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On Deflated Singular Matrices Associated with the 5D Discrete Number Operator

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Abstract

An algebraic approach to solving an eigenvalue–eigenvector problem for the $5D$ discrete number operator $\mathcal{N}^{(5)}$, which governs the eigenvectors of the $5D$ discrete Fourier transform, is discussed. The main idea of this hybrid approach is to combine the use of the algebraic properties of the raising and lowering difference operators, that factorize the discrete number operator $\mathcal{N}^{(5)}$, with the repeated application of the matrix deflation method, which delivers at the first step a matrix of size 1 smaller than the matrix, associated with the operator $\mathcal{N}^{(5)}$.

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1 Introduction

The purpose of the present paper is to develop an algebraic approach to solving an eigenvalue problem for the $5D$ discrete number operator $\mathcal{N}^{(5)}$, which governs the eigenvectors of the $5D$ discrete Fourier transform. This discrete number operator $\mathcal{N}^{(5)}$ is defined

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as a product of the lowering $\mathbf{b}_5$ and raising $\mathbf{b}_5^{\intercal}$ difference operators, constructed with the aid of the standard intertwining relations [1–4]

$$\mathbf{b}_5\,\mathbf{\Phi}^{(5)} = \mathrm{i}\,\mathbf{\Phi}^{(5)}\,\mathbf{b}_5\,, \qquad \mathbf{b}_5^{\intercal}\,\mathbf{\Phi}^{(5)} = -\,\mathrm{i}\,\mathbf{\Phi}^{(5)}\,\mathbf{b}_5^{\intercal}, \qquad N \geq 3\,, \tag{1.1}$$

with the 5 discrete Fourier transform operator $\mathbf{\Phi}^{(5)}$, represented by an 5×5 unitary symmetric matrix with entries

$$\Phi^{(5)}_{m,n} := \frac{1}{\sqrt{5}}\,q^{\,mn} \equiv \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & q & q^2 & q^3 & q^4 \\ 1 & q^2 & q^4 & q & q^3 \\ 1 & q^3 & q & q^4 & q^2 \\ 1 & q^4 & q^3 & q^2 & q \end{bmatrix}, \tag{1.2}$$

where the parameter $q = \exp(2\pi\mathrm{i}/5)$ is the 5th primitive root of unity and indices $m, n \in \{0, 1, 2, 3, 4\}$. The explicit form of these lowering and raising difference operators is

$$\mathbf{b}_5 = c\begin{bmatrix} 0 & 1 & 0 & 0 & -1 \\ -1 & s_1 & 1 & 0 & 0 \\ 0 & -1 & s_2 & 1 & 0 \\ 0 & 0 & -1 & s_3 & 1 \\ 1 & 0 & 0 & -1 & s_4 \end{bmatrix}, \quad \mathbf{b}_5^{\intercal} = c\begin{bmatrix} 0 & -1 & 0 & 0 & 1 \\ 1 & s_1 & -1 & 0 & 0 \\ 0 & 1 & s_2 & -1 & 0 \\ 0 & 0 & 1 & s_3 & -1 \\ -1 & 0 & 0 & 1 & s_4 \end{bmatrix}, \tag{1.3}$$

where $c = \sqrt{5/16\pi}$ and we introduced for brevity $s_k := 2\sin\dfrac{2\pi}{5}k,\;\; 0 \leq k \leq 4$. From the relations (1.1) it follows at once that the $5D$ discrete number operator, defined as

$$\mathcal{N}^{(5)} = \mathbf{b}_5^{\intercal}\,\mathbf{b}_5 = c^2\begin{bmatrix} 2 & s_4 & -1 & -1 & s_4 \\ s_4 & 4-c_2 & c_1 s_2 & -1 & -1 \\ -1 & c_1 s_2 & 4-c_1 & c_0 s_2 & -1 \\ -1 & -1 & c_0 s_2 & 4-c_1 & c_1 s_2 \\ s_4 & -1 & -1 & c_1 s_2 & 4-c_2 \end{bmatrix}, \tag{1.4}$$

with $c_k := 2\cos\dfrac{2\pi}{5}k$, commutes with the DFT operator $\mathbf{\Phi}^{(5)}$.

It should be noted at this point that the matrix (1.4) has been studied in detail in [4], where it has been established in particular that the lowest eigenvector $\vec{f_0}$ of the $\mathcal{N}^{(5)}$, associated with the eigenvalue $\lambda_0 = 0$, can be found from the simpler (than the defining equation $\mathcal{N}^{(5)}\vec{f_0} = 0$) difference equation $\mathbf{b}_5\vec{f_0} = 0$. Thus an explicit form of the 5-column eigenvector $\vec{f_0}$ has been found to be

$$\vec{f_0} \,=\, d_0^{-1}(x_0, x_1, 1, 1, x_1)^T, \tag{1.5}$$

where $x_0 = s_1 - 2c_2$, $x_1 = 1 + s_2$ and d_0 is a normalization constant,

$$d_0 \,=\, \left(\frac{2\pi}{5}\right)^{1/4}\Big[x_0^2 + 2(x_1^2 + 1)\Big]^{1/2}. \tag{1.6}$$

The eigenvectors $\vec{f}_k$, $1 \le k \le 4$, associated with the other 4 eigenvalues λ_k, have been constructed with the aid of the formula

$$\vec{f}_k := \Big(\prod_{j=1}^{k} \lambda_j\Big)^{-1/2} \Big(\mathbf{b}_5^{\intercal}\Big)^{j} \vec{f}_0 \,. \tag{1.7}$$

But the explicit forms of the eigenvalues λ_k, associated with those 4 eigenvectors $\vec{f}_k$, have been found in [4] by using *Mathematica*. Hence it is the main goal of this work to formulate a systematic approach to the evaluation of those nonzero eigenvalues λ_k without resorting to the help of any computer programs. We thus plan to give a detailed account of how one can solve the eigenvalue–eigenvector problem for the discrete number operator $\mathcal{N}^{(\mathbf{5})}$ by "using the power method via a bridge, called deflation, which delivers a square matrix, of size 1 smaller" than $\mathcal{N}^{(\mathbf{5})}$, where we have quoted from [6]. The basic strategy of our approach in this work is to combine the use of the algebraic properties of the lowering and raising difference operators $\mathbf{b}_5$ and $\mathbf{b}_5^{\intercal}$, with the repeated application of the matrix deflation method.

