{a+1}$.

Definition 2.10 (See [5]). The nabla Laplace transform of $u : \mathbb{N}_a \to \mathbb{R}$ is defined by

$$\mathcal{N}_a\big[u(t)\big] = \sum_{t=a}^{\infty} (1-s)^{t-1} u(t),$$

for those values of s for which the infinite series converges.

Definition 2.11 (See [7]). The nabla convolution product of u, $v : \mathbb{N}_a \to \mathbb{R}$ is defined by

$$(u *_a v)(t) = \sum_{s=a}^{t} u(t - \rho(s) + a)v(s), \quad t \in \mathbb{N}_a.$$

Theorem 2.12 (See [5,7,9]). *Assume* u, $v : \mathbb{N}_a \to \mathbb{R}$.

(i) *For* $\mu \in \mathbb{R} \setminus \{\ldots, -1, 0\}$, $\mathcal{N}_a\big[(t-\rho(a))^{\overline{\mu-1}}\big] = \Gamma(\mu)(1-s)^{a-1}s^{-\mu}$.

(ii) *For* $\mu \in \mathbb{R} \setminus \{\ldots, -1, 0\}$, $\mathcal{N}_a\big[H_{\mu-1}(t, \rho(a))\big] = (1-s)^{a-1}s^{-\mu}$.

(iii) $\mathcal{N}_a\big[(u *_a v)(t)\big] = \mathcal{N}_1\big[u(t+a)\big]\mathcal{N}_a\big[v(t)\big]$.

(iv) *For* $\nu > 0$, $\mathcal{N}_a\big[\big(\nabla_a^{-\nu} u\big)(t)\big] = s^{-\nu}\mathcal{N}_a\big[u(t)\big]$.

(v) *For* $0 < \nu \leq 1$, $\mathcal{N}_{a+1}\big[\big(\nabla_a^{\nu} u\big)(t)\big] = s^{\nu}\mathcal{N}_a\big[u(t)\big] - (1-s)^{a-1}u(a)$.

(vi) $\mathcal{N}_a\big[E_{p,\alpha,\beta}(t,\rho(a))\big] = (1-s)^{a-1}\dfrac{s^{\alpha-\beta-1}}{s^\alpha - p}$, $|s^{-\alpha}p| < 1.$

(vii) $\mathcal{N}_1\big[E_{p,\alpha,\beta}(t+a,a)\big] = \dfrac{s^{\alpha-\beta-1}}{s^\alpha - p}$, $|s^{-\alpha}p| < 1.$

Proof. The proofs of (i) - (v) are available in [5,7,9]. To prove (vi), consider

$$\begin{aligned}
\mathcal{N}_a\big[E_{p,\alpha,\beta}(t,\rho(a))\big] &= \mathcal{N}_a\left[\sum_{k=0}^{\infty} p^k H_{\alpha k+\beta}(t,\rho(a))\right] \\
&= \sum_{k=0}^{\infty} p^k \mathcal{N}_a\big[H_{\alpha k+\beta}(t,\rho(a))\big] \\
&= \sum_{k=0}^{\infty} p^k (1-s)^{a-1} s^{-(\alpha k+\beta+1)} \quad \text{[Using (ii)]} \\
&= (1-s)^{a-1} s^{-\beta-1} \sum_{k=0}^{\infty} \big(s^{-\alpha}p\big)^k \\
&= (1-s)^{a-1} s^{-\beta-1} \frac{1}{1-s^{-\alpha}p} \\
&= (1-s)^{a-1} \frac{s^{\alpha-\beta-1}}{s^\alpha - p}.
\end{aligned}$$

The proof is completed. To prove (vii), consider

$$\begin{aligned}
\mathcal{N}_1\big[E_{p,\alpha,\beta}(t+a,a)\big] &= \mathcal{N}_1\left[\sum_{k=0}^{\infty} p^k H_{\alpha k+\beta}(t+a,a)\right] \\
&= \sum_{k=0}^{\infty} p^k \mathcal{N}_1\big[H_{\alpha k+\beta}(t+a,a)\big] \\
&= \sum_{k=0}^{\infty} p^k \mathcal{N}_1\left[\frac{t^{\overline{\alpha k+\beta}}}{\Gamma(\alpha k+\beta+1)}\right] \\
&= \sum_{k=0}^{\infty} p^k s^{-(\alpha k+\beta+1)} \quad \text{[Using (i)]} \\
&= s^{-\beta-1} \sum_{k=0}^{\infty} \big(s^{-\alpha}p\big)^k \\
&= s^{-\beta-1} \frac{1}{1-s^{-\alpha}p} \\
&= \frac{s^{\alpha-\beta-1}}{s^\alpha - p}.
\end{aligned}$$

The proof is completed. □

3 Hilfer Nabla Fractional Difference

First, we propose the nabla analogue of Hilfer fractional derivative.

Definition 3.1. Let $u : \mathbb{N}_a \to \mathbb{R}$, $0 \leq \beta \leq 1$, and choose $N \in \mathbb{N}_1$ such that $N - 1 < \alpha \leq N$. The α^{th}-order and β^{th}-type Hilfer nabla difference of u is defined by

$$\big(\nabla_a^{\alpha,\beta} u\big)(t) = \Big(\nabla_{a+N}^{-\beta(N-\alpha)} \nabla^N \nabla_a^{-(1-\beta)(N-\alpha)} u\Big)(t), \quad t \in \mathbb{N}_{a+N}. \tag{3.1}$$

The type β allows to interpolate continuously from the Riemann–Liouville case $\nabla_a^{\alpha,0} \equiv \nabla_a^{\alpha}$ to the Caputo case $\nabla_a^{\alpha,1} \equiv \nabla_{*a}^{\alpha}$.

