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Properties of Solutions to a Discrete Analog of the Bernoulli Equation in the Case of Nonregressivity Using Time Scale Calculus

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Abstract

Using the framework of the time scale calculus, we focus on a discrete analog of the classic Bernoulli differential equation. We describe the derivation of the discrete equation, $y^\sigma = y/[2-(1+\mu p)^\alpha + f\mu y^\alpha]^{1/\alpha}$, and its relationship to the classic Bernoulli differential equation. We then explore the solution behavior that occur as the parameters vary, resulting in three distinct bifurcation diagrams. The analysis of these bifurcation diagrams lead to a complete characterization of solution behavior in the nonregressive case. One case in particular yields periodic solutions and limit cycles which are not exhibited in any of the other cases.

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1 Introduction

There is a rich history of examining discrete analogs of various important differential equations. Many different mechanisms for constructing the corresponding discrete analog of a given differential equation have been used throughout the literature. For example, the well-known logistic recurrence relation $y_{n+1} = ry_n(1 - y_n)$ takes its form from the logistic differential equation $y' = ry(1 - y)$ by replacing y' with y_{n+1} and y with y_n. Another strategy often employed stems from the framework of difference equations. Here, the forward difference operator replaces the derivatives in the differential equation to create a discrete analog of the equation: $y = xy' + f(y') \implies y = x\Delta y + f(\Delta y)$ (see, for example, [8] for this particular instance involving the Clairaut equation).

Sometimes subtle changes to the equation itself in conjunction with difference equation theory is used to preserve certain properties exhibited in the differential equation. For example, the discrete analog of the Riccati differential equation, $z' + q(t) + \frac{z^2}{p(t)} = 0$, is often given as $\Delta z + q(t) + \frac{z^2}{z + p(t-1)} = 0$. By constructing the discrete Riccati equation in this fashion, a number of important properties that the original differential equation enjoys are persevered in the discrete analog. In particular, disconjugacy of the corresponding linear, second-order, self-adjoint difference equation can be determined by examining solutions of this discrete Riccati equation. This parallels similar results found in the differential equations case (see [6, §6.5]).

The above is certainly not an exhaustive list of techniques used to establish a discrete analog of a differential equation, but gives a small hint at the many different strategies that have been used. For this present work, we focus on using the framework of the time scale calculus to derive a discrete analog of the so-called Bernoulli differential equation. With the discrete Bernoulli equation in hand, we will focus on solutions in the nonregressive case. (See [7] for an application dealing with switched linear circuits in the nonregressive case.)

We introduce the core ideas of the time scale calculus in Section 2. We then derive the discrete analog of the Bernoulli differential equation in Section 3. In Section 4, we proceed to outline all underlying assumptions, and the analysis of solution behavior and the main results are found in Section 5. Finally, in Section 6, we examine a particular example which yields periodic solutions and limit cycles.

2 A Brief Introduction to the Time Scale Calculus

In this section we collect the basic definitions and notation of the time scale calculus that are used throughout this work. We will not give any examples or motivation for these time scale results, as that is treated in many thorough introductions already available (see, for example, [1], [4], and [5]).

Definition 2.1. A *time scale*, denoted $\mathbb{T}$, is a nonempty, closed subset of $\mathbb{R}$.

Definition 2.2. Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$, we define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, and the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$. In this definition we use the convention that $\inf \emptyset = \sup(\mathbb{T})$ and $\sup \emptyset = \inf(\mathbb{T})$. For a function $y : \mathbb{T} \to \mathbb{R}$, we use the notation $y^\sigma := y \circ \sigma$.

Definition 2.3. A time scale $\mathbb{T}$ is said to be an *isolated time scale* provided $\sigma(t) > t$ and $\rho(t) < t$ for all $t \in \mathbb{T}$.

Definition 2.4. The *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$.

Definition 2.5. If $\sup(\mathbb{T}) = m$ such that $\rho(m) < m$, then we define $\mathbb{T}^\kappa := \mathbb{T}\backslash\{m\}$; otherwise, define $\mathbb{T}^\kappa := \mathbb{T}$.

Definition 2.6. Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$. If $\sigma(t) > t$, the Δ-*derivative* of f is defined to be

$$f^\Delta(t) := \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} = \frac{f^\sigma(t) - f(t)}{\mu(t)}.$$

Otherwise, we define it to be

$$f^\Delta(t) := \lim_{s \to t} \frac{f(t) - f(s)}{t - s},$$

provided this limit exists.

Theorem 2.7. *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are Δ-differentiable such that $gg^\sigma \neq 0$. Then,*

$$\left(\frac{f}{g}\right)^\Delta = \frac{f^\Delta g - f g^\Delta}{g g^\sigma}.$$

Definition 2.8. The set of *positively regressive* functions on a time scale $\mathbb{T}$ is given by $\mathcal{R}^+ := \{p : \mathbb{T} \to \mathbb{R} | 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}\}$.

1. For $p, q \in \mathcal{R}^+$, we define *circle plus addition*, denoted $\oplus$, and *circle minus*, denoted $\ominus$, as follows:

$$(p \oplus q)(t) := p(t) + q(t) + \mu(t)p(t)q(t) \quad \text{and} \quad (\ominus p)(t) := \frac{-p(t)}{1 + p(t)\mu(t)},$$

2. For $\alpha \in \mathbb{R}$, $f \in \mathcal{R}^+$, we define *circle dot multiplication* by:

$$\alpha \odot f := \begin{cases} \dfrac{(1 + \mu(t)f(t))^\alpha - 1}{\mu(t)}, & \mu(t) > 0, \\ \alpha f(t), & \mu(t) = 0. \end{cases}$$

Theorem 2.9. *Let $\mathbb{T}$ be arbitrary, and assume $\alpha \in \mathbb{R}$. If $\alpha \in \mathbb{N}$, assume $x(t) \neq 0$ for all $t \in \mathbb{T}$. If $\alpha \notin \mathbb{N}$, assume that $x(t)x^\sigma(t) > 0$ for all $t \in \mathbb{T}$. Then,* $\dfrac{(x^\alpha)^\Delta}{x^\alpha} = \alpha \odot \dfrac{x^\Delta}{x}$.

3 Derivation of the Discrete Bernoulli Equation

The idea behind the derivation is to start with a non-zero solution to a first-order linear equation, and apply an appropriate substitution which yields a logistic equation. From there, another substitution is applied which yields the Bernoulli equation. This derivation follows the main techniques presented in [3] and [5, §2.6]; however, in this work we start with a different first-order, linear equation not considered elsewhere.

As discussed in [2], there are two versions of the linear, first-order time scale dynamic equation. Following the lead of Bohner and Peterson in [4] and [5], much of the literature on the time scale calculus cite these two linear equations in the forms:

$$y^{\Delta} = p(t)y + f(t) \quad \text{and} \quad x^{\Delta} = -p(t)x^{\sigma} + f(t). \tag{3.1}$$

Note that when we apply $\mathbb{T} = \mathbb{R}$ in (3.1), we obtain two different differential equations: $y' = p(t)y + f(t)$ and $x' = -p(t)x + f(t)$. Of course, the only difference is the negative in front of the arbitrary coefficient function p, and so the solutions of each equation are related. Here the solutions are given by $y(t) = \exp\left(-\int p(t)dt\right)$ and $x(t) = \exp\left(\int p(t)dt\right)$, respectively. We note that the only difference between the solutions is a negative in the exponential. Certainly, in the first-order linear differential equation case, we are able to easily obtain these exact solutions. Hence, (and rightfully so) we do not make a distinction between these cases when $\mathbb{T} = \mathbb{R}$.

But in the arbitrary time scale case, especially in the nonregressive case (i.e., when $1 + \mu(t)p(t) = 0$ for at least one $t \in \mathbb{T}$), we no longer have this close relationship between the solutions of the two forms of the first-order equation.

Hence, we will focus on linear first-order equations of the form

$$y^{\Delta} + p(t)y = f(t) \quad \text{and} \quad x^{\Delta} + p(t)x^{\sigma} = f(t).$$

Notice the only difference is in the first equation. But, when we apply $\mathbb{T} = \mathbb{R}$ to these equations, we obtain the *same* differential equation: $y' + p(t)y = f(t)$. We can think about both as a generalization of the same differential equation.