It remains only to add that the limited aim of this work is to restrict our attention to the 5D DFT and the motivation for selecting this special dimension $N = 5$ of the general discrete Fourier transform $\mathbf{\Phi}^{(\mathbf{N})}$ is twofold [4]. First, this dimension is large enough to contain a multiple eigenvalue and therefore one has to handle the same degeneracy problem as in the more general case. Second, this dimension is small enough in order to have calculational advantages that appear in the process of resolving the eigenvalue–eigenvector problem for the discrete number operator $\mathcal{N}^{(\mathbf{5})}$. We hope that this study will deepen our understanding of the case with an arbitrary ND discrete Fourier transform and help us to provide some rigorous proofs, still needed for general values of N.

2 Deflated Matrices: First Step

It is the main goal of this work to evaluate the eigenvalues of the 5×5 symmetric matrix

$$M_5 \,=\, c^{-2}\mathcal{N}^{(\mathbf{5})} \,=\, \begin{bmatrix} 2 & s_4 & -1 & -1 & s_4 \\ s_4 & 4-c_2 & c_1 s_2 & -1 & -1 \\ -1 & c_1 s_2 & 4-c_1 & 2s_2 & -1 \\ -1 & -1 & 2s_2 & 4-c_1 & c_1 s_2 \\ s_4 & -1 & -1 & c_1 s_2 & 4-c_2 \end{bmatrix} . \tag{2.1}$$

As a product of two non-invertible (singular) 5×5 matrices $\mathbf{b_5}$ and $\mathbf{b}_5^{\intercal}$, the matrix M_5 is also singular. This means that the matrix M_5 has at least one zero eigenvalue $\nu_0 = \lambda_0/c^2 = 0$. But it is not hard to verify that the rank of the matrix M_5 is 4, therefore four remaining nonzero eigenvalues $\nu_k = \lambda_k/c^2, 1 \le k \le 4$, of the matrix M_5 are distinct. The 5-column eigenvector $\vec{f}_0$, associated with zero eigenvalue $\nu_0 = 0$, is defined by (1.5). An additional 4 eigenvectors and eigenvalues can be explicitly constructed using the *matrix deflation method*, which delivers a square matrix, of size 1 smaller than the M_5, whose eigenvalues are the remaining eigenvalues of the matrix M_5 [6]. This can be achieved in the following way.

As a consequence of the fact that the rank of the matrix M_5 is 4, the first row (or equivalently the first column, since the matrix M_5 is symmetric) is a linear combination of the four other rows.

Lemma 2.1. *The first row of the matrix M_5 is a linear combination of the four other rows,*

$$I\,row = \alpha\, II\,row + \beta\, III\,row + \gamma\, IV\,row + \delta\, V\,row, \tag{2.2}$$

where the coefficients α, β, γ and δ can be expressed in terms of the components of the lowest eigenvector of the matrix M_5, corresponding to the zero eigenvalue, as

$$\alpha \,=\, \delta \,=\, -\,\frac{x_1}{x_0}\,, \qquad \beta \,=\, \gamma \,=\, -\,\frac{1}{x_0}\,, \tag{2.3}$$

and $x_0 = s_1 - 2c_2$ and $x_1 = 1 + s_2$ are the same as in (1.5).

Proof. The identity (2.2) evidently breaks up into five equations

$$\begin{aligned} 2 &= \alpha\, s_4 \,-\beta\, -\gamma\, + \delta s_4, \\ s_4 &= \alpha(4-c_2)\, + \beta c_1 s_2\, -\gamma\, -\delta, \\ -1 &= \alpha\, c_1 s_2\, + \beta(4-c_1)\, + \gamma 2 s_2 -\, \delta, \\ -1 &= -\,\alpha\, + \beta 2 s_2 + \gamma(4-c_1)\, + \delta c_1 s_2, \\ s_4 &= -\,\alpha\, -\beta\, + \gamma c_1 s_2\, + \delta(4-c_2)\,, \end{aligned} \tag{2.4}$$

for finding the coefficients α, β, γ and δ. From the third and fourth lines in (2.4) it follows that $\gamma = \beta$ and $\delta = \alpha$, therefore the second and fifth lines in (2.4) represent the same equation. The system (2.4) thus reduces to a system of three equations namely

$$\begin{aligned} 1 &= \alpha\, s_4\, -\beta\,, \\ s_4 &= \alpha\,(3-c_2)\, + \beta\,(c_1 s_2 - 1)\,, \\ -1 &= \alpha\,(c_1 s_2 - 1)\, + \beta\,(4 - c_1 + 2 s_2)\,, \end{aligned} \tag{2.5}$$

the second and third equations of which are linearly dependent. So substituting now $\beta = -(\alpha\, s_1 + 1)$ from the first equation in (2.5) into the second one, one finally finds that

$$\alpha \,=\, \frac{1+s_2}{2c_2 - s_1} \,=\, -\,c_1^4\,(1+s_1)\,, \quad \beta \,=\, c_1^3\,(s_2 - 2)\,. \tag{2.6}$$

Finally, from (2.6) it is now evident that the relations $\alpha = -\,\dfrac{x_1}{x_0}$, $\beta = -\,\dfrac{1}{x_0}$ do hold. □

Remark 2.2. It is worth noting at this point that the use of the link between the coefficients α, β, γ and δ of the linear combination (2.2) and the components of the lowest eigenvector of the matrix M_5, corresponding to the zero eigenvalue, is a crucial element of the matrix deflation method, which we repeatedly employ in this work (for a detailed discussion of this point see Problems 8 and 9, page 63, in [6]).

Let us partition now the matrix M_5 as

$$M_5 \;=\; \begin{bmatrix} a_{11} & a_4 \\ a_4^{\intercal} & M_4 \end{bmatrix}, \tag{2.7}$$

where $a_{11} = 2$, $a_4 = -(s_1, 1, 1, s_1)$ is a 4-row vector, $a_4^{\intercal} = -(s_1, 1, 1, s_1)^{\intercal}$ is a 4-column vector, $s_1 = -\, s_4$ and M_4 is a 4×4 submatrix of the matrix M_5, which is obtained from M_5 by deleting its first row and first column. Let us consider also an invertible matrix S_5 and its inverse matrix S_5^{-1} of the form

$$S_5 \;=\; \begin{bmatrix} 1 & b \\ 0_4^{\intercal} & I_4 \end{bmatrix}, \qquad S_5^{-1} \;=\; \begin{bmatrix} 1 & -\,b \\ 0_4^{\intercal} & I_4 \end{bmatrix}, \tag{2.8}$$

where b is a 4-row vector with components (b_1, b_2, b_3, b_4), $0_4^{\intercal}$ is the zero 4-column vector and I_4 is the identity 4×4 matrix. Then the matrix

$$M_5^{'} := S_5^{-1}\, M_5\, S_5 \;=\; \begin{bmatrix} a_{11} - b\, a_4^{\intercal} & (a_{11} - b\, a_4^{\intercal})\, b + a_4 - b\, M_4 \\ a_4^{\intercal} & M_4 + a_4^{\intercal}\, b \end{bmatrix} \tag{2.9}$$