Remark 3.2. It follows from Remark 2.4 that if $u : \mathbb{N}_a \to \mathbb{R}$, then

$$\big(\nabla_a^{-(1-\beta)(N-\alpha)} u\big) : \mathbb{N}_a \to \mathbb{R},$$

implying that

$$\big(\nabla^N \nabla_a^{-(1-\beta)(N-\alpha)} u\big) : \mathbb{N}_{a+N} \to \mathbb{R},$$

further implying that

$$\big(\nabla_{a+N}^{-\beta(N-\alpha)} \nabla^N \nabla_a^{-(1-\beta)(N-\alpha)} u\big) : \mathbb{N}_{a+N} \to \mathbb{R}.$$

That is, if $u : \mathbb{N}_a \to \mathbb{R}$, then $\big(\nabla_a^{\alpha,\beta} u\big) : \mathbb{N}_{a+N} \to \mathbb{R}$.

Now, we obtain two important properties of Hilfer nabla fractional differences.

Proposition 3.3 (Power Rule)**.** *Let $\mu \in \mathbb{R}$, $0 < \alpha \leq 1$, $0 \leq \beta \leq 1$ such that the following fractional nabla Taylor monomials are well defined. Then,*

$$\nabla_a^{\alpha,\beta} H_\mu(t, \rho(a)) = H_{\mu-\alpha}(t, \rho(a)) - H_{\beta(1-\alpha)-1}(t, \rho(a)), \quad t \in \mathbb{N}_{a+1}.$$

Proof. We use Lemma 2.7 to prove the result. Consider

$$\begin{aligned}
\nabla_a^{\alpha,\beta} H_\mu(t, \rho(a)) &= \nabla_{a+1}^{-\beta(1-\alpha)} \nabla \nabla_a^{-(1-\beta)(1-\alpha)} H_\mu(t, \rho(a)) \\
&= \nabla_{a+1}^{-\beta(1-\alpha)} \nabla H_{\mu+(1-\beta)(1-\alpha)}(t, \rho(a)) \\
&= \nabla_{a+1}^{-\beta(1-\alpha)} H_{\mu+(1-\beta)(1-\alpha)-1}(t, \rho(a)) \\
&= \nabla_a^{-\beta(1-\alpha)} H_{\mu-\alpha-\beta(1-\alpha)}(t, \rho(a)) \\
&\quad - H_{\beta(1-\alpha)-1}(t, \rho(a)) H_{\mu-\alpha-\beta(1-\alpha)}(a, \rho(a)) \\
&= H_{\mu-\alpha-\beta(1-\alpha)+\beta(1-\alpha)}(t, \rho(a)) - H_{\beta(1-\alpha)-1}(t, \rho(a)) \\
&= H_{\mu-\alpha}(t, \rho(a)) - H_{\beta(1-\alpha)-1}(t, \rho(a)).
\end{aligned}$$

The proof is completed. □

Proposition 3.4 (Composition Rule). *Assume* $u : \mathbb{N}_a \to \mathbb{R}$, $0 < \alpha \leq 1$ *and* $0 \leq \beta \leq 1$. *Denote by* $\gamma = \alpha + \beta - \alpha\beta$. *Then,*

1. $\big(\nabla_a^{\alpha,\beta} u\big)(t) = \big(\nabla_{a+1}^{-\beta(1-\alpha)} \nabla_a^{\gamma} u\big)(t), \quad t \in \mathbb{N}_{a+1}.$
2. $\big(\nabla_{a+1}^{-\gamma} \nabla_a^{\gamma} u\big)(t) = \big(\nabla_{a+1}^{-\alpha} \nabla_a^{\alpha,\beta} u\big)(t), \quad t \in \mathbb{N}_{a+1}.$
3. $\big(\nabla_a^{\gamma} \nabla_a^{-\alpha} u\big)(t) = \big(\nabla_a^{\beta(1-\alpha)}\big)(t), \quad t \in \mathbb{N}_{a+1}.$
4. $\big(\nabla_a^{\alpha,\beta} \nabla_a^{-\alpha} u\big)(t) = \big(\nabla_{a+1}^{-\beta(1-\alpha)} \nabla_a^{\beta(1-\alpha)} u\big)(t), \quad t \in \mathbb{N}_{a+1}.$

Proof. Clearly $0 < \gamma \leq 1$. We use Lemma 2.6 to prove the results. Consider

$$\begin{aligned}
\big(\nabla_a^{\alpha,\beta} u\big)(t) &= \nabla_{a+1}^{-\beta(1-\alpha)} \big(\nabla \nabla_a^{-(1-\beta)(1-\alpha)} u\big)(t) \\
&= \nabla_{a+1}^{-\beta(1-\alpha)} \big(\nabla_a^{1-(1-\beta)(1-\alpha)} u\big)(t) \\
&= \big(\nabla_{a+1}^{-\beta(1-\alpha)} \nabla_a^{\gamma} u\big)(t).
\end{aligned}$$

The proof of (1) is completed. Consider

$$\begin{aligned}
\big(\nabla_{a+1}^{-\gamma} \nabla_a^{\gamma} u\big)(t) &= \big(\nabla_{a+1}^{-\gamma} \nabla \nabla_a^{-(1-\gamma)} u\big)(t) \\
&= \big(\nabla_{a+1}^{-(\alpha+\beta-\alpha\beta)} \nabla \nabla_a^{-(1-\alpha-\beta+\alpha\beta)} u\big)(t) \\
&= \Big(\nabla_{a+1}^{-\alpha} \nabla_{a+1}^{-\beta(1-\alpha)} \nabla \nabla_a^{-(1-\beta)(1-\alpha)} u\Big)(t) \\
&= \big(\nabla_{a+1}^{-\alpha} \nabla_a^{\alpha,\beta} u\big)(t).
\end{aligned}$$