The time scale Bernoulli equation derived in [5, §2.6], uses the first-order linear equation $x^{\Delta} + p(t)x^{\sigma} = f(t)$ as its base. We instead derive our Bernoulli equation starting with the first-order linear equation $y^{\Delta} + p(t)y = f(t)$. As discussed above, this linear equation is not the one typically cited in the time scale literature, and so our derivation is distinct from the results obtained in [3] and [5, §2.6].

Remark 3.1. Throughout the derivation, we implicitly assume that each step makes mathematical sense in the context of real-valued solutions: no zeros in the denominator, no even roots of negative numbers, etc. In Section 4, we formalize the specific assumptions that ensure each step of this derivation is valid.

Let $\mathbb{T}$ be a time scale. The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined as in Definition 2.4. We let $p, f : \mathbb{T} \to \mathbb{R}$, $\alpha > 0$. (For clarity, we omit the arguments on $\mu, p,$ and f in the following derivation.)

In both the arbitrary $\mathbb{T}$ and $\mathbb{R}$ cases, we start with a first-order linear equation. For an arbitrary $\mathbb{T}$, the first-order linear equation we start with is $v^\Delta + (\alpha \odot p)v = f$. Notice that if we take $\mathbb{T} = \mathbb{R}$, this reduces to the differential equation $v' + \alpha p v = f$. We assume that $v(t)$ is a solution such that $v(t) \neq 0$.

Following the technique of [5, §2.6], let $x(t) = \dfrac{1}{v(t)}$. We start the derivation by taking the derivative of the substitution, applying the quotient rule, and then rewriting as an equation in terms of x. Below, the derivation in the arbitrary $\mathbb{T}$ and $\mathbb{R}$ cases is given side-by-side:

Arbitrary $\mathbb{T}$:

$$v^\Delta + (\alpha \odot p)v = f$$

Let $x(t) := \dfrac{1}{v(t)}$.

$$x^\Delta = \frac{x((\alpha \odot p) - fx)}{1 - \mu(\alpha \odot p) + f\mu x}$$

Let $x(t) = y^\alpha(t)$ and divide by y^α:

$$\frac{(y^\alpha)^\Delta}{y^\alpha} = \frac{(\alpha \odot p) - fy^\alpha}{1 - \mu(\alpha \odot p) + f\mu y^\alpha}$$

Apply Theorem 2.9:

$$\alpha \odot \frac{y^\Delta}{y} = \frac{(\alpha \odot p) - fy^\alpha}{1 - \mu(\alpha \odot p) + f\mu y^\alpha}$$

$\mathbb{T} = \mathbb{R}$:

$$v' + \alpha p v = f$$

Let $x(t) := \dfrac{1}{v(t)}$.

$$x' = x(\alpha p - fx)$$

Let $x(t) = y^\alpha(t)$ and rearrange to match the $\mathbb{T}$ form:

$$\alpha \frac{y'}{y} = \alpha p - fy^\alpha$$

At this point, for the case $\mathbb{T} = \mathbb{R}$, we are able to solve for y' and obtain a classic Bernoulli differential equation of the form:

$$y' = py - \frac{f}{\alpha} y^{\alpha+1}. \tag{3.2}$$

In the arbitrary time scale case, we assume $\mathbb{T}$ is isolated (since we are ultimately interested in studying the discrete equation in this present work). Thus, using Definition 2.8.2 in the case $\mu > 0$, and solving for y^Δ yields the time scale dynamic equation:

$$y^\Delta = \frac{y}{\mu}\left[\frac{1}{[2 - (1 + \mu p)^\alpha + f\mu y^\alpha]^{1/\alpha}} - 1\right]. \tag{3.3}$$

It is in this sense that we say that (3.3) is a discrete analog of the Bernoulli equation (3.2).

Definition 3.2. The Bernoulli equation on an isolated time scale is given by

$$y^{\Delta} = \frac{y}{\mu(t)} \left[\frac{1}{[2 - (1 + \mu(t)p(t))^{\alpha} + f(t)\mu(t)y^{\alpha}]^{1/\alpha}} - 1 \right]. \tag{3.4}$$

We will refer to this equation as the *discrete Bernoulli equation*.

Remark 3.3. Despite the similarity in derivation, a quick glance at the resulting discrete equation (3.4) shows a stark dissimilarity from the differential equation analog, (3.2). While it is enough for our purposes to explore the equation (3.4) simply because it is the result of the time scale derivation, there are other observations that lead us to the conclusion that this is a possible discretization of the Bernoulli equation.

There are several lines that one could discuss on this topic, but we mention one focused on the autonomous situation and the critical points of (3.2) and (3.4). For the autonomous case, we assume $p \neq 0, f \neq 0, \mu > 0, \alpha > 0$ are constant. The critical points of the differential equation (3.2) are found to be $y = 0$ and $y^{\alpha} = \frac{p\alpha}{f}$. Whereas, the critical points of the discrete equation (3.4) are $y = 0$ and $y^{\alpha} = \frac{(1 + \mu p)^{\alpha} - 1}{f\mu}$ (under appropriate assumptions on the parameters, of course). Except for the case $\alpha = 1$ (which corresponds to a logistic type equation), these yield different values for the critical points. However, a quick application of L'Hôpital's rule shows that for small μ, the non-zero critical point of the discrete equation is close to the non-zero critical point of the differential equation:

$$\lim_{\mu \to 0^+} \frac{(1 + \mu p)^{\alpha} - 1}{f\mu} = \frac{p\alpha}{f}.$$

4 Assumptions and Preliminaries

Since the focus of this work is on the discrete Bernoulli equation given by (3.4), we assume throughout that $\mathbb{T}$ is an isolated time scale which is unbounded above. Then, $\mathbb{T} = \{t_0, t_1, t_2, \cdots\}$ such that

$$t_0 < t_1 < t_2 < \cdots \quad \text{and} \quad \lim_{k \to \infty} t_k = \infty.$$

Often, we will employ subscript notation: $y_k := y(t_k)$, which implies $y_{k+1} = y(t_{k+1}) = y^{\sigma}(t_k) = y_k^{\sigma}$.

Our primary focus is on the autonomous case. Hence, we assume that p, f, and μ are constant. Further, we study the nonregressive case: the assumption that $1+\mu p = 0$. Note that surveying the literature on the time scale calculus, the vast majority of results are predicated on a regressivity assumption. The dynamics that result in the nonregressive case, as we study here, are rarely discussed. (However, as noted above, an application dealing with switched linear circuits that focuses specifically on the nonregressive case is given in [7].)

Convention 4.1. We consider real-valued solutions, and so throughout we follow the convention that the n-th root is real-valued, and not the complex-valued root obtained using the least complex argument. In other words, for $x \in \mathbb{R}$ and $n \in \mathbb{N}$,

- if n is odd, $x^{1/n}$ is the unique real number y such that $y^n = x$;
- if n is even and $x \geq 0$, then $x^{1/n}$ is the unique positive real number y such that $y^n = x$;
- if n is even and $x < 0$, then $x^{1/n}$ is undefined.

Assumption 4.2. The primary assumptions on the parameters are as follows:

- Let $\mu \in \mathbb{R}$ such that $\mu > 0$. Hence, the time scale considered is $\mathbb{T} = \mu\mathbb{Z} := \{n\mu : n \in \mathbb{Z}\}$. (H1)
- Assume $f \in \mathbb{R}$. (H2)
- Assume $\alpha \in \mathbb{Q} \cap (0, \infty)$. Let $m, n \in \mathbb{N}$ such that $\alpha = \dfrac{m}{n}$ is in lowest terms. (H3)
- Take $p = -\dfrac{1}{\mu}$ which implies that $1 + \mu p = 0$. (H4)

Remark 4.3. Note that assumption (H4) establishes the nonregressive case we focus on.