is similar to the matrix M_5 (denoted $M_5^{'} \sim M_5$) via the transforming (similarity) matrix S_5. The key idea of the deflation method is to choose the similarity matrix S_5 in such a way that only zero entries shall appear in the first row of the matrix $M_5^{'}$. From (2.9) it is evident that this can be achieved if one defines the 4-row vector b in (2.8) in such a way that two identities

$$a_{11} \;=\; b\, a_4^{\intercal}\,, \qquad a_4 \;=\; b\, M_4\,, \tag{2.10}$$

are valid. These two identities for defining components of the 4-row vector b, written in extended form, represent a system of five equations, which are identical to equations (2.4) for the coefficients of the linear combination (2.2). Therefore

$$b \;=\; (\alpha, \beta, \beta, \alpha) \;\equiv\; \beta\, (x_1, 1, 1, x_1) \tag{2.11}$$

and the relation (2.9) simplifies to

$$M_5^{'} = S_5^{-1}\, M_5\, S_5 \;=\; \begin{bmatrix} 0 & 0_4 \\ a_4^{\intercal} & M_4^{'} \end{bmatrix}, \tag{2.12}$$

where the extended form of the 4×4 matrix $M_4^{'} := M_4 + a_4^{\intercal}\, b$ is

$$M_4^{'} \;=\; \begin{bmatrix} 6 - \alpha & (1 + \alpha) s_1 & -2\, c_2\, \beta & \beta \\ c_1 s_2 - \alpha & 4 + \alpha & 2 s_2 - \beta & -(\alpha + 1) \\ -(\alpha + 1) & 2 s_2 - \beta & 4 + \alpha & c_1 s_2 - \alpha \\ \beta & -2\, c_2\, \beta & (1 + \alpha) s_1 & 6 - \alpha \end{bmatrix}. \tag{2.13}$$

Since similar matrices must have the same trace [9], one readily verifies that in the case of similar matrices $M_5^{'}$ and M_5,

$$Tr\,M_5^{'} \,=\, Tr\,M_5 \,=\, Tr\,M_4^{'} \,=\, 20\,. \tag{2.14}$$

Another point to observe is that since the matrix $M_5^{'}$ is deflated one row, the dimension of the row space of this square matrix is evidently 4, so that the dimension of its column space should be the same [8]. To make that explicit, we employ the fact that the first column of the matrix $M_5^{'}$ represents a linear combination of the four other columns, that is,

$$I\,col. = u\,II\,col. + v\,III\,col. + w\,IV\,col. + z\,V\,col. \tag{2.15}$$

Written in the extended form, the identity (2.15) breaks up into four equations

$$\begin{aligned} s_4 \,&=\, (6-\alpha)u + (1+\alpha)s_1 v - 2c_2\beta w + \beta z\,,\\ -1 \,&=\, (c_1 s_2 - \alpha)u + (4+\alpha)v + (2s_2 - \beta)w - (\alpha+1)z\,,\\ -1 \,&=\, -(\alpha+1)u + (2s_2-\beta)v + (4+\alpha)w + (c_1 s_2 - \alpha)z\,,\\ s_4 \,&=\, \beta u - 2c_2\beta v + (1+\alpha)s_1 w + (6-\alpha)z\,, \end{aligned} \tag{2.16}$$

for finding the coefficients u, v, w and z. The first equation in (2.16) minus the fourth one, and then the second equation minus the third one yield two homogeneous equations for the two combinations $u-z$ and $v-w$ of the form

$$\begin{aligned} (6+c_1)\,(u-z) \,+\, (c_1 s_2 + 1)(v-w) \,&=\, 0\,,\\ (c_1 s_2 + 1)\,(u-z) \,+\, (4 - c_1 - 2s_2)\,(v-w) \,&=\, 0\,. \end{aligned} \tag{2.17}$$

This system of two homogeneous equations is not compatible, so that it has only trivial solutions $u-z=0$ and $v-w=0$. Thus the coefficients $z=u$ and $w=v$, and equations (2.16) reduce to only two linearly independent equations

$$\begin{aligned} s_4 \,&=\, (5+2\beta-c_2)\,u \,+\, (c_1 s_2 + 1 - 4c_2\beta)\,v\,,\\ -1 \,&=\, (c_1 s_2 - 2\alpha - 1)\,u \,+\, (4 - c_1 + 2s_2 - 2\beta)\,v\,, \end{aligned} \tag{2.18}$$

for the coefficients u and v. The first equation in (2.18) minus the second one, multiplied by the factor s_1, reveals that the coefficients u and v are interrelated, $u = x_1 v$, where $x_1 = 1 + s_2$ (recall that in the case of the similarity transformation (2.9) via the similarity matrix S_5 the coefficients α and β are connected in the same way, $\alpha = x_1\beta$). Substituting the relation $u = x_1 v$ back into the first equation in (2.18), one finally gets that

$$v \,=\, -\Big[c_1^2 s_2 (6s_1 \,+\, 5)\Big]^{-1}. \tag{2.19}$$

Having found all the coefficients of the linear combination (2.15) explicitly,

$$(u,v,w,z) \,=\, (x_1 v, v, v, x_1 v) \,=\, (x_1, 1, 1, x_1)\,v\,, \tag{2.20}$$

where v is given by (2.19), we are now in a position to consider the similarity transformation of the matrix $M_5^{'}$ via the similarity matrix

$$T_5 = \begin{bmatrix} 1 & 0_4 \\ -c^{\mathsf{T}} & I_4 \end{bmatrix} . \tag{2.21}$$

Then the matrix

$$M_5^{''} := T_5^{-1} M_5^{'} T_5 = \begin{bmatrix} 1 & 0_4 \\ c^{\mathsf{T}} & I_4 \end{bmatrix} \begin{bmatrix} 0 & 0_4 \\ a_4^{\mathsf{T}} & M_4^{'} \end{bmatrix} \begin{bmatrix} 1 & 0_4 \\ -c^{\mathsf{T}} & I_4 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 0_4 \\ a_4^{\mathsf{T}} - M_4^{'} c^{\mathsf{T}} & M_4^{'} \end{bmatrix} = \begin{bmatrix} 0 & 0_4 \\ 0_4^{\mathsf{T}} & M_4^{'} \end{bmatrix}, \tag{2.22}$$

is similar to the matrix $M_5^{'}$, $M_5^{''} \sim M_5^{'}$, provided that 4-column vector c^{T} in (2.21) is chosen in such a way that the identity $a_4^{\mathsf{T}} = M_4^{'} c^{\mathsf{T}}$ holds. But the identity $a_4^{\mathsf{T}} = M_4^{'} c^{\mathsf{T}}$, regarded as a system of 4 defining inhomogeneous equations for 4 unknown components of the vector c^{T}, does coincide with the system (2.16) for finding the coefficients u, v, w and z of the linear combination (2.15). Therefore if one chooses the 4-column vector c^{T} as