The proof of (2) is completed. Consider

$$\begin{aligned}
\big(\nabla_a^{\gamma} \nabla_a^{-\alpha} u\big)(t) &= \big(\nabla \nabla_a^{-(1-\gamma)} \nabla_a^{-\alpha} u\big)(t) \\
&= \big(\nabla \nabla_a^{-(1-\gamma+\alpha)} u\big)(t) \\
&= \big(\nabla_a^{1-(1-\gamma+\alpha)} u\big)(t) \\
&= \big(\nabla_a^{\beta(1-\alpha)}\big)(t).
\end{aligned}$$

The proof of (3) is completed. Consider

$$\begin{aligned}
\big(\nabla_a^{\alpha,\beta} \nabla_a^{-\alpha} u\big)(t) &= \big(\nabla_{a+1}^{-\beta(1-\alpha)} \nabla \nabla_a^{-(1-\beta)(1-\alpha)} \nabla_a^{-\alpha} u\big)(t) \\
&= \big(\nabla_{a+1}^{-\beta(1-\alpha)} \nabla \nabla_a^{-(1-\beta)(1-\alpha)-\alpha} u\big)(t) \\
&= \big(\nabla_{a+1}^{-\beta(1-\alpha)} \nabla \nabla_a^{\beta(1-\alpha)-1} u\big)(t) \\
&= \big(\nabla_{a+1}^{-\beta(1-\alpha)} \nabla_a^{1+\beta(1-\alpha)-1} u\big)(t) \\
&= \big(\nabla_{a+1}^{-\beta(1-\alpha)} \nabla_a^{\beta(1-\alpha)} u\big)(t).
\end{aligned}$$

The proof of (4) is completed. □

Consider the initial value problem

$$\begin{cases} \big(\nabla_a^{\alpha,\beta} u\big)(t) = f(t, u(t)), \quad t \in \mathbb{N}_{a+1}, \\ \Big[\big(\nabla_a^{-(1-\gamma)} u\big)(t)\Big]_{t=a} = u(a) = u_0, \end{cases} \tag{3.2}$$

where $0 < \alpha \leq 1$, $0 \leq \beta \leq 1$, $\gamma = \alpha + \beta - \alpha\beta$ and $f : \mathbb{N}_a \times \mathbb{R} \to \mathbb{R}$.

Theorem 3.5. *u is a solution of the initial value problem (3.2) if, and only if, u is a solution of the Volterra summation equation*

$$u(t) = u_0 H_{\gamma-1}(t, \rho(a)) + \sum_{s=a+1}^{t} H_{\alpha-1}(t, \rho(s)) f(s, u(s)), \quad t \in \mathbb{N}_a. \tag{3.3}$$

Proof. Consider the nabla fractional difference equation

$$\big(\nabla_a^{\alpha,\beta} u\big)(t) = f(t, u(t)), \quad t \in \mathbb{N}_{a+1}. \tag{3.4}$$

Applying the operator $\nabla_{a+1}^{-\alpha}$ on both sides of (3.4), and using (2) of Proposition 3.4, we obtain

$$\big(\nabla_{a+1}^{-\gamma} \nabla_a^{\gamma} u\big)(t) = \nabla_{a+1}^{-\alpha} f(t, u(t)), \quad t \in \mathbb{N}_{a+1}. \tag{3.5}$$

Now, consider

$$\begin{aligned} &\big(\nabla_{a+1}^{-\gamma} \nabla_a^{\gamma} u\big)(t) \\ &= \nabla_{a+1}^{-\gamma} \nabla \big(\nabla_a^{-(1-\gamma)} u\big)(t) \\ &= \big(\nabla \nabla_a^{-\gamma} \nabla_a^{-(1-\gamma)} u\big)(t) - H_{\gamma-1}(t, \rho(a)) \big(\nabla_a^{-(1-\gamma)} u\big)(a) \text{ [Using Lemma 2.6 (2)]} \\ &= \big(\nabla_a^{1-\gamma} \nabla_a^{-(1-\gamma)} u\big)(t) - u(a) H_{\gamma-1}(t, \rho(a)) \\ &= u(t) - u_0 H_{\gamma-1}(t, \rho(a)). \end{aligned} \tag{3.6}$$

Using (3.6) in (3.5), we achieve (3.3). Thus, the first part of the proof is completed. Conversely, suppose u is a solution of the Volterra summation equation (3.3). Applying the operator ∇_a^{γ} on both sides of (3.3), and using (3) of Proposition 3.4 and Lemma 2.7, we obtain

$$\begin{aligned} &\big(\nabla_a^{\gamma} u\big)(t) \\ &= u_0 \nabla_a^{\gamma} H_{\gamma-1}(t, \rho(a)) + \nabla_a^{\gamma} \nabla_{a+1}^{-\alpha} f(t, u(t)) \\ &= u_0 H_{-1}(t, \rho(a)) + \nabla_a^{\gamma} \Big[\nabla_a^{-\alpha} f(t, u(t)) - H_{\alpha-1}(t, \rho(a)) f(a, u(a))\Big] \\ &= \nabla_a^{\gamma} \nabla_a^{-\alpha} f(t, u(t)) - H_{\alpha-\gamma-1}(t, \rho(a)) f(a, u(a)) \ [\because H_{-1}(t, \rho(a)) = 0] \\ &= \nabla_a^{\beta(1-\alpha)} f(t, u(t)) - H_{\alpha-\gamma-1}(t, \rho(a)) f(a, u(a)). \end{aligned} \tag{3.7}$$