Corollary 4.4. *Assume* (H1)–(H3). *Then, the discrete Bernoulli equation* (3.4) *is equivalent to the recurrence relation form*

$$y^\sigma = \frac{y}{[2 - (1 + \mu p)^\alpha + f\mu y^\alpha]^{1/\alpha}}. \tag{4.1}$$

If, in addition (H4) *holds, then the recurrence relation reduces to*

$$y^\sigma = \frac{y}{[2 + f\mu y^\alpha]^{1/\alpha}}. \tag{4.2}$$

Proof. Since $\mathbb{T} = \mu\mathbb{Z}$ is isolated, we apply $y^\Delta = (y^\sigma - y)/\mu$ from Definition 2.6 to (3.4) and, by solving for y^σ, obtain the equivalent recurrence relation form (4.1). Applying the nonregressivity condition of (H4) immediately gives (4.2). □

Notice that the recurrence relations, (4.1) and (4.2), are defined by expressions that are not continuous. Thus, we must be aware of y-values that could cause a "zero in the denominator." In particular, there is a danger of certain initial conditions leading to an undefined y^σ, and hence a solution that has a bounded interval of existence. The following lemma establishes the initial conditions that lead to such a bounded interval of existence, and will play a key role in the bifurcation diagrams in Section 5.

Lemma 4.5. *Assume* (H1)–(H4) *such that* $f \neq 0$. *From* (H3), $\alpha = \dfrac{m}{n}$. *If* m *or* n *is even, then further assume* $f < 0$. *Define* $(T_k)_{k=1}^{\infty}$ *by*

$$T_k := \left[\frac{-1}{f\mu}\left(\frac{2^k}{2^k-1}\right)\right]^{\frac{1}{\alpha}}. \tag{4.3}$$

If $y = T_1$, *then the next value,* y^σ, *of* (4.2) *is undefined. If* $y = T_k$ *with* $k \geq 2$, *then* $y^\sigma = T_{k-1}$.

Proof. The assumption that $f < 0$ if either m or n is even ensures two things: first, the T_k are well-defined, and second, that when the T_k are well-defined, it follows that $T_k^\alpha = \dfrac{-1}{f\mu}\left(\dfrac{2^k}{2^k-1}\right)$. (See Remark 4.6 for further clarification.)

If $y = T_1 = \left(\dfrac{-2}{f\mu}\right)^{\frac{1}{\alpha}}$, then the denominator of (4.2) is zero, and hence y^σ is undefined.

Fix $k \in \mathbb{N}$ such that $k \geq 2$, and assume $y = T_k$. Focusing on the denominator of the recurrence relation (4.2), it follows that

$$[2 + f\mu y^\alpha]^{1/\alpha} = [2 + f\mu T_k^\alpha]^{1/\alpha} = \left[2 - \frac{2^k}{2^k-1}\right]^{1/\alpha} = \left[\frac{2^k-2}{2^k-1}\right]^{1/\alpha}.$$

Thus,

$$\begin{aligned} y^\sigma = \frac{y}{[2+f\mu y^\alpha]^{1/\alpha}} &= \frac{T_k}{[2+f\mu T_k^\alpha]^{1/\alpha}} = \left[\frac{2^k-1}{2^k-2}\right]^{\frac{1}{\alpha}} T_k \\ &= \left[\frac{-1}{f\mu}\left(\frac{2^k-1}{2^k-2}\right)\left(\frac{2^k}{2^k-1}\right)\right]^{\frac{1}{\alpha}} = \left[\frac{-1}{f\mu}\left(\frac{2^{k-1}}{2^{k-1}-1}\right)\right]^{\frac{1}{\alpha}} = T_{k-1}, \end{aligned}$$

completing the proof. □

Remark 4.6. To clarify the necessity of the assumptions of Lemma 4.5, we examine what occurs if $f > 0$ and either m or n is even:

If m is even and $f > 0$, then $\dfrac{-1}{f\mu}\left(\dfrac{2^k}{2^k-1}\right) < 0$, and $1/\alpha = \dfrac{n}{m}$ is an even root; hence, T_k is undefined following Convention 4.1.

If n is even and $f > 0$, the issue is a bit more subtle since the formula for T_k is well-defined; however, the conclusion of the lemma does not follow in this situation. To see this, assume n is even, m is odd, and that $y = T_1$. The conclusion of the lemma claims that y^σ is undefined, which would be achieved by obtaining a zero in the denominator. However, examining the denominator of the recurrence relation (4.2) yields

$$2 + f\mu y^\alpha = 2 + f\mu T_1^\alpha = 2 + f\mu\left[\left(\frac{-2}{f\mu}\right)^{\frac{n}{m}}\right]^{\frac{m}{n}} = 2 + f\mu\left|\frac{-2}{f\mu}\right| = 2 + f\mu\frac{2}{f\mu} = 4,$$

where the absolute value is necessary because of the even n-th power.

Remark 4.7. An analog of Lemma 4.5 can be developed for the regressive case as well. However, a closed form representation of the (T_k) sequence is difficult to obtain, and so we fall back on a recurrence relation form:

$$T_1 := \left(\frac{(1+\mu p)^\alpha - 2}{f\mu}\right)^{1/\alpha}, \qquad T_{k+1} := T_k\left(\frac{2-(1+\mu p)^\alpha}{1-f\mu T_k^\alpha}\right)^{1/\alpha}.$$

Further, since $(1+\mu p)^\alpha \neq 0$ in the regressive case, more sophisticated assumptions are required on the parameters in order to ensure that the sequence (T_k) is well-defined following Convention 4.1. This leads to certain choices of the parameters that can generate a finite sequence of T_k's (i.e., situations such that T_k is defined for, say, $k = 1, 2, 3$, but not for $k \geq 4$). This, in turn, leads to interesting solution behavior in the regressive case that does not occur in the nonregressive case; however, it falls outside the scope of this present work.

Note that in Lemma 4.5 the ability to define the sequence (T_k) in the even root/even power cases rests on the sign of f. This leads to natural cases for the bifurcation diagrams depending on whether $f < 0$, $f = 0$, or $f > 0$. The case with $f = 0$ is trivial:

Lemma 4.8. *Assume* (H1)–(H4) *with* $f = 0$*. Then, solutions of* (4.2) *exist for all initial conditions* $y_0 \in \mathbb{R}$*, and all solutions are monotonic and approach* 0.

Proof. If $f = 0$, then the recurrence relation (4.2) reduces to $y^\sigma = (1/2)^{1/\alpha} y$. Thus, with initial condition $y_0 \in \mathbb{R}$, iteration yields the explicit solution $y_k = (1/2)^{k/\alpha} y_0$. Since $0 < (1/2)^{1/\alpha} < 1$, the conclusion follows immediately. □

5 Bifurcation Diagrams

Assuming (H1)–(H4), the discrete Bernoulli equation from Definition 3.2 and the equivalent recurrence relation form Corollary 4.4 are:

$$y^\Delta = \frac{y}{\mu}\left[\frac{1}{[2+f\mu y^\alpha]^{1/\alpha}} - 1\right] \quad \Longleftrightarrow \quad y^\sigma = \frac{y}{[2+f\mu y^\alpha]^{1/\alpha}}. \tag{5.1}$$

The remainder of the work analyzes the solution behavior of (5.1) in all cases of the parameters.

5.1 Main Lemmas

As stated in assumption (H3), we assume that $\alpha \in \mathbb{Q} \cap (0, \infty)$ such that $\alpha = \dfrac{m}{n}$, $m, n \in \mathbb{N}$, is in lowest terms. Since α is expressed in lowest terms, there are three cases to consider: 1) m odd, n odd; 2) m even, n odd; and 3) m odd, n even. As one can imagine, there is much in common in regard to solution behavior between the three cases; however, there are several differences that occur. And in the case of m odd, n even, dramatically different behavior is found.

In this section, we collect the primary results that establish the bulk of the behavior for the solutions of the discrete Bernoulli equation (5.1). In particular, this section establishes many results in the $f < 0$ case (although some situations for $f > 0$ are dealt with here as well). Then, in Sections 5.2, 5.3, and 5.4 we will develop the case-specific solution behavior which will then allow us to create bifurcation diagrams, as well as completely analyze solution behavior.

Lemma 5.1. *Assume* (H1)–(H4). *Then $y = 0$ is a critical point of* (5.1). *If, in addition $f < 0$, then $c = \left(\frac{-1}{f\mu}\right)^{1/\alpha}$ is also a critical point.*

Proof. Clearly $y = 0$ is a critical point since $y^\sigma = 0/2^{1/\alpha} = 0$.