$$c^{\mathsf{T}} = v\,(x_1, 1, 1, x_1)^{\mathsf{T}} , \tag{2.23}$$

then the identity $a_4^{\mathsf{T}} = M_4^{'} c^{\mathsf{T}}$ is valid and $M_5^{''} \sim M_5^{'}$. Moreover, since similarity is known to be transitive, that is, $A \sim B$ and $B \sim C$ imply $A \sim C$, then the matrix $M_5^{''}$ is similar to the matrix M_5 via the similarity matrix $S_5 T_5$,

$$M_5^{''} \sim M_5 . \tag{2.24}$$

It remains but to recall that a similarity transformation of a $n \times n$ real matrix corresponds to representation of a linear transformation on $\mathbb{R}^n$ in another basis (see, for example, page 44 in [6]). Thus the explicit form of the matrix $M_5^{''}$ in (2.22) tells us how the initial matrix M_5 looks in the basis, which represents a direct sum of the null space of the matrix M_5, associated with the zero eigenvalue $\nu_0 = 0$ of the matrix M_5, and its orthogonal complement $span\{\vec{f}_1, \vec{f}_2, \vec{f}_3, \vec{f}_4\}$, spanned by the yet unknown eigenvectors $\vec{f}_k$, $1 \leq k \leq 4$, of the matrix M_5, associated with four remaining distinct eigenvalues ν_k of the M_5.

Remark 2.3. It is to be remarked at this point that the deflated submatrix $M_4^{'}$ of the matrix $M_5^{''}$ actually first appears as a result of the similarity transformation (2.12) of the initial matrix M_5 via the similarity transform S_5. So the next similarity transformation (2.22) of the matrix $M_5^{'}$ simply nullifies all the nonzero entries of the first column of the matrix $M_5^{'}$, without changing its submatrix $M_4^{'}$. This means that to find the deflated form $M_4^{'}$ of the matrix M_5 it is just sufficient to employ the similarity transformation (2.12) and then to interpret it as the first step towards to the desired *triangularization* of the matrix M_5.[1] This observation will be essentially used in the next section, where we are going to implement the same approach for solving the eigenvalue problem for the matrix $M_4^{'}$, one size smaller than the initial matrix M_5.

3 Deflated Matrices: Second Step

In order to employ the same matrix deflating technique for finding the eigenvalues of the submatrix $M_4^{'}$, one needs to know explicitly at least one of its eigenvalues $\nu_k,\ 1 \leq k \leq 4$ [6]. It is important to realize that the significant feature of our approach is that from the outset we restrict ourselves to finding those eigenvectors $\vec{f_k}$ and associated eigenvalues $\nu_k,\ 0 \leq k \leq 4$, of the initial matrix M_5, which are successively ordered according to the formula (1.7). This means that the first of the four eigenvalues ν_k of the matrix $M_4^{'}$ can be found in the following way. From (1.7) it follows at once that

$$\vec{f_1} \,=\, \lambda_1^{-1/2}\,\mathbf{b}_5^{\intercal}\,\vec{f_0} \,=\, \frac{2s_1}{d_0\sqrt{\nu_1}}\,(0, x_1, c_1, -\,c_1, -x_1)^{\intercal}\,, \tag{3.1}$$

where the eigenvalue $\lambda_1 = c^2\nu_1$ of the discrete number operator $\mathcal{N}^{(\mathbf{5})}$ is yet to be found. To define ν_1, one evaluates then the action of the matrix M_5 on the eigenvector $\vec{f_1}$ and readily confirms that

$$M_5\,\vec{f_1} \,=\, \nu_1\,\vec{f_1}, \qquad \nu_1 = s_2(s_2 + c_1) + 5\,, \tag{3.2}$$

upon using the relations $c_1c_2 = -1$, ${c_1}^2 = 1 - c_1$ and ${s_2}^2 = 2 - c_1$.

Another point to note is that the key assumption about the initial matrix M_5 was that it is a singular one. This restriction permitted us to express the first row of the matrix M_5 as the linear combination (2.2) of the four other rows of the M_5 and find then the explicit form of the similarity matrix S_5. Within the matrix deflating method, the complication, introduced by the fact that the $M_4^{'}$ is an invertible matrix, is commonly met by the replacement

$$M_4^{'} \,\Rightarrow\, P_4 := \,M_4^{'} \,-\, \nu_1 I_4, \tag{3.3}$$

where ν_1 is defined as in (3.2) and therefore the 4×4 singular matrix P_4 has the following extended form:

$$P_4 \,=\, \begin{bmatrix} -\,c_2\alpha & (1+\alpha)s_1 & -2\,c_2\,\beta & \beta \\ c_1s_2 - \alpha & (2-c_2)\alpha - 2 & 2s_2 - \beta & -(\alpha+1) \\ -(\alpha+1) & 2s_2 - \beta & (2-c_2)\alpha - 2 & c_1s_2 - \alpha \\ \beta & -2\,c_2\,\beta & (1+\alpha)s_1 & -\,c_2\alpha \end{bmatrix}. \tag{3.4}$$

It becomes now evident that one may use the same pattern, as in the previous section, as follows.

[1] Recall that a square matrix A is said to be *triangularizable* if there is an invertible matrix S such that $S^{-1}A\,S = T$ is upper triangular. Note that the diagonal entries of the triangular matrix T will necessarily be the eigenvalues of A. Every complex square matrix is triangularizable (see page 271 in [8]).

Lemma 3.1. *The first row of the singular matrix P_4 is a linear combination of the three other rows,*

$$I\,row = u\,II\,row + v\,III\,row + w\,IV\,row\,, \tag{3.5}$$

where the coefficients u, v and w can be expressed in terms of the components of the eigenvector $\vec{f_1}$ of the matrix M_5, which is associated with the eigenvalue ν_1, as

$$v \;=\; -\,u \;=\; \frac{c_1}{x_1} \;=\; s_1 + c_2, \quad w \;=\; 1\,, \tag{3.6}$$

where $x_1 = 1 + s_2$.