Applying the operator $\nabla_{a+1}^{-\beta(1-\alpha)}$ on both sides of (3.7), and using (1) of Proposition 3.4 and Lemma 2.7, we obtain

$$\big(\nabla_a^{\alpha,\beta} u\big)(t)$$

$$
\begin{aligned}
&= \nabla_{a+1}^{-\beta(1-\alpha)} \nabla_a^{\beta(1-\alpha)} f(t,u(t)) - f(a,u(a)) \nabla_{a+1}^{-\beta(1-\alpha)} H_{\alpha-\gamma-1}(t,\rho(a)) \\
&= \nabla_{a+1}^{-\beta(1-\alpha)} \nabla \nabla_a^{-(1-\beta(1-\alpha))} f(t,u(t)) \\
&\quad - f(a,u(a)) \Big[\nabla_a^{-\beta(1-\alpha)} H_{\alpha-\gamma-1}(t,\rho(a)) - H_{\beta(1-\alpha)-1}(t,\rho(a)) H_{\alpha-\gamma-1}(a,\rho(a)) \Big] \\
&= \nabla \nabla_a^{-\beta(1-\alpha)} \nabla_a^{-(1-\beta(1-\alpha))} f(t,u(t)) - f(a,u(a)) H_{\beta(1-\alpha)-1}(t,\rho(a)) \\
&\quad - f(a,u(a)) \Big[H_{\alpha-\gamma-1+\beta(1-\alpha)}(t,\rho(a)) - H_{\beta(1-\alpha)-1}(t,\rho(a)) \Big] \\
&= \nabla_a^{1-\beta(1-\alpha)} \nabla_a^{-(1-\beta(1-\alpha))} f(t,u(t)) - f(a,u(a)) H_{-1}(t,\rho(a)) \\
&= f(t,u(t)). \; [\because H_{-1}(t,\rho(a)) = 0]
\end{aligned}
$$

Thus, we have (3.4). Finally, we show that the initial condition also holds. Applying the operator $\nabla_a^{-(1-\gamma)}$ on both sides of (3.3), and using (3) of Proposition 3.4 and Lemma 2.7, we obtain

$$
\begin{aligned}
&\big(\nabla_a^{-(1-\gamma)} u\big)(t) \\
&= u_0 \nabla_a^{-(1-\gamma)} H_{\gamma-1}(t,\rho(a)) + \nabla_a^{-(1-\gamma)} \nabla_{a+1}^{-\alpha} f(t,u(t)) \\
&= u_0 H_0(t,\rho(a)) + \nabla_a^{-(1-\gamma)} \Big[\nabla_a^{-\alpha} f(t,u(t)) - H_{\alpha-1}(t,\rho(a)) f(a,u(a)) \Big] \\
&= u_0 + \nabla_a^{-(1-\gamma)} \nabla_a^{-\alpha} f(t,u(t)) - H_{\alpha-\gamma}(t,\rho(a)) f(a,u(a)) \; [\because H_0(t,\rho(a)) = 1] \\
&= u_0 + \nabla_a^{-(1-\gamma+\alpha)} f(t,u(t)) - H_{\alpha-\gamma}(t,\rho(a)) f(a,u(a)).
\end{aligned} \tag{3.8}
$$

Taking $t = a$ in (3.8), we get

$$
\Big[\big(\nabla_a^{-(1-\gamma)} u\big)(t) \Big]_{t=a} = u_0.
$$

The second part of the proof is completed. □

4 Linear Hilfer Nabla Fractional Difference Equations

First, we obtain the discrete Laplace transform of $\nabla_a^{\alpha,\beta}$.

Proposition 4.1. *Let* $0 < \alpha \leq 1$ *and* $0 \leq \beta \leq 1$. *Denote by* $\gamma = \alpha + \beta - \alpha\beta$. *Then,*

$$
\mathcal{N}_{a+1}\big[\big(\nabla_a^{\alpha,\beta} u\big)(t)\big] = s^{\alpha} \mathcal{N}_a\big[u(t)\big] - \frac{(1-s)^{a-1}}{s^{\gamma-\alpha}} u(a).
$$

Proof. Put $w(t) = \big(\nabla_a^{-(1-\beta)(1-\alpha)} u\big)(t)$ for $t \in \mathbb{N}_a$ and $v(t) = \big(\nabla w\big)(t)$ for $t \in \mathbb{N}_{a+1}$. Consider

$$
\begin{aligned}
\mathcal{N}_{a+1}\big[\big(\nabla_a^{\alpha,\beta} u\big)(t)\big] &= \mathcal{N}_{a+1}\big[\big(\nabla_{a+1}^{-\beta(1-\alpha)} v\big)(t)\big] \\
&= \frac{1}{s^{\beta(1-\alpha)}} \mathcal{N}_{a+1}\big[v(t)\big]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{s^{\beta(1-\alpha)}} \mathcal{N}_{a+1}\big[\big(\nabla w\big)(t)\big] \\
&= \frac{1}{s^{\beta(1-\alpha)}} \left[s\mathcal{N}_a\big[w(t)\big] - (1-s)^{a-1}w(a)\right] \\
&= \frac{1}{s^{\beta(1-\alpha)}} \left[s\mathcal{N}_a\big[\big(\nabla_a^{-(1-\beta)(1-\alpha)}u\big)(t)\big] - (1-s)^{a-1}u(a)\right] \\
&= \frac{1}{s^{\beta(1-\alpha)}} \left[\frac{s}{s^{(1-\beta)(1-\alpha)}}\mathcal{N}_a\big[u(t)\big] - (1-s)^{a-1}u(a)\right] \\
&= s^{\alpha}\mathcal{N}_a\big[u(t)\big] - \frac{(1-s)^{a-1}}{s^{\gamma-\alpha}}u(a).
\end{aligned}
$$

The proof is completed. □

Remark 4.2. Observe that Proposition 4.1 coincides with (v) of Theorem 2.12 for $\beta = 0$ (the case of Riemann–Liouville).