Assume $f < 0$ and $y = c = (-1/(f\mu))^{1/\alpha}$. Note that $-1/(f\mu) > 0$, and thus $y^\alpha = -1/(f\mu)$ which implies $[2 + f\mu y^\alpha]^{\frac{1}{\alpha}} = 1$. Therefore, $y^\sigma = y/[2 + f\mu y^\alpha]^{1/\alpha} = c/1 = c$. □

Lemma 5.2. *Assume* (H1)–(H4) *with $f < 0$. Define $(T_k)_1^\infty$ as in Lemma 4.5. Then, (T_k) is decreasing and approaches the critical point $c = (-1/(f\mu))^{1/\alpha}$.*

Proof. Using basic limit properties,

$$\lim_{k\to\infty} T_k = \lim_{k\to\infty} \left[\frac{-1}{f\mu}\left(\frac{2^k}{2^k - 1}\right)\right]^{\frac{1}{\alpha}} = \left[\frac{-1}{f\mu}\left(\lim_{k\to\infty} \frac{2^k}{2^k - 1}\right)\right]^{\frac{1}{\alpha}} = \left(\frac{-1}{f\mu}\right)^{\frac{1}{\alpha}} = c.$$

Further, since $2^k/(2^k - 1)$ is decreasing, and $-1/(f\mu) > 0$ and $1/\alpha > 0$, it follows that T_k is also decreasing. □

In the next three lemmas, at first glance, it may seem that the assumption $y > 0$ is overly restrictive. However, we will see various symmetry properties of the solutions in each of the cases on α. This will allow us to quickly establish analog results for the $y < 0$ case as appropriate.

Lemma 5.3. *Assume* (H1)–(H4) *with $f < 0$. Define $(T_k)_1^\infty$ as in Lemma 4.5. If $y > 0$, then the next value, y^σ, of the solution to* (5.1) *satisfies:*

1. *If $T_2 < y < T_1$, then $y^\sigma > T_1$.*
2. *If $T_{k+1} < y < T_k$ for $k \geq 2$, then $T_k < y^\sigma < T_{k-1}$.*

Proof. Part 1: Assume $T_2 < y < T_1$. Then,

$$T_2 < y < T_1 \implies \left(\frac{-1}{f\mu} \cdot \frac{4}{3}\right)^{\frac{1}{\alpha}} < y < \left(\frac{-2}{f\mu}\right)^{\frac{1}{\alpha}} \implies \frac{-4}{3f\mu} < y^\alpha < \frac{-2}{f\mu}$$

$$\begin{aligned}
&\implies \frac{-4}{3} > f\mu y^{\alpha} > -2 \quad \text{(since } f\mu < 0\text{)}\\
&\implies \frac{2}{3} > 2 + f\mu y^{\alpha} > 0 \implies 0 < [2 + f\mu y^{\alpha}]^{1/\alpha} < \left(\frac{2}{3}\right)^{\frac{1}{\alpha}}\\
&\implies \left(\frac{3}{2}\right)^{\frac{1}{\alpha}} < \frac{1}{[2 + f\mu y^{\alpha}]^{1/\alpha}} \implies \left(\frac{3}{2}\right)^{\frac{1}{\alpha}} y < \frac{y}{[2 + f\mu y^{\alpha}]^{1/\alpha}} = y^{\sigma}.
\end{aligned}$$

Finally, since $y > T_2$, we have

$$y^{\sigma} > \left(\frac{3}{2}\right)^{\frac{1}{\alpha}} y > \left(\frac{3}{2}\right)^{\frac{1}{\alpha}} T_2 = \left(\frac{3}{2}\right)^{\frac{1}{\alpha}} \left(\frac{-1}{f\mu} \cdot \frac{4}{3}\right)^{\frac{1}{\alpha}} = \left(\frac{-2}{f\mu}\right)^{\frac{1}{\alpha}} = T_1.$$

Part 2: Fix $k \in \mathbb{N}$ with $k \geq 2$, and assume $T_{k+1} < y < T_k$. Then,

$$\begin{aligned}
T_{k+1} < y < T_k &\implies \left[\frac{-1}{f\mu}\left(\frac{2^{k+1}}{2^{k+1}-1}\right)\right]^{\frac{1}{\alpha}} < y < \left[\frac{-1}{f\mu}\left(\frac{2^{k}}{2^{k}-1}\right)\right]^{\frac{1}{\alpha}}\\
&\implies \frac{-1}{f\mu}\left(\frac{2^{k+1}}{2^{k+1}-1}\right) < y^{\alpha} < \frac{-1}{f\mu}\left(\frac{2^{k}}{2^{k}-1}\right)\\
&\implies -\left(\frac{2^{k+1}}{2^{k+1}-1}\right) > f\mu y^{\alpha} > -\left(\frac{2^{k}}{2^{k}-1}\right) \quad \text{(since } f\mu < 0\text{)}\\
&\implies \frac{2(2^{k}-1)}{2^{k+1}-1} > 2 + f\mu y^{\alpha} > \frac{2(2^{k-1}-1)}{2^{k}-1} \quad \text{(adding 2, simplifying)}\\
&\implies \left(\frac{2^{k+1}-1}{2(2^{k}-1)}\right)^{\frac{1}{\alpha}} < \frac{1}{[2 + f\mu y^{\alpha}]^{1/\alpha}} < \left(\frac{2^{k}-1}{2(2^{k-1}-1)}\right)^{\frac{1}{\alpha}}\\
&\implies \left(\frac{2^{k+1}-1}{2(2^{k}-1)}\right)^{\frac{1}{\alpha}} y < \frac{y}{[2 + f\mu y^{\alpha}]^{1/\alpha}} < \left(\frac{2^{k}-1}{2(2^{k-1}-1)}\right)^{\frac{1}{\alpha}} y\\
&\implies \left(\frac{2^{k+1}-1}{2(2^{k}-1)}\right)^{\frac{1}{\alpha}} y < y^{\sigma} < \left(\frac{2^{k}-1}{2(2^{k-1}-1)}\right)^{\frac{1}{\alpha}} y.
\end{aligned}$$

By assumption $y > T_{k+1}$, and thus

$$\begin{aligned}
y^{\sigma} &> \left(\frac{2^{k+1}-1}{2(2^{k}-1)}\right)^{\frac{1}{\alpha}} y\\
&> \left(\frac{2^{k+1}-1}{2(2^{k}-1)}\right)^{\frac{1}{\alpha}} T_k\\
&= \left(\frac{2^{k+1}-1}{2(2^{k}-1)}\right)^{\frac{1}{\alpha}} \left[\frac{-1}{f\mu}\left(\frac{2^{k+1}}{2^{k+1}-1}\right)\right]^{\frac{1}{\alpha}}\\
&= \left[\frac{-1}{f\mu}\left(\frac{2^{k}}{2^{k}-1}\right)\right]^{\frac{1}{\alpha}} = T_k.
\end{aligned}$$

Similarly, since $y < T_k$,

$$y^\sigma < \left(\frac{2^k - 1}{2(2^{k-1} - 1)}\right)^{\frac{1}{\alpha}} y < \left(\frac{2^k - 1}{2(2^{k-1} - 1)}\right)^{\frac{1}{\alpha}} T_k = \left[\frac{-1}{f\mu}\left(\frac{2^{k-1}}{2^{k-1} - 1}\right)\right]^{\frac{1}{\alpha}} < T_{k-1},$$

completing the proof. □

Lemma 5.4. *Assume* (H1)–(H4) *with* $f < 0$. *Define* $c := (-1/(f\mu))^{1/\alpha}$. *If* $0 < y < c$, *then the next value,* y^σ, *of the solution to* (5.1) *satisfies* $0 < y^\sigma < c$. *Further, in this case, the solution* (y_k) *is decreasing and approaches* 0.

Proof. Since $0 < y < c$, there exists $0 < \epsilon < 1$ such that $0 < y < \epsilon c$. Define $\gamma := \left(\frac{1}{2 - \epsilon}\right)^{1/\alpha}$ and note that $0 < \gamma < 1$.