Proof. Written in the extended form, the identity (3.5) breaks up into four equations

$$\begin{aligned} -\,c_2\alpha \;&=\; (c_1s_2 - \alpha)\,u \,-\, (\alpha+1)\,v \,+\, \beta\,w\,, \\ (1+\alpha)s_1 \;&=\; [(2-c_2)\alpha - 2]\,u \,+ (2s_2 - \beta)\,v \,-\, 2c_2\beta\,w\,, \\ -2c_2\beta \;&=\; (2s_2 - \beta)\,u \,+\, [(2-c_2)\alpha - 2]\,v \,+\, (1+\alpha)s_1\,w\,, \\ \beta \;&=\; -(\alpha+1)\,u \,+\, (c_1s_2 - \alpha)\,v \,-\, c_2\alpha\,w\,, \end{aligned} \tag{3.7}$$

for the coefficients u, v and w of the linear combination (3.5). To find them explicitly, let us sum first the first and fourth equations in (3.7). This gives

$$A(1-w) \;=\; B(u+v), \qquad A = \beta - c_2\alpha,\; B = c_1s_2 - 2\alpha - 1. \tag{3.8}$$

Similarly, the sum of the second and third equations can be written as

$$C(1-w) \;=\; D(u+v), \qquad C = (1+\alpha-\beta)s_1 - 1,\; D = 2(s_2-1)+(2-c_2)\alpha-\beta. \tag{3.9}$$

Let us assume now that the unknown w in equations (3.8) and (3.9) is not equal to 1, and hence the sum $u+v$ is not equal to zero. Then from (3.8) and (3.9) it follows that $B/A = D/C$, or, equivalently, $BC = AD$. But it is not hard to verify directly that $BC \neq AD$. Consequently, $w = 1$ and $u + v = 0$. Finally, substituting $w = 1$ and $u = -v$ into the first equation in the system (3.7), one concludes that the coefficients u, v and w are uniquely defined as in (3.6). It remains only to add that the link between the coefficients u, v and w and the components of the eigenvector $\vec{f_1}$ of the matrix M_5, associated with the eigenvalue ν_1 of the M_5, becomes transparent when one compares (3.6) with the definition (3.1) of the eigenvector $\vec{f_1}$ (*cf* lemma 2.1). □

Having found the coefficients of the linear combination (3.5), we consider now the similarity transformation of the matrix P_4 via the 4×4 similarity matrix (*cf* (2.8))

$$S_4 \;=\; \begin{bmatrix} 1 & b \\ 0_3^{\intercal} & I_3 \end{bmatrix}, \tag{3.10}$$

where b is a 3-row vector with components (b_1, b_2, b_3). To this end let us partition first the matrix P_4 as

$$P_4 \,=\, \begin{bmatrix} a_{11} & a_3 \\ d_3^{\intercal} & P_3 \end{bmatrix}, \tag{3.11}$$

where $a_{11} \,=\, -\,c_2\alpha$, $a_3 \,=\, [(1+\alpha)s_1, -2c_2\beta, \beta]$ is a 3-row vector, $d_3^{\intercal} \,=\, [c_1 s_2 - \alpha, -\,(\alpha+1), \beta]^{\intercal}$ is a 3-column vector, and P_3 is a 3×3 submatrix of the matrix P_4, which is obtained from P_4 by deleting its first row and first column. Then the matrix

$$P_4^{'} := S_4^{-1}\, P_4\, S_4 \,=\, \begin{bmatrix} a_{11} - b\, d_3^{\intercal} & (a_{11} - b\, d_3^{\intercal})\, b + a_3 - b\, P_3 \\ d_3^{\intercal} & d_3^{\intercal}\, b + P_3 \end{bmatrix} \tag{3.12}$$

is similar to the matrix P_4, $P_4^{'} \sim P_4$, via the similarity matrix S_4. As in the case of (2.8), the similarity matrix S_4 is defined in such a way that only zero entries will appear in the first row of the matrix $P_4^{'}$. Therefore from (3.12) it is evident that this can be achieved only if one defines the 3-row vector b in (3.10) in such a way that two identities

$$a_{11} \,=\, b\, d_3^{\intercal}\,, \qquad a_3 \,=\, b\, P_3\,, \tag{3.13}$$

are valid. These two identities for defining components of the 3-row vector b, written in extended form, represent a system of four equations, which are identical to equations (3.7) for the coefficients of the linear combination (3.5). Therefore

$$b \,=\, (u, v, w) \,=\, \frac{1}{x_1}\,(-\,c_1, c_1, x_1) \tag{3.14}$$

and the relation (3.12) simplifies to

$$P_4^{'} = S_4^{-1}\, P_4\, S_4 \,=\, \begin{bmatrix} 0 & 0_3 \\ d_3^{\intercal} & d_3^{\intercal}\, b + P_3 \end{bmatrix}, \tag{3.15}$$

where the matrix $P_4^{'}$ has only zero entries in the first row. Taking into account that the matrix $d_3^{\intercal} b$ is equal to

$$d_3^{\intercal} b \,=\, \begin{bmatrix} (\alpha - c_1 s_2)(s_1 + c_2) & (c_1 s_2 - \alpha)(s_1 + c_2) & c_1 s_2 - \alpha \\ (\alpha+1)(s_1+c_2) & -(\alpha+1)(s_1+c_2) & -(\alpha+1) \\ 1 - 3c_2\beta & 3c_2\beta - 1 & \beta \end{bmatrix} \tag{3.16}$$

by definition, one readily verifies that the extended form of the 3×3 matrix $P_3 + d_3^{\intercal} b$ is

$$P_3 + d_3^{\intercal} b \,=\, \begin{bmatrix} c_1 - 5 - 2c_1\alpha & c_1 s_2 + 4 - 3\,c_1 & c_1 s_2 - 2\alpha - 1 \\ 2s_1 + c_2 - 2\,c_1\alpha & c_2 s_1 - c_1 & c_1 s_2 - 2\alpha - 1 \\ 1 - 5c_2\beta & c_1 - \alpha & c_1(\alpha - 1) \end{bmatrix}. \tag{3.17}$$

Note that since the dimension of the row space of the matrix $P_4^{'}$ is evidently 3, the dimension of its column space should be the same. Quite similar to the case of the matrix $M_5^{'}$ in the previous section, this can be made explicit by using one more similarity transformation of the $P_4^{'}$, this time via a similarity matrix of the type (2.21). But as it has been emphasized above (see the remark (2.3)), the submatrix $P_3 + d_3^{\intercal} b$ of the matrix $P_4^{'}$ will not be affected by this similarity transformation and therefore one may actually skip over it. This can be elucidated in the following way.