Consider the initial value problem

$$
\begin{cases} \big(\nabla_a^{\alpha,\beta}u\big)(t) + \lambda u(t) = h(t), & t \in \mathbb{N}_{a+1}, \\ \Big[\big(\nabla_a^{-(1-\gamma)}u\big)(t)\Big]_{t=a} = u(a), \end{cases} \tag{4.1}
$$

where $0 < \alpha \leq 1$, $0 \leq \beta \leq 1$, $\gamma = \alpha + \beta - \alpha\beta$, $|\lambda| < 1$, and $h : \mathbb{N}_a \to \mathbb{R}$.

Theorem 4.3. *The solution of the initial value problem* (4.1) *is uniquely determined on* $\mathbb{N}_a$.

Proof. It follows from Theorem 3.5 that u is a solution of the initial value problem (4.1) if, and only if, u is a solution of the Volterra summation equation

$$
u(t) = u_0 H_{\gamma-1}(t,\rho(a)) + \sum_{s=a+1}^{t} H_{\alpha-1}(t,\rho(s))\left[h(s) - \lambda u(s)\right], \quad t \in \mathbb{N}_a. \tag{4.2}
$$

Rearranging the terms of (4.2), we obtain

$$
\begin{aligned}
u(t) = \frac{1}{1+\lambda}\Big[&u_0 H_{\gamma-1}(t,\rho(a)) + \sum_{s=a+1}^{t} H_{\alpha-1}(t,\rho(s))h(s) \\
&- \lambda \sum_{s=a+1}^{t-1} H_{\alpha-1}(t,\rho(s))u(s)\Big], \quad t \in \mathbb{N}_a.
\end{aligned} \tag{4.3}
$$

This iteration scheme ensures us that the initial value problem (4.1) has a unique solution defined on $\mathbb{N}_a$. □

Theorem 4.4. *The unique solution of the initial value problem* (4.1) *is given by*

$$
\begin{aligned}
u(t) = [\lambda u(a) - h(a)]E_{-\lambda,\alpha,\alpha-1}(t,\rho(a)) + u(a)E_{-\lambda,\alpha,\gamma-1}(t,\rho(a)) \\
+ \big(E_{-\lambda,\alpha,\alpha-1}(\cdot,a) *_a h\big)(t), \quad t \in \mathbb{N}_a,
\end{aligned} \tag{4.4}
$$

Proof. We take the Laplace transform based at $a+1$ on both sides of the nabla fractional difference equation in (4.1) to get that

$$\mathcal{N}_{a+1}\big[\big(\nabla_a^{\alpha,\beta}u\big)(t)\big] + \lambda\mathcal{N}_{a+1}\big[u(t)\big] = \mathcal{N}_{a+1}\big[h(t)\big]. \tag{4.5}$$

Now, we use Theorem 2.12 and Proposition 4.1 to rewrite the left and right hand parts of (4.5) in terms of $\mathcal{N}_a\big[u(t)\big]$ and $\mathcal{N}_a\big[h(t)\big]$. We have

$$\mathcal{N}_{a+1}\big[\big(\nabla_a^{\alpha,\beta}u\big)(t)\big] = s^\alpha\mathcal{N}_a\big[u(t)\big] - \frac{(1-s)^{a-1}}{s^{\gamma-\alpha}}u(a), \tag{4.6}$$

$$\mathcal{N}_{a+1}\big[u(t)\big] = \mathcal{N}_a\big[u(t)\big] - (1-s)^{a-1}u(a), \tag{4.7}$$

and

$$\mathcal{N}_{a+1}\big[h(t)\big] = \mathcal{N}_a\big[h(t)\big] - (1-s)^{a-1}h(a). \tag{4.8}$$

Use (4.6), (4.7) and (4.8) in (4.5) and rearranging the terms, we obtain

$$\begin{aligned}\mathcal{N}_a\big[u(t)\big] = [\lambda u(a) - h(a)]\frac{(1-s)^{a-1}}{s^\alpha+\lambda} + u(a)\frac{(1-s)^{a-1}s^{\alpha-\gamma}}{(s^\alpha+\lambda)}& \\ + \frac{1}{s^\alpha+\lambda}\mathcal{N}_a\big[h(t)\big].&\end{aligned} \tag{4.9}$$

From Theorem 2.12, we observe that

$$\mathcal{N}_a\big[E_{-\lambda,\alpha,\alpha-1}(t,\rho(a))\big] = \frac{(1-s)^{a-1}}{s^\alpha+\lambda}, \tag{4.10}$$

$$\mathcal{N}_a\big[E_{-\lambda,\alpha,\gamma-1}(t,\rho(a))\big] = \frac{(1-s)^{a-1}s^{\alpha-\gamma}}{s^\alpha+\lambda}, \tag{4.11}$$

$$\mathcal{N}_1\big[E_{-\lambda,\alpha,\alpha-1}(t+a,a)\big] = \frac{1}{s^\alpha+\lambda}. \tag{4.12}$$