Since $f < 0$, it follows that $2^{1/\alpha} > [2 + f\mu y^\alpha]^{1/\alpha} > (2 - \epsilon)^{1/\alpha}$ because

$$\begin{aligned} 0 < y < \epsilon c \implies 0 < y^\alpha < \frac{-\epsilon}{f\mu} &\implies 0 > f\mu y^\alpha > -\epsilon \\ &\implies 2 > 2 + f\mu y^\alpha > 2 - \epsilon \\ &\implies 2^{1/\alpha} > [2 + f\mu y^\alpha]^{1/\alpha} > (2 - \epsilon)^{1/\alpha}. \end{aligned}$$

Reciprocating and multiplying by $y > 0$ yields:

$$y \cdot \left(\frac{1}{2}\right)^{\frac{1}{\alpha}} < \frac{y}{[2 + f\mu y^\alpha]^{1/\alpha}} < y \cdot \left(\frac{1}{2 - \epsilon}\right)^{\frac{1}{\alpha}} \implies 0 < y^\sigma < \gamma y.$$

Now let $0 < y_0 < c$. Thus, there exists $0 < \epsilon_0 < 1$ such that $0 < y_0 < \epsilon_0 c$. Define $\gamma := (1/(2 - \epsilon_0))^{1/\alpha}$. Then, $y_1 = y_0^\sigma < \gamma y_0$, $y_2 = y_1^\sigma < \gamma y_1 < \gamma^2 y_0$, etc. Continuing inductively, we find $0 < y_k < \gamma^k y_0$, and since $0 < \gamma < 1$, the conclusion follows. □

Lemma 5.5. *Assume* (H1)–(H4) *with* $f > 0$. *If* $y > 0$, *then the next value,* y^σ, *of the solution to* (5.1) *satisfies* $y^\sigma > 0$. *Further, in this case, the solution* (y_k) *is decreasing and approaches* 0.

Proof. Define $\gamma := (1/2)^{1/\alpha}$ and note that $0 < \gamma < 1$.

Assuming $f > 0$ and $y > 0$, it follows that $[2 + f\mu y^\alpha]^{1/\alpha} > 2^{1/\alpha}$ since

$$y > 0 \implies y^\alpha > 0 \implies f\mu y^\alpha > 0 \implies 2 + f\mu y^\alpha > 2 \implies [2 + f\mu y^\alpha]^{1/\alpha} > 2^{1/\alpha}.$$

Reciprocating and multiplying by $y > 0$ yields:

$$0 < \frac{y}{[2 + f\mu y^\alpha]^{1/\alpha}} < y \cdot \left(\frac{1}{2}\right)^{\frac{1}{\alpha}} \implies 0 < y^\sigma < \gamma y.$$

Let $y_0 > 0$. Then, $y_1 = y_0^\sigma < \gamma y_0$, $y_2 = y_1^\sigma < \gamma y_1 < \gamma^2 y_0$, etc. Continuing inductively, we find that $0 < y_k < \gamma^k y_0$, and since $0 < \gamma < 1$, the conclusion follows. □

Remark 5.6. Taking the above lemmas together (Lemmas 4.5, 5.1, 5.2, 5.3, and 5.4), at least in the case of $f < 0$, we have established the behavior of y^σ for all $y \in [0, T_1]$. In particular:

- Lemma 5.1 and 5.4 establish the results for $y \in [0, c]$ where $c = (-1/(f\mu))^{1/\alpha}$.
- Since Lemma 5.2 guarantees that $\lim_{k\to\infty} T_k = c$, it follows for any $y \in (c, T_1)\backslash\{T_k : k \in \mathbb{N}\}$, there exists $k \in \mathbb{N}$, such that $T_{k+1} < y < T_k$, and Lemma 5.3 applies.
- Finally, Lemma 4.5 establishes the case for $y = T_k$, $k \in \mathbb{N}$.

For all other values of y (and the situation when $f > 0$), we will need to examine the specific cases on $\alpha = \dfrac{m}{n}$.

5.2 Case: m odd, n odd

In this case, the powers $\alpha = m/n$ and $1/\alpha = n/m$ are both odd roots, and so following Convention 4.1, powers of α and $1/\alpha$ will always be defined. Thus, we seek to determine solution behavior for any value of $y \in \mathbb{R}$.

The following theorem establishes the solution behavior for all possible initial conditions in the case of m, n both odd, and $f < 0$:

Theorem 5.7. *Assume* (H1)–(H4) *with both m, n odd, and $f < 0$. Let $y \in \mathbb{R}$. Define $c := (-1/(f\mu))^{1/\alpha}$, and $(T_k)_1^\infty$ as in Lemma 4.5. Then, the next value, y^σ, of the solution to the discrete Bernoulli equation* (5.1) *satisfies:*

1. *If $y = 0$, then $y^\sigma = 0$, and if $y = c$, then $y^\sigma = c$.*

---

2. *If $0 < y < c$, then $0 < y^\sigma < c$. Further, in this case, the solution (y_k) is decreasing and approaches 0.*

---

3. *If $T_{k+1} < y < T_k$ for $k \geq 2$, then $T_k < y^\sigma < T_{k-1}$.*
4. *If $T_2 < y < T_1$, then $y^\sigma > T_1$.*
5. *If $y > T_1$, then $y^\sigma < 0$.*
6. *If $y < 0$, then $y^\sigma < 0$. Further, in this case, the solution (y_k) is increasing and approaches 0.*

---

7. *If $y = T_k$ for $k \geq 2$, then $y^\sigma = T_{k-1}$.*
8. *If $y = T_1$, then y^σ is undefined.*

Remark 5.8. Before proving this result, we note that the eight statements of this theorem completely describe solution behavior for all initial conditions $y_0 \in \mathbb{R}$ in the case m, n odd and $f < 0$. To examine the behavior of a solution with a particular initial condition $y_0 \in \mathbb{R}$, we apply the appropriate statement for y_0 to obtain information about $y_0^\sigma = y_1$. We then repeat the process by applying the theorem to y_1 to obtain information about $y_1^\sigma = y_2$, etc. There are four possible asymptotic outcomes for a given initial condition:

- Part 1 describes initial conditions that correspond to equilibrium solutions.
- Part 2 describes initial conditions that yield a solution that decreases to 0.
- Parts 3-6 describe initial conditions that are positive and increase for a finite number of terms, eventually become negative, and then increase to 0.
- Parts 7-8 describe initial conditions that eventually lead to an undefined y^σ, and hence correspond to solutions with bounded intervals of existence.

Note that all initial conditions $y_0 \in \mathbb{R}$ are accounted for because of Lemma 5.2 and Remark 5.6. Example 5.9 below illustrates the typical application of this theorem to analyze the solution behavior for particular initial conditions.

Proof of Theorem 5.12. Much of the theorem has already been proven in the lemmas of the previous section. In particular, Part 1 follows from Lemma 5.1; Part 2 from Lemma 5.4; Parts 3-4 from Lemma 5.3; and Parts 7-8 from Lemma 4.5. Hence, Parts 5 and 6 remain to be justified.

Part 5: Note that $y > T_1 > 0$. Thus,

$$\begin{aligned} y > T_1 &\implies y > \left(\frac{-2}{f\mu}\right)^{\frac{1}{\alpha}} \implies y^\alpha > \frac{-2}{f\mu} \implies f\mu y^\alpha < -2 \quad \text{(since } f\mu < 0\text{)} \\ &\implies 2 + f\mu y^\alpha < 0 \\ &\implies [2 + f\mu y^\alpha]^{1/\alpha} < 0 \quad \text{(since } 1/\alpha \text{ is an odd power, odd root)} \\ &\implies \frac{1}{[2 + f\mu y^\alpha]^{1/\alpha}} < 0 \implies y^\sigma = \frac{y}{[2 + f\mu y^\alpha]^{1/\alpha}} < 0. \end{aligned}$$

Part 6: Assume $y < 0$ Following an almost identical approach as in the proof of Lemma 5.5, we define $\gamma := (1/2)^{1/\alpha}$ and note that $0 < \gamma < 1$. Assuming $f < 0$ and $y < 0$, it follows that $[2 + f\mu y^\alpha]^{1/\alpha} > 2^{1/\alpha}$. Reciprocating and multiplying by $y < 0$ yields $0 > y^\sigma > \gamma y$.