Let us consider the similarity 5×5 matrix Σ_5 and its inverse Σ_5^{-1} of the form

$$\Sigma_5 = \begin{bmatrix} 1 & 0_4 \\ 0_4^{\intercal} & S_4 \end{bmatrix}, \qquad \Sigma_5^{-1} = \begin{bmatrix} 1 & 0_4 \\ 0_4^{\intercal} & S_4^{-1} \end{bmatrix}, \tag{3.18}$$

where the 4×4 matrix S_4 is defined as in (3.10). Then the similarity transformation of the matrix $M_5^{'}$, defined by (2.12), via the similarity matrix Σ_5 can be evaluated as

$$M_5^{(2)} := \Sigma_5^{-1} M_5^{'} \Sigma_5 =$$

$$= \begin{bmatrix} 1 & 0_4 \\ 0_4^{\intercal} & S_4^{-1} \end{bmatrix} \begin{bmatrix} 0 & 0_4 \\ a_4^{\intercal} & M_4^{'} \end{bmatrix} \begin{bmatrix} 1 & 0_4 \\ 0_4^{\intercal} & S_4 \end{bmatrix} = \begin{bmatrix} 0 & 0_4 \\ S_4^{-1} a_4^{\intercal} & S_4^{-1} M_4^{'} S_4 \end{bmatrix}. \tag{3.19}$$

Since $M_4^{'} = P_4 + \nu_1 I_4$ by (3.3), the submatrix $S_4^{-1} M_4^{'} S_4$ in (3.18) reduces to

$$S_4^{-1} M_4^{'} S_4 = S_4^{-1} \Big(P_4 + \nu_1 I_4 \Big) S_4 = S_4^{-1} P_4 S_4 + \nu_1 I_4 = P_4^{'} + \nu_1 I_4 \,, \tag{3.20}$$

where $P_4^{'}$ is explicitly given in (3.15). Therefore the matrix $M_5^{(2)}$ can be finally partitioned as

$$M_5^{(2)} = \begin{bmatrix} 0 & 0_4 \\ S_4^{-1} a_4^{\intercal} & P_4^{'} + \nu_1 I_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0_3 \\ * & \nu_1 & 0_3 \\ *_3^{\intercal} & d_3^{\intercal} & P_3^{'} \end{bmatrix}, \tag{3.21}$$

where $* = \Big(S_4^{-1} a_4^{\intercal} \Big)_1$, $*_3^{\intercal} = \Big[\Big(S_4^{-1} a_4^{\intercal} \Big)_2, \Big(S_4^{-1} a_4^{\intercal} \Big)_3, \Big(S_4^{-1} a_4^{\intercal} \Big)_4 \Big]^{\intercal}$ is a 3-column vector, $a_4^{\intercal}$ is defined as in (2.7) and [2]

$$P_3^{'} := d_3^{\intercal}\, b + P_3 + \nu_1 I_3 \,. \tag{3.22}$$

The extended form of the 3×3 matrix $P_3^{'}$ is

$$P_3^{'} = \begin{bmatrix} 2 + c_1(s_2 - 2\alpha) & c_1 s_2 + 4 - 3\, c_1 & c_1 s_2 - 2\alpha - 1 \\ 2(s_1 - c_1\alpha) + c_2 & 7 - 2s_2 - 2c_1 & c_1 s_2 - 2\alpha - 1 \\ 1 - 5c_2\beta & c_1 - \alpha & 2(3 - \alpha) - c_1 \end{bmatrix}. \tag{3.23}$$

After the second successive step in deflating the initial matrix M_5 one thus arrives at the matrix $M_5^{(2)}$, which has only zero entries above the diagonal on the first two rows. Evidently, this can be interpreted as the second step towards to the desired form of triangularizing the matrix M_5.

[2]Note that we use the same symbol $*$ as in book [5] to denote those matrix entries in (3.21), which are irrelevant to the matter being discussed at the moment.

4 Deflated Matrices: Third Step

To proceed further with the task under consideration we find now, with the aid of (1.7), that the eigenvector $\vec{f_2}$ has the form

$$\vec{f_2} \,=\, \lambda_2^{-1/2}\, \mathbf{b}_5^{\intercal}\, \vec{f_1} \,=\, \kappa\, c_2 s_1 x_1\, (-2c_1, 1, 1, 1, 1)^{\intercal}\,, \tag{4.1}$$

where $\kappa = -2/d_0\sqrt{\nu_1\nu_2}$ and ν_2 is yet to be found. One evaluates then the action of the matrix M_5 on the eigenvector $\vec{f_2}$ and confirms that

$$M_5\, \vec{f_2} \,=\, \nu_2\, \vec{f_2}, \qquad \nu_2 = s_1(s_1 - c_2)\,. \tag{4.2}$$

To deflate the invertible matrix $P_3^{'}$ from (3.23), we introduce the auxiliary singular matrix $Q_3 := P_3^{'} - \nu_2 I_3$. One may then employ the same technique as in the two previous sections, in order to find a square matrix of size 1 smaller than Q_3. But it turns out that this particular case with Q_3 is even simpler to deal with than those, considered in the preceding sections, and the reason is the simplicity of the components of the eigenvector $\vec{f_2}$. This can be detailed in the following way.

From the (3.23) and (4.2) it follows that the extended form of the 3×3 matrix Q_3 is

$$Q_3 \,=\, \begin{bmatrix} c_2 - 2c_1(s_1+\alpha) & c_1 s_2 + 4 - 3\,c_1 & c_1 s_2 - 2\alpha - 1 \\ 2s_1 + c_2 - 2c_1\alpha & c_2 s_1 - 2s_2 + 4 - 3c_1 & c_1 s_2 - 2\alpha - 1 \\ 1 - 5c_2\beta & c_1 - \alpha & 2\beta + c_2 s_1 + 3 \end{bmatrix}. \tag{4.3}$$

Since the matrix Q_3 is singular, its columns are not linearly independent. In particular, one readily verifies that

$$I\,col. + II\,col. + III\,col. \,=\, 0\,. \tag{4.4}$$

Written in the extended form, the identity (4.4) breaks up into three constraints namely

$$\sum_{l=1}^{3} (Q_3)_{kl} \,=\, 0\,, \quad 1 \le k \le 3\,, \tag{4.5}$$

which interrelate all three entries $(Q_3)_{kl}$, $1 \le k, l \le 3$, of the matrix Q_3 from the each row. Note also that this particular form of the identity (4.4) is of the matrix M_5, associated with the eigenvalue ν_2, is of the type (4.1). The same pattern thus recurs here as in the cases of the identities (2.2) and (3.5), upon which we have already commented in the remark 2.2.