Using (4.10)–(4.12) in (4.9), we have

$$\begin{aligned}\mathcal{N}_a\big[u(t)\big] = [\lambda u(a) - h(a)]\mathcal{N}_a\big[E_{-\lambda,\alpha,\alpha-1}(t,\rho(a))\big] + u(a)\mathcal{N}_a\big[E_{-\lambda,\alpha,\gamma-1}(t,\rho(a))\big]& \\ + \mathcal{N}_1\big[E_{-\lambda,\alpha,\alpha-1}(t+a,a)\big]\mathcal{N}_a\big[h(t)\big].&\end{aligned} \tag{4.13}$$

Applying the inverse nabla Laplace transform $\mathcal{N}_a^{-1}$ on both sides of (4.13), we obtain (4.4). The proof is completed. □

Remark 4.5. An alternative form of (4.4) is given by

$$\begin{aligned}u(t) = u(a)\left[\lambda E_{-\lambda,\alpha,\alpha-1}(t,\rho(a)) + E_{-\lambda,\alpha,\gamma-1}(t,\rho(a))\right]& \\ + \sum_{s=a+1}^{t} E_{-\lambda,\alpha,\alpha-1}(t,\rho(s))h(s), \quad t\in\mathbb{N}_a.&\end{aligned}$$

Consider the initial value problem

$$\begin{cases} \big(\nabla_a^{\alpha,\beta} u\big)(t) = h(t), \quad t \in \mathbb{N}_{a+1}, \\ \Big[\big(\nabla_a^{-(1-\gamma)} u\big)(t)\Big]_{t=a} = u(a), \end{cases} \tag{4.14}$$

where $0 < \alpha \leq 1$, $0 \leq \beta \leq 1$, $\gamma = \alpha + \beta - \alpha\beta$, and $h : \mathbb{N}_a \to \mathbb{R}$.

Corollary 4.6. *The unique solution of the initial value problem* (4.14) *is given by*

$$u(t) = u(a) H_{\gamma-1}(t, \rho(a)) + \sum_{s=a+1}^{t} H_{\alpha-1}(t, \rho(s)) h(s), \quad t \in \mathbb{N}_a. \tag{4.15}$$

Proof. The proof follows from Theorem 3.5. □

Consider the initial value problem

$$\begin{cases} \big(\nabla_a^{\alpha,\beta} u\big)(t) + \lambda u(t) = 0, \quad t \in \mathbb{N}_{a+1}, \\ \Big[\big(\nabla_a^{-(1-\gamma)} u\big)(t)\Big]_{t=a} = u(a), \end{cases} \tag{4.16}$$

where $0 < \alpha \leq 1$, $0 \leq \beta \leq 1$, $\gamma = \alpha + \beta - \alpha\beta$, and $|\lambda| < 1$.

Corollary 4.7. *The unique solution of the initial value problem* (4.16) *is given by*

$$u(t) = u(a) \left[\lambda E_{-\lambda,\alpha,\alpha-1}(t, \rho(a)) + E_{-\lambda,\alpha,\gamma-1}(t, \rho(a))\right], \quad t \in \mathbb{N}_a. \tag{4.17}$$

Proof. The proof follows from Theorem 4.4. □

Jia et al. [15] have obtained the following asymptotic behaviour of nabla discrete Mittag–Leffler functions.

Theorem 4.8 (See [15])**.** *For* $0 < p < 1$ *and* $0 < \mu < 1$*, we have*

$$\lim_{t \to \infty} E_{p,\mu,\mu-1}(t, \rho(a)) = +\infty,$$

and

$$\lim_{t \to \infty} E_{-p,\mu,\mu-1}(t, \rho(a)) = 0.$$

It follows from Theorem 4.8 and Corollary 4.7 that the unique solution of the initial value problem

$$\begin{cases} \big(\nabla_a^{\alpha,\beta} u\big)(t) + \lambda u(t) = 0, \quad t \in \mathbb{N}_{a+1}, \quad 0 < \alpha \leq 1, \quad 0 \leq \beta \leq 1, \\ \Big[\big(\nabla_a^{-(1-\gamma)} u\big)(t)\Big]_{t=a} = u(a) = 1, \end{cases} \tag{4.18}$$

tends to zero if $0 < \lambda < 1$ and tends to $+\infty$ if $-1 < \lambda < 0$, as $t \to \infty$.

Appendix

Remark 4.9. Here we verify that the unique solution of the initial value problem (4.16) is given by (4.17). Denote by

$$g(t) = \lambda E_{-\lambda,\alpha,\alpha-1}(t,\rho(a)) + E_{-\lambda,\alpha,\gamma-1}(t,\rho(a)), \quad t \in \mathbb{N}_a.$$

We show that g satisfies the homogeneous linear Hilfer nabla fractional difference equation

$$\big(\nabla_a^{\alpha,\beta} u\big)(t) + \lambda u(t) = 0, \quad t \in \mathbb{N}_{a+1}. \tag{4.19}$$