Let $y_0 < 0$. Then, $y_1 = y_0^\sigma > \gamma y_0$, $y_2 = y_1^\sigma > \gamma y_1 > \gamma^2 y_0$, etc. Continuing inductively, we find $0 > y_k > \gamma^k y_0$, and since $0 < \gamma < 1$, the conclusion follows. □

Example 5.9 (Case: m odd, n odd, $f < 0$)**.** Consider $\mu = 1$, $\alpha = 1/3$ (i.e., $m = 1$, $n = 3$), and $f = -1$. With these values of the parameters, the first few values of the sequence $(T_k)_1^\infty$ defined in Lemma 4.5 are approximately:

$$(T_k) \approx (8, 2.37037, 1.49271, 1.21362, 1.09992, 1.04837, \cdots).$$

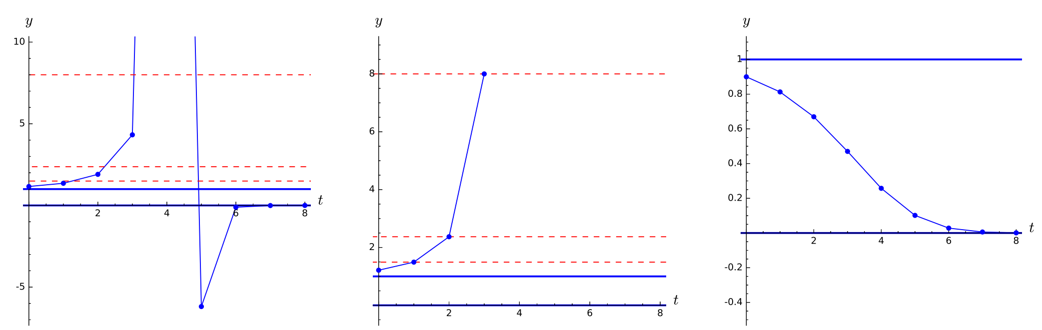
Consider the initial condition $y_0 = 1.16$, and notice $T_5 \approx 1.09992 < y_0 < 1.21362 \approx T_4$. So, following the steps outlined in Remark 5.8, we will apply Part 3 of the theorem to determine bounds on y_1. We then repeat the process:

$$
\begin{array}{lcl}
 & & T_5 < y_0 < T_4 \\
\text{Theorem 5.7.3} & \Longrightarrow & T_4 < y_1 < T_3 \\
\text{Theorem 5.7.3} & \Longrightarrow & T_3 < y_2 < T_2 \\
\text{Theorem 5.7.3} & \Longrightarrow & T_2 < y_3 < T_1 \\
\text{Theorem 5.7.4} & \Longrightarrow & y_4 > T_1 \\
\text{Theorem 5.7.5} & \Longrightarrow & y_5 < 0 \\
\text{Theorem 5.7.6} & \Longrightarrow & y_6 < 0 \text{ and the solution increases with } \lim_{k\to\infty} y_k = 0.
\end{array}
$$

Now consider the initial condition $y_0 = T_4 = (16/15)^3 \approx 1.21362$. Applying Theorem 5.7.7, we find that $y_1 = T_3$, $y_2 = T_2$, $y_3 = T_1$, and then y_4 is undefined by Part 8. So, with this initial condition, we have a solution with bounded interval of existence.

Finally, consider the initial condition $y_0 = 0.9$. Since the critical point $c = 1$, we are in the case $0 < y_0 < c$, and Part 2 guarantees that this solution will be decreasing and approach 0.

The plot of each of these solutions is given in Figure 5.1, which graphically confirms the analytic results produced by Theorem 5.7. In the plots, the solid curves are the critical points, and the dashed curves are the first several T_k's.



$y_0 = 1.16$ $y_0 = T_4 \approx 1.21$ $y_0 = 0.9$

Figure 5.1: Solutions for initial conditions considered in Example 5.9. The solid lines are the critical points, and the dashed lines are the first several T_k's.

In Theorem 5.7, we have completely characterized solutions in the case $f < 0$. What about when $f > 0$? The following lemma shows that for $f > 0$, the solutions behave identically except for a certain reflection. More specifically, if (y_k) is a solution of (5.1) for a fixed $f < 0$, then $(-y_k)$ is a solution for the equation replacing f with $-f > 0$.

Lemma 5.10. *Assume $f < 0$ and that (y_k) is a solution of* (5.1)*, then $(-y_k)$ is a solution of*

$$y^\sigma = \frac{y}{[2 + (-f)\mu y^\alpha]^{1/\alpha}}.$$

Proof. The key observation in this case is that $y^\alpha = y^{m/n}$ where m is an odd power, and n is an odd root. Hence, for any $y \in \mathbb{R}$, $(-y)^\alpha = -y^\alpha$.

Assume (y_k) is a solution of (5.1). Define (x_k) by $x_k := -y_k$, and note that

$$\begin{aligned} x_{k+1} = -y_{k+1} &= -\left(\frac{y_k}{[2 + f\mu y_k^\alpha]^{1/\alpha}}\right) \\ &= \frac{-y_k}{[2 + (-f)\mu(-y_k)^\alpha]^{1/\alpha}} = \frac{x_k}{[2 + (-f)\mu x_k^\alpha]^{1/\alpha}}, \end{aligned}$$

and hence (x_k) satisfies $y^\sigma = \dfrac{y}{[2 + (-f)\mu y^\alpha]^{1/\alpha}}$. □

Remark 5.11. By design, in the proof of Theorem 5.7.6 above, we chose to simply restate the proof of Lemma 5.10 with the appropriate sign changes. However, now that we have established Lemma 5.10, we see that Theorem 5.7.6 follows directly from Lemma 5.5. It was for the sake of clarity that we chose to delay Lemma 5.10 until after Example 5.9.

In light of Lemma 5.10, the analog to Theorem 5.7 for $f > 0$ is given by:

Theorem 5.12. *Assume* (H1)–(H4) *with both m and n odd, and $f > 0$. Let $y \in \mathbb{R}$. Define $c := (-1/(f\mu))^{1/\alpha}$, and $(T_k)_1^\infty$ as in Lemma 4.5. (Note that $c < 0$, $T_k < 0$, and (T_k) is increasing since $f > 0$.) Then, the next value, y^σ, of the solution to the discrete Bernoulli equation* (5.1) *satisfies:*

1. *If $y = 0$, then $y^\sigma = 0$, and if $y = c$, then $y^\sigma = c$.*

---

2. *If $c < y < 0$, then $c < y^\sigma < 0$. Further, in this case, the solution (y_k) is increasing and approaches 0.*

---

3. *If $T_k < y < T_{k+1}$ for $k \geq 2$, then $T_{k-1} < y^\sigma < T_k$.*
4. *If $T_1 < y < T_2$, then $y^\sigma < T_1$.*
5. *If $y < T_1$, then $y^\sigma > 0$.*
6. *If $y > 0$, then $y^\sigma > 0$. Further, in this case, the solution (y_k) is decreasing and approaches 0.*

---

7. *If $y = T_k$ for $k \geq 2$, then $y^\sigma = T_{k-1}$.*
8. *If $y = T_1$, then y^σ is undefined.*

Proof. Note that Part 6 follows directly from Lemma 5.5. For all other parts, assume $f > 0$. Then $-f < 0$, and so Theorem 5.7 applies using $-f$ in the recurrence relation (5.1). Then, apply Lemma 5.10 to that solution, which then establishes the result in the $f > 0$ case as desired. □

Now with Theorem 5.7 and 5.12 (along with the trivial case of Lemma 4.8), we are able to give the complete bifurcation diagram for the case $\alpha = \dfrac{m}{n}$ with m odd, n odd in Figure 5.2.