As a direct consequence of the identity (4.4), one may consider now the similarity transformation of the matrix $Q_3^{\intercal}$, which is similar to the matrix Q_3 and has the same eigenvalues as Q_3 (see, for example, page 135 in [6]). So let us partition $Q_3^{\intercal}$ as

$$Q_3^{\intercal} \,=\, \begin{bmatrix} a_{11} & a_2 \\ d_2^{\intercal} & Q_2 \end{bmatrix}, \tag{4.6}$$

where $a_{11} = (Q_3)_{11}$, $a_2 = [(Q_3)_{21}, (Q_3)_{31}]$ is a 2-row vector, $d_2^\intercal = [(Q_3)_{12}, (Q_3)_{13}]^\intercal$ is a 2-column vector, and Q_2 is a 2×2 submatrix of the matrix $Q_3^\intercal$, which is obtained from $Q_3^\intercal$ by deleting its first row and first column. So the similarity transformation of the matrix $Q_3^\intercal$ via the similarity matrix of the form

$$S_3 = \begin{bmatrix} 1 & b \\ 0_2^\intercal & I_2 \end{bmatrix} \tag{4.7}$$

can be represented as

$$\begin{aligned} \left(Q_3^\intercal\right)' := S_3^{-1}\, Q_3^\intercal\, S_3 &= \begin{bmatrix} 1 & -b \\ 0_2^\intercal & I_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_2 \\ d_2^\intercal & Q_2 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0_2^\intercal & I_2 \end{bmatrix} = \\ &= \begin{bmatrix} a_{11} - b\, d_2^\intercal & (a_{11} - b\, d_2^\intercal)\, b + a_2 - b\, Q_2 \\ d_2^\intercal & Q_2 + d_2^\intercal b \end{bmatrix}, \end{aligned} \tag{4.8}$$

where $b = (b_1, b_2)$ is a 2-column vector. From (4.8) it becomes then evident that the matrix $\left(Q_3^\intercal\right)'$ may be reduced to the simpler form

$$\left(Q_3^\intercal\right)' = \begin{bmatrix} 0 & 0_2 \\ d_2^\intercal & Q_2 + d_2^\intercal b \end{bmatrix} \tag{4.9}$$

with only zero entries in the first row, by choosing the 2-row vector b in such a way that the two identities

$$a_{11} = b\, d_2^\intercal\,, \qquad a_2 = b\, Q_2\,, \tag{4.10}$$

hold. Indeed, if the components b_1 and b_2 of the vector b are such that $b_1 = b_2 = -1$, then

$$\begin{aligned} a_{11} - \ b\, d_2^\intercal &= (Q_3)_{11} + (1,1)\, [(Q_3)_{12}, (Q_3)_{13}]^\intercal = \\ &= (Q_3)_{11} + (Q_3)_{12} + (Q_3)_{13} = 0 \end{aligned} \tag{4.11}$$

by the first line in (4.5). Similarly,

$$\begin{aligned} a_2 - b\, Q_2 = \Big((Q_3)_{21}, (Q_3)_{31}\Big) + (1,1) \begin{bmatrix} (Q_3)_{22} & (Q_3)_{32} \\ (Q_3)_{23} & (Q_3)_{33} \end{bmatrix} = \\ = \Big((Q_3)_{21}, (Q_3)_{31}\Big) + \Big((Q_3)_{22} + (Q_3)_{23}, (Q_3)_{32} + (Q_3)_{33}\Big) = (0,0), \end{aligned} \tag{4.12}$$

because of the second and third identities in (4.5).

We are now in a position to evaluate the similarity transformation of the matrix $M_5^{(2)}$ (see (3.21)),

$$M_5^{(3)} := U_5^{-1} M_5^{(2)}\, U_5\,, \tag{4.13}$$

via the transforming matrix U_5, defined as

$$U_5 = \begin{bmatrix} 1 & 0 & 0_3 \\ 0 & 1 & 0_3 \\ 0_3^\intercal & 0_3^\intercal & S_q\, S_3 \end{bmatrix}, \tag{4.14}$$

where S_3 is given in (4.7) and S_q is the matrix that interrelates the similar matrices Q_3 and $Q_3^{\intercal}$, that is, $Q_3^{\intercal} = S_q^{-1} Q_3 S_q$. From (3.21) and (4.14) it follows that thus introduced matrix $M_5^{(3)}$ is equal to

$$M_5^{(3)} = \begin{bmatrix} 1 & 0 & 0_3 \\ 0 & 1 & 0_3 \\ 0_3^{\intercal} & 0_3^{\intercal} & S_3^{-1} S_q^{-1} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0_3 \\ * & \nu_1 & 0_3 \\ *_3^{\intercal} & d_3^{\intercal} & P_3^{'} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0_3 \\ 0 & 1 & 0_3 \\ 0_3^{\intercal} & 0_3^{\intercal} & S_q S_3 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 0 & 0_3 \\ * & \nu_1 & 0_3 \\ *_3^{\intercal} & d_3^{\intercal} & S_3^{-1} S_q^{-1} P_3^{'} S_q S_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0_3 \\ * & \nu_1 & 0_3 \\ *_3^{\intercal} & d_3^{\intercal} & \nu_2 I_3 + \left(Q_3^{\intercal}\right)^{'} \end{bmatrix}, \qquad (4.15)$$

where we have taken into account that $P_3^{'} = Q_3 + \nu_2 I_3$ and $\left(Q_3^{\intercal}\right)^{'} = S_3^{-1} Q_3^{\intercal} S_3$. Finally, substituting (4.9) into the submatrix $\left(Q_3^{\intercal}\right)^{'} + \nu_2 I_3$ in (4.15), one arrives at the desired form of the matrix

$$M_5^{(3)} = \begin{bmatrix} 0 & 0 & 0 & 0_2 \\ * & \nu_1 & 0 & 0_2 \\ * & * & \nu_2 & 0_2 \\ *_2^{\intercal} & *_2^{\intercal} & d_2^{\intercal} & Q_2^{'} \end{bmatrix}, \qquad (4.16)$$

where $\nu_1 = s_2(s_2 + c_1) + 5,\ \nu_2 = s_1(s_1 - c_2)$ and the extended form of the 2×2 upper triangular matrix $Q_2^{'} := Q_2 + d_2^{\intercal} b + \nu_2 I_2$ is

$$Q_2^{'} = \begin{bmatrix} s_1(s_1 + c_2) & -s_2(s_2 + c_1^2) \\ 0 & s_2(s_2 - c_1) + 5 \end{bmatrix}. \qquad (4.17)$$

One thus concludes that the third successive step in triangularizing the initial matrix M_5 results in the matrix $M_5^{(3)}$, which has only zero entries above the main diagonal on the first *three rows*.