To see this, take

$$\begin{aligned}\big(\nabla_a^{\alpha,\beta} g\big)(t) &= \nabla_a^{\alpha,\beta}\left[\lambda E_{-\lambda,\alpha,\alpha-1}(t,\rho(a)) + E_{-\lambda,\alpha,\gamma-1}(t,\rho(a))\right]\\ &= \lambda \nabla_a^{\alpha,\beta} E_{-\lambda,\alpha,\alpha-1}(t,\rho(a)) + \nabla_a^{\alpha,\beta} E_{-\lambda,\alpha,\gamma-1}(t,\rho(a))\\ &= \lambda S_1 + S_2,\end{aligned} \tag{4.20}$$

where

$$S_1 = \nabla_a^{\alpha,\beta} E_{-\lambda,\alpha,\alpha-1}(t,\rho(a)), \quad S_2 = \nabla_a^{\alpha,\beta} E_{-\lambda,\alpha,\gamma-1}(t,\rho(a)).$$

Consider

$$\begin{aligned} S_1 &= \nabla_a^{\alpha,\beta} E_{-\lambda,\alpha,\alpha-1}(t,\rho(a))\\ &= \nabla_{a+1}^{-\beta(1-\alpha)} \nabla \left[\nabla_a^{-(1-\beta)(1-\alpha)} E_{-\lambda,\alpha,\alpha-1}(t,\rho(a))\right]\\ &= \nabla_{a+1}^{-\beta(1-\alpha)} \left[\nabla E_{-\lambda,\alpha,\alpha-1+(1-\beta)(1-\alpha)}(t,\rho(a))\right] \text{ [Using Lemma 2.9 (1)]}\\ &= \nabla_{a+1}^{-\beta(1-\alpha)} \left[\nabla E_{-\lambda,\alpha,-\beta(1-\alpha)}(t,\rho(a))\right]\\ &= \nabla_{a+1}^{-\beta(1-\alpha)} E_{-\lambda,\alpha,-\beta(1-\alpha)-1}(t,\rho(a)) \text{ [Using Lemma 2.9 (2)]}\\ &= \nabla_a^{-\beta(1-\alpha)} E_{-\lambda,\alpha,-\beta(1-\alpha)-1}(t,\rho(a)) - H_{\beta(1-\alpha)-1}(t,\rho(a)) E_{-\lambda,\alpha,-\beta(1-\alpha)-1}(a,\rho(a))\\ &= E_{-\lambda,\alpha,-\beta(1-\alpha)-1+\beta(1-\alpha)}(t,\rho(a))\\ &\quad - \frac{1}{1+\lambda} H_{\beta(1-\alpha)-1}(t,\rho(a)) \text{ [Using Lemma 2.9 (1)]}\\ &= E_{-\lambda,\alpha,-1}(t,\rho(a)) - \frac{1}{1+\lambda} H_{\beta(1-\alpha)-1}(t,\rho(a))\\ &= -\lambda E_{-\lambda,\alpha,\alpha-1}(t,\rho(a))\\ &\quad - \frac{1}{1+\lambda} H_{\beta(1-\alpha)-1}(t,\rho(a)). \text{ [Using Lemma 2.9 (3)]} \end{aligned} \tag{4.21}$$

Now, consider

$$\begin{aligned} S_2 &= \nabla_a^{\alpha,\beta} E_{-\lambda,\alpha,\gamma-1}(t,\rho(a))\\ &= \nabla_{a+1}^{-\beta(1-\alpha)} \nabla \left[\nabla_a^{-(1-\beta)(1-\alpha)} E_{-\lambda,\alpha,\gamma-1}(t,\rho(a))\right]\\ &= \nabla_{a+1}^{-\beta(1-\alpha)} \left[\nabla E_{-\lambda,\alpha,\gamma-1+(1-\beta)(1-\alpha)}(t,\rho(a))\right] \text{ [Using Lemma 2.9 (1)]} \end{aligned}$$

$$
\begin{aligned}
&= \nabla_{a+1}^{-\beta(1-\alpha)} \left[\nabla E_{-\lambda,\alpha,0}(t,\rho(a))\right] \\
&= \nabla_{a+1}^{-\beta(1-\alpha)} E_{-\lambda,\alpha,-1}(t,\rho(a)) \text{ [Using Lemma 2.9 (2)]} \\
&= -\lambda \nabla_{a+1}^{-\beta(1-\alpha)} E_{-\lambda,\alpha,\alpha-1}(t,\rho(a)) \text{ [Using Lemma 2.9 (3)]} \\
&= -\lambda \Big[\nabla_{a}^{-\beta(1-\alpha)} E_{-\lambda,\alpha,\alpha-1}(t,\rho(a)) - H_{\beta(1-\alpha)-1}(t,\rho(a)) E_{-\lambda,\alpha,\alpha-1}(a,\rho(a))\Big] \\
&= -\lambda E_{-\lambda,\alpha,\alpha-1+\beta(1-\alpha)}(t,\rho(a)) \\
&\quad + \frac{\lambda}{1+\lambda} H_{\beta(1-\alpha)-1}(t,\rho(a)) \text{ [Using Lemma 2.9 (1)]} \\
&= -\lambda E_{-\lambda,\alpha,\gamma-1}(t,\rho(a)) + \frac{\lambda}{1+\lambda} H_{\beta(1-\alpha)-1}(t,\rho(a)).
\end{aligned} \tag{4.22}
$$

Using (4.21) and (4.22) in (4.20), we obtain

$$
\begin{aligned}
\left(\nabla_a^{\alpha,\beta} g\right)(t) &= \lambda \left[-\lambda E_{-\lambda,\alpha,\alpha-1}(t,\rho(a)) - \frac{1}{1+\lambda} H_{\beta(1-\alpha)-1}(t,\rho(a))\right] \\
&\quad + \left[-\lambda E_{-\lambda,\alpha,\gamma-1}(t,\rho(a)) + \frac{\lambda}{1+\lambda} H_{\beta(1-\alpha)-1}(t,\rho(a))\right] \\
&= -\lambda \left[\lambda E_{-\lambda,\alpha,\alpha-1}(t,\rho(a)) + E_{-\lambda,\alpha,\gamma-1}(t,\rho(a))\right] \\
&= -\lambda g(t).
\end{aligned}
$$