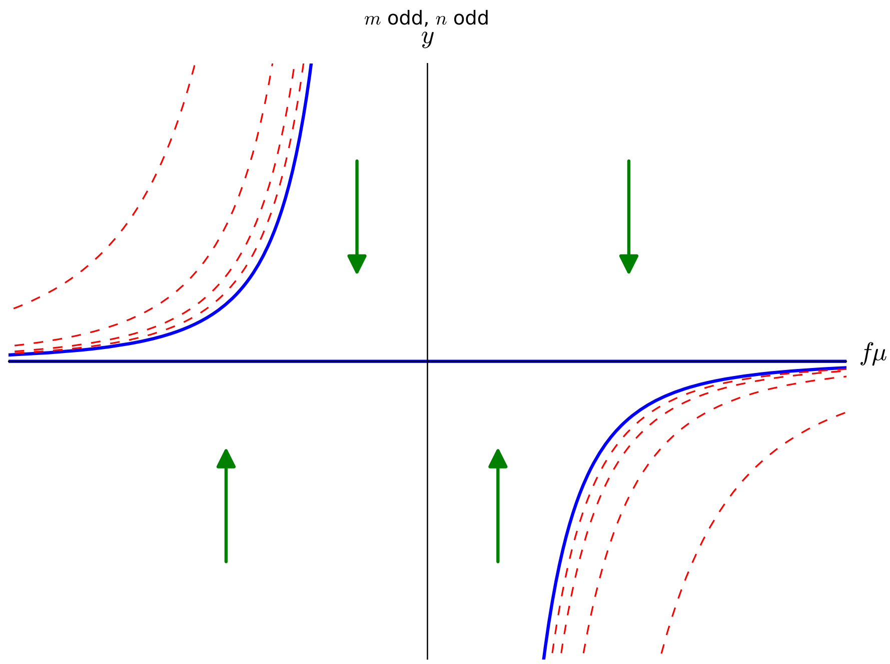


Figure 5.2: Bifurcation diagram for the m odd, n odd case. The solid curves are the critical points, and the dashed curves are the first several T_k's. Arrows indicate monotonic solutions approaching 0.

5.3 Case: m even, n odd

Just like in the last case, a symmetry argument can be given to reduce the number of cases we must consider individually. Note that the symmetry here is not the same as the symmetry of solutions found in the previous case.

Lemma 5.13. *Assume* (H1)–(H4) *with m even, and n odd. If (y_k) is a solution of* (5.1)*, then so too is $(-y_k)$.*

Proof. The key observation in this case is that $y^\alpha = y^{m/n}$ and m is an even power. Hence, for any $y \in \mathbb{R}$, $y^\alpha = (-y)^\alpha$.

Assume (y_k) is a solution of (5.1). Define (x_k) by $x_k := -y_k$, and note that

$$x_{k+1} = -y_{k+1} = -\left(\frac{y_k}{[2 + f\mu y_k^\alpha]^{1/\alpha}}\right) = \frac{-y_k}{[2 + f\mu(-y_k)^\alpha]^{1/\alpha}} = \frac{x_k}{[2 + f\mu x_k^\alpha]^{1/\alpha}},$$

and hence (x_k) satisfies (5.1). □

With the symmetry of solutions guaranteed by Lemma 5.13, along with the lemmas of Section 5.1, the majority of the following theorem has already been proven.

Theorem 5.14. *Assume* (H1)–(H4) *with m even, n odd, and $f < 0$. Let $y \in \mathbb{R}$. Define $c := (-1/(f\mu))^{1/\alpha}$, and $(T_k)_1^\infty$ as in Lemma 4.5. Then, the next value, y^σ, of the solution to the discrete Bernoulli equation* (5.1) *satisfies:*

1a. If $y = 0$, then $y^\sigma = 0$; and, if $y = c$, then $y^\sigma = c$.

1b. If $y = -c$, then $y^\sigma = -c$.

2a. If $0 < y < c$, then $0 < y^\sigma < c$. Further, in this case, the solution (y_k) is decreasing and approaches 0.

2b. If $-c < y < 0$, then $-c < y^\sigma < 0$. Further, in this case, the solution (y_k) is increasing and approaches 0.

3a. If $T_{k+1} \le y < T_k$ for $k \ge 2$, then $T_k \le y^\sigma < T_{k-1}$.

4a. If $T_2 \le y < T_1$, then $y^\sigma \ge T_1$.

5a. If $y \ge T_1$, then y^σ is undefined.

3b. If $-T_k \le y < -T_{k+1}$ for $k \ge 2$, then $-T_{k-1} \le y^\sigma < -T_k$.

4b. If $-T_1 \le y < -T_2$, then $y^\sigma \le -T_1$.

5b. If $y \le -T_1$, then y^σ is undefined.

Proof. Again, much of the theorem has already been proven in the lemmas. In particular, Part 1a follows from Lemma 5.1; Part 2a from Lemma 5.4; and Parts 3a-4a from Lemmas 4.5 and 5.3. Further, Parts 1b, 2b, and 3b-5b follow immediately from Lemma 5.13 applied to the solutions in the corresponding "a" part. Thus, the only part to establish here is Part 5a.

Part 5a: Note that $\alpha = \dfrac{m}{n}$ with m even, and n odd. By assumption, $y \geq T_1 = \left(\dfrac{-2}{f\mu}\right)^{1/\alpha}$. Raising both sides of the inequality to the α power, multiplying by $f\mu < 0$, and adding 2 results in:

$$y \geq \left(\frac{-2}{f\mu}\right)^{1/\alpha} \implies y^\alpha \geq \frac{-2}{f\mu} \implies 2 + f\mu y^\alpha \leq 0.$$

The denominator of the recurrence relation (5.1) is $[2+f\mu y^\alpha]^{1/\alpha}$. If $2+f\mu y^\alpha = 0$, then the denominator is 0, and hence y^σ is undefined. If $2 + f\mu y^\alpha < 0$, then $[2 + f\mu y^\alpha]^{1/\alpha}$ is undefined because $1/\alpha = \dfrac{n}{m}$ and m is even (as per Convention 4.1). □

We now turn our attention to the situation when $f > 0$. As already noted in Remark 4.6, when m is even, the sequence (T_k) is not defined. This leads to the following result:

Theorem 5.15. *Assume* (H1)–(H4) *with m even, n odd, and $f > 0$. Let $y \in \mathbb{R}$. Then, the next value, y^σ, of the solution to the discrete Bernoulli equation* (5.1) *satisfies:*

1. *If $y = 0$, then $y^\sigma = 0$.*

2. *If $y > 0$, then $y^\sigma > 0$. Further, in this case, the solution (y_k) is decreasing and approaches 0.*

3. *If $y < 0$, then $y^\sigma < 0$. Further, in this case, the solution (y_k) is increasing and approaches 0.*

Proof. Delightfully, the lemmas we have already proven establish all parts here: Part 1 follows from Lemma 5.1, Part 2 follows from Lemma 5.5, and Part 3 follows from Lemma 5.13. □

Example 5.16 (Case: m even, n odd, $f < 0$)**.** Figure 5.3 shows several solutions for various initial conditions in the m even, n odd, $f < 0$ case. The details are omitted here as the procedure mimics what was illustrated in Example 5.9 for the m odd, n odd case. In this example, Theorem 5.14 is applied to initial conditions satisfying: (a) $T_4 < y_0 < T_3$, (b) $0 < y_0 < c$, (c) $-c < y_0 < 0$, and (d) $-T_4 < y_0 < -T_5$.

Now with Theorem 5.14 and 5.15 (along with the trivial case of Lemma 4.8), we are able to give the complete bifurcation diagram for the case $\alpha = \dfrac{m}{n}$ with m even, n odd in Figure 5.4.

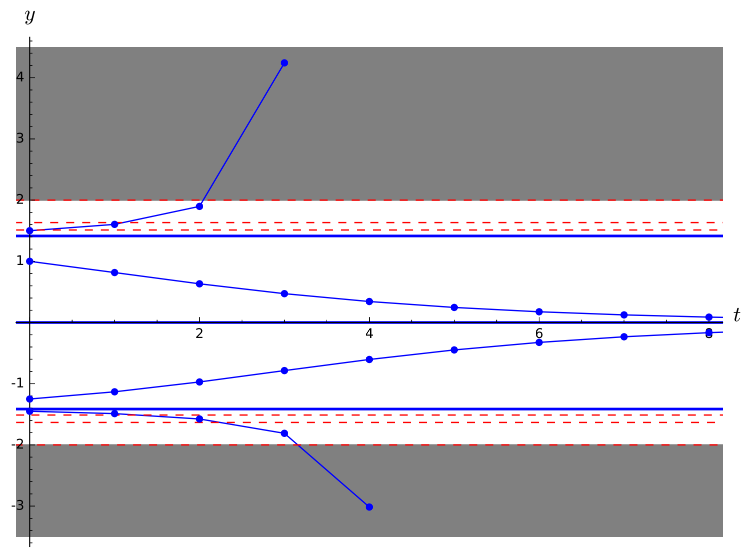


Figure 5.3: Solutions for initial conditions considered in Example 5.16. The solid lines are the critical points, the dashed lines are the first several T_k's and $-T_k$'s, and the shaded regions are where y^σ is undefined.