5 Deflated Matrices: Fourth Step

To complete the process of triangularizing the initial matrix M_5, we find first from (1.7) and (4.1) that the eigenvector $\vec{f}_3$ has the form

$$\vec{f}_3 = \lambda_3^{-1/2} \mathbf{b}_5^{\intercal} \vec{f}_2 = \kappa^{'} c_2 x_1 s_1^2 (0, 1 - s_2, c_1, -c_1, s_2 - 1)^{\intercal}, \qquad (5.1)$$

where $\kappa^{'} = \kappa / \sqrt{\nu_3}$ and the eigenvalue ν_3 is yet to be found. One evaluates then the action of the matrix M_5 on the eigenvector $\vec{f}_3$ and confirms that

$$M_5 \vec{f}_3 = \nu_3 \vec{f}_3, \qquad \nu_3 = s_1(s_1 + c_2). \qquad (5.2)$$

Note that having defined the eigenvalues ν_1, ν_2 and ν_3, the only remaining eigenvalue ν_4 may be readily found in the following way. The point is that the trace of the initial matrix M_5 is equal to 20 (see (2.1) and recall that $c_1 + c_2 = -1$). Hence from (3.2), (4.2) and (5.2) it follows at once that

$$tr\, M_5 \;=\; \sum_{k=1}^{4} \nu_k \;=\; 20, \qquad \nu_4 = s_2(s_2 - c_1) + 5. \tag{5.3}$$

Needless to say we would have arrived at the same expression for the eigenvalue ν_4 if we had defined first the eigenvector $\vec{f}_4$ via formula $\vec{f}_4 \;=\; \lambda_4^{-1/2} \mathbf{b}_5^{\mathsf{T}}\, \vec{f}_3$ and then checked that $M_5\, \vec{f}_4 = \nu_4\, \vec{f}_4$.

It becomes therefore evident that the two diagonal elements of the upper triangular matrix $Q_2^{'}$ in (4.17), which is under consideration in this section, are just the eigenvalues ν_3 and ν_4, that is, the matrix $Q_2^{'}$ can be written as

$$Q_2^{'} \;=\; \begin{bmatrix} \nu_3 & -\,s_2(s_2 + c_1^2) \\ 0 & \nu_4 \end{bmatrix} . \tag{5.4}$$

Hence it is straightforward to bring the matrix $M_5^{(3)}$ into its final lower triangular form by the similarity transformation with the transforming matrix V_5, partitioned as

$$V_5 \;=\; \begin{bmatrix} I_3 & 0_{32} \\ 0_{23} & \Sigma_q \end{bmatrix} , \tag{5.5}$$

where 0_{23} and 0_{32} are the 2×3 and 3×2 zero matrices, respectively, and the 2×2 matrix Σ_q represents the similarity matrix, interconnecting the matrices $Q_2^{'}$ and its transpose, the lower tridiagonal matrix

$$\left(Q_2^{'}\right)^{\mathsf{T}} \;=\; \begin{bmatrix} \nu_3 & 0 \\ -\,s_2(s_2 + c_1^2) & \nu_4 \end{bmatrix} . \tag{5.6}$$

Indeed, it is not hard to verify now that the similarity transformation of the matrix $M_5^{(3)}$ via the similarity matrix V_5 results in the 5×5 lower tridiagonal matrix

$$M_5^{(4)} = V_5^{-1} M_5^{(3)}\, V_5 \tag{5.7}$$

with diagonal elements $(0, \nu_1, \nu_2, \nu_3, \nu_4)$. Since similarity is transitive, the lower triangular matrix $M_5^{(3)}$ is similar to the initial matrix M_5 and therefore the diagonal entries of $M_5^{(3)}$, described by the single overall formula [4]

$$\nu_k \;=\; (c_1 s_k + c_k)\, s_{2k} + 5(1 - \delta_{k0})\,, \qquad 0 \le k \le 4\,, \tag{5.8}$$

are at the same time the desired eigenvalues the matrix M_5.

To close the last section, it may be worth mentioning a graph that compares the previously defined eigenvalues $\lambda_k = c^2\nu_k$, $0 \le k \le 4$, of the discrete number operator $\mathcal{N}^{(\mathbf{5})}$ with the first 5 eigenvalues of the number operator $\mathbf{N} = \mathbf{a}^\dagger\mathbf{a}$ for the linear harmonic oscillator in quantum mechanics [7], which is considered to be a continuous prototype of the discrete number operator $\mathcal{N}^{(\mathbf{5})}$ [4].

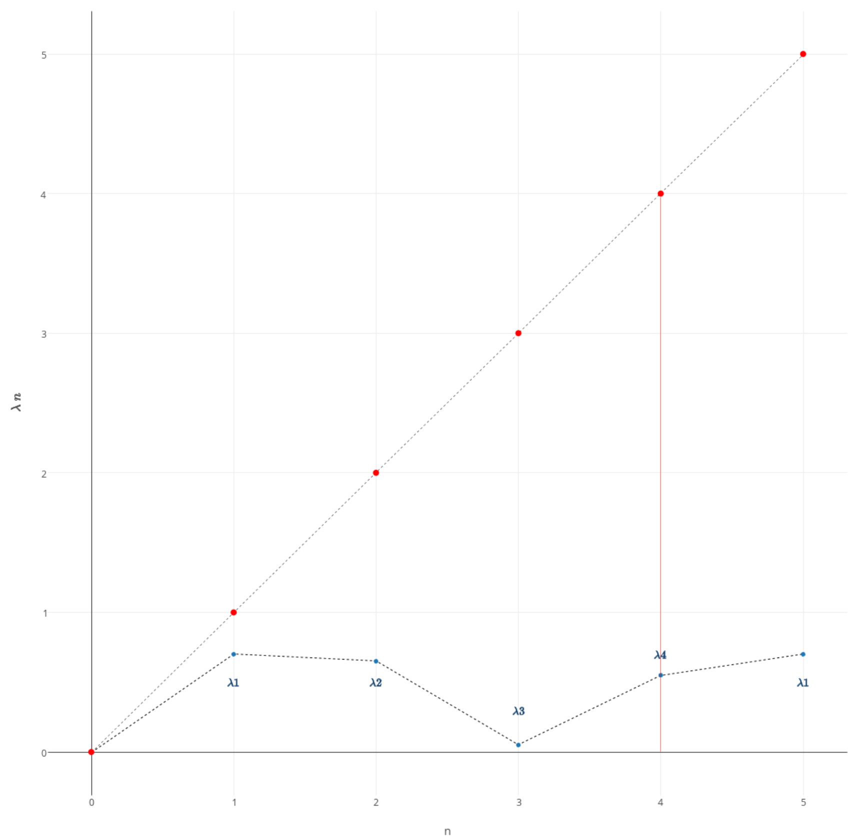

Figure 5.1: The first 5 eigenvalues of the number operator $\mathbf{N} = \mathbf{a}^\dagger\mathbf{a}$ for the linear harmonic oscillator in quantum mechanics vs. the eigenvalues of the discrete number operator $\mathcal{N}^{(\mathbf{5})}$ under study.

This concludes our use of the matrix deflation method for finding the eigenvalues and eigenvectors of the discrete number operator $\mathcal{N}^{(5)}$.

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