Thus, g satisfies (4.19). So, the general solution of (4.19) is given by

$$
u(t) = Cg(t), \quad t \in \mathbb{N}_a, \tag{4.23}
$$

for some arbitrary constant C. To find C, we take $t = a$ in (4.23). Then, we have

$$
\begin{aligned}
u(a) &= Cg(a) \\
&= C\left[\lambda E_{-\lambda,\alpha,\alpha-1}(a,\rho(a)) + E_{-\lambda,\alpha,\gamma-1}(a,\rho(a))\right] \\
&= C\left[\frac{\lambda}{1+\lambda} + \frac{1}{1+\lambda}\right] \\
&= C.
\end{aligned}
$$

Consequently, the unique solution of the initial value problem (4.16) is given by (4.17). The verification is completed.

Remark 4.10. Here we verify that the unique solution of the initial value problem (4.1) is given in Remark 4.5. In Remark 4.9, we have already showed that the general solution of (4.19) is given by (4.23). Denote by

$$
x(t) = \sum_{s=a+1}^{t} E_{-\lambda,\alpha,\alpha-1}(t,\rho(s))h(s), \quad t \in \mathbb{N}_{a+1}.
$$

So, it is enough to show that x is a particular solution of the nonhomogeneous linear Hilfer nabla fractional difference equation

$$\big(\nabla_a^{\alpha,\beta} u\big)(t) + \lambda u(t) = h(t), \quad t \in \mathbb{N}_{a+1}. \tag{4.24}$$

To see this, take

$$\begin{aligned} x(t) &= \sum_{s=a+1}^{t} E_{-\lambda,\alpha,\alpha-1}(t,\rho(s))h(s) \\ &= \sum_{s=a+1}^{t} \left[\sum_{k=0}^{\infty} (-\lambda)^k H_{\alpha k+\alpha-1}(t,\rho(s))\right] h(s) \\ &= \sum_{k=0}^{\infty} (-\lambda)^k \left[\sum_{s=a+1}^{t} H_{\alpha k+\alpha-1}(t,\rho(s))h(s)\right] \\ &= \sum_{k=0}^{\infty} (-\lambda)^k \left[\big(\nabla_{a+1}^{-(\alpha k+\alpha)} h\big)(t)\right]. \end{aligned} \tag{4.25}$$

Consider

$$\begin{aligned} &\big(\nabla_a^{-(1-\beta)(1-\alpha)} x\big)(t) \\ &= \sum_{k=0}^{\infty} (-\lambda)^k \left[\big(\nabla_a^{-(1-\beta)(1-\alpha)} \nabla_{a+1}^{-(\alpha k+\alpha)} h\big)(t)\right] \\ &= \sum_{k=0}^{\infty} (-\lambda)^k \left[\big(\nabla_{a+1}^{-(1-\beta)(1-\alpha)} \nabla_{a+1}^{-(\alpha k+\alpha)} h\big)(t)\right] \\ &= \sum_{k=0}^{\infty} (-\lambda)^k \left[\big(\nabla_{a+1}^{-(\alpha k+\alpha+(1-\beta)(1-\alpha))} h\big)(t)\right] \ \text{[Using Lemma 2.6 (1)]}. \end{aligned}$$

Further, consider

$$\begin{aligned} \big(\nabla\nabla_a^{-(1-\beta)(1-\alpha)} x\big)(t) &= \sum_{k=0}^{\infty} (-\lambda)^k \left[\big(\nabla\nabla_{a+1}^{-(\alpha k+\alpha+(1-\beta)(1-\alpha))} h\big)(t)\right] \\ &= \sum_{k=0}^{\infty} (-\lambda)^k \left[\big(\nabla_{a+1}^{1-(\alpha k+\alpha+(1-\beta)(1-\alpha))} h\big)(t)\right]. \end{aligned}$$

Now, consider

$$\begin{aligned} &\big(\nabla_a^{\alpha,\beta} x\big)(t) \\ &= \big(\nabla_{a+1}^{-\beta(1-\alpha)} \nabla\nabla_a^{-(1-\beta)(1-\alpha)} x\big)(t) \\ &= \sum_{k=0}^{\infty} (-\lambda)^k \left[\big(\nabla_{a+1}^{-\beta(1-\alpha)} \nabla_{a+1}^{1-(\alpha k+\alpha+(1-\beta)(1-\alpha))} h\big)(t)\right] \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty}(-\lambda)^k \left[\left(\nabla_{a+1}^{1-(\alpha k+\alpha+(1-\beta)(1-\alpha)+\beta(1-\alpha))}h\right)(t)\right] \text{ [Using Lemma 2.6 (1)]}\\
&= \sum_{k=0}^{\infty}(-\lambda)^k \left[\left(\nabla_{a+1}^{-\alpha k}h\right)(t)\right]\\
&= \sum_{k=1}^{\infty}(-\lambda)^k \left[\left(\nabla_{a+1}^{-\alpha k}h\right)(t)\right] + h(t)\\
&= -\lambda\sum_{k=0}^{\infty}(-\lambda)^k \left[\left(\nabla_{a+1}^{-\alpha k-\alpha}h\right)(t)\right] + h(t)\\
&= -\lambda x(t) + h(t) \text{ [Using (4.25)]}.
\end{aligned}
$$

The verification is completed.

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