5.4 Case: m odd, n even

In this case, $\alpha = \dfrac{m}{n}$ with n even, and thus the y^α term in the denominator of the recurrence relation (5.1) is an even root. So, following Convention 4.1, y^α will only be defined provided $y \geq 0$. In the case $f > 0$, as already noted in Remark 4.6, when n is even, the sequence (T_k) is not defined. This leads to the following result when $f > 0$:

Theorem 5.17. *Assume* (H1)–(H4) *with m odd, n even, and $f > 0$. Let $y \in \mathbb{R}$. Then, the next value, y^σ, of the solution to the discrete Bernoulli equation* (5.1) *satisfies:*

1. *If $y = 0$, then $y^\sigma = 0$.*

2. *If $y > 0$, then $y^\sigma > 0$. Further, in this case, the solution (y_k) is decreasing and approaches* 0.

Proof. Part 1 follows from Lemma 5.1, and Part 2 from Lemma 5.4. □

We now turn our attention to the one case which exhibits the most intricate and interesting behavior: m odd, n even, and $f < 0$. While the statement of the theorem looks very similar to the theorems of the previous cases, it turns out that certain initial conditions will lead to complicated long-term behavior. This behavior is discussed in Remark 5.19, and explored in Section 6 for a specific example.

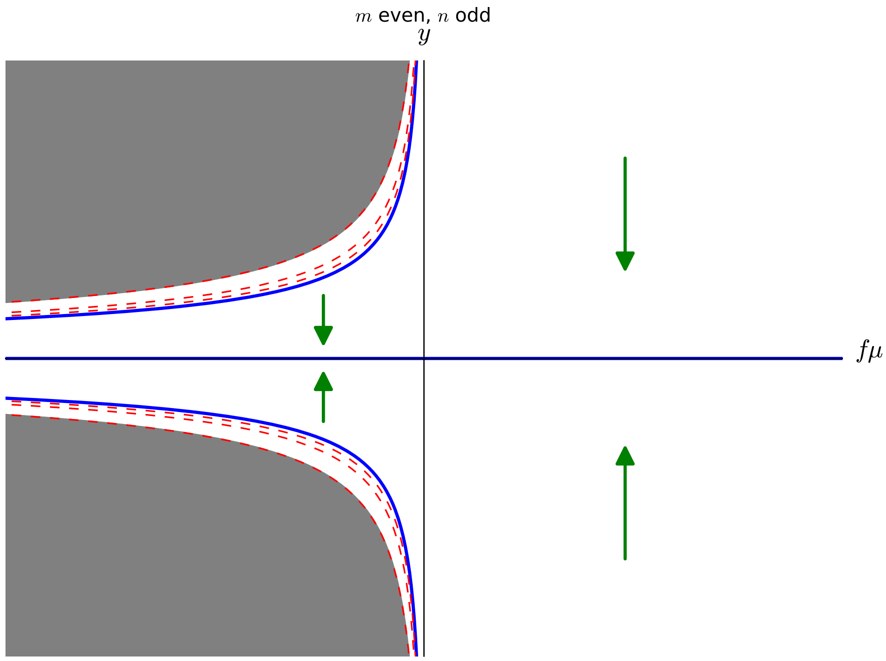


Figure 5.4: Bifurcation diagram for the m even, n odd case. The solid curves are the critical points, and the dashed curves are the first several T_k's. Arrows indicate monotonic solutions approaching 0. Shaded regions are where y^σ is undefined.

Theorem 5.18. *Assume* (H1)–(H4) *with both m odd, m odd, and $f < 0$. Let $y \in \mathbb{R}$. Define $c_1 := (-1/(f\mu))^{1/\alpha}$, $c_2 := (-3/(f\mu))^{1/\alpha}$, and $(T_k)_1^\infty$ as in Lemma 4.5. Then, the next value, y^σ, of the solution to the discrete Bernoulli equation* (5.1) *satisfies:*

1. *If $y \in \{0, c_1, c_2\}$, then $y^\sigma = y$*
2. *If $0 < y < c_1$, then $0 < y^\sigma < c_1$. Further, in this case, the solution (y_k) is decreasing and approaches* 0.
3. *If $T_{k+1} < y < T_k$ for $k \geq 2$, then $T_k < y^\sigma < T_{k-1}$.*
4. *If $T_2 < y < T_1$, then $y^\sigma > T_1$.*
5. *If $T_1 < y < c_2$, then $y^\sigma > T_1$.*
6. *If $y > c_2$, then $c_1 < y^\sigma < c_2$.*
7. *If $y = T_k$ for $k \geq 2$, then $y^\sigma = T_{k-1}$.*
8. *If $y = T_1$, then y^σ is undefined.*

Proof. As we have seen several times, the lemmas justify much of this theorem. In particular, Part 2 from Lemma 5.4; Parts 3-4 from Lemma 5.3; and Parts 7-8 from Lemma 4.5. Hence, Parts 1, 5, and 6 remain to be justified.

Part 1: From Lemma 5.1, we know that 0 and c_1 are critical points for (5.1). With m odd, and n even, we show that c_2 is also a critical point.

Assume $f < 0$ and $y = c_2 = (-3/(f\mu))^{1/\alpha}$. Note that $-3/(f\mu) > 0$, and thus $y^\alpha = -3/(f\mu)$ which implies $2 + f\mu y^\alpha = -1$. Since $\dfrac{1}{\alpha} = \dfrac{n}{m}$ and n is an even power, we have that $[2 + f\mu y^\alpha]^{1/\alpha} = 1$. Therefore, $y^\sigma = y/[2 + f\mu y^\alpha]^{1/\alpha} = c_2/1 = c_2$.

Part 5: Note that

$$\begin{aligned} T_1 < y < c_2 &\implies \left(\frac{-2}{f\mu}\right)^{\frac{1}{\alpha}} < y < \left(\frac{-3}{f\mu}\right)^{\frac{1}{\alpha}} \implies \frac{-2}{f\mu} < y^\alpha < \frac{-3}{f\mu} \\ &\implies -2 > f\mu y^\alpha > -3 \quad (\text{since } f\mu < 0) \\ &\implies 0 > 2 + f\mu y^\alpha > -1 \\ &\implies 0 < [2 + f\mu y^\alpha]^{1/\alpha} < 1 \quad (\text{since } 1/\alpha \text{ is an even power}) \\ &\implies \frac{1}{[2 + f\mu y^\alpha]^{1/\alpha}} > 1 \implies y^\sigma = \frac{y}{[2 + f\mu y^\alpha]^{1/\alpha}} > y > T_1. \end{aligned}$$

Part 6: Assume $y > c_2$. Then, there exists $\gamma > 1$ such that $y = \gamma^{1/\alpha} c_2 > c_2$. Then,

$$y^\sigma = \frac{\gamma^{1/\alpha} c_2}{[2 + f\mu(\gamma^{1/\alpha} c_2)^\alpha]^{1/\alpha}} = \frac{\gamma^{1/\alpha} c_2}{\left[2 + f\mu\gamma\left(\frac{-3}{f\mu}\right)\right]^{1/\alpha}} = \left(\frac{\gamma}{2 - 3\gamma}\right)^{\frac{1}{\alpha}} c_2.$$

Consider $g(\gamma) = \gamma/(2-3\gamma)$. Applying basic calculus techniques, it follows that $g'(\gamma) = 2/(3\gamma - 2)^2 > 0$, so g is increasing; further, $g(1) = -1$ and $\lim_{\gamma\to\infty} g(\gamma) = -1/3$. Thus, $-1 < g(\gamma) < -1/3$ for $\gamma > 1$. Since $1/\alpha$ is an even power, it follows that:

$$\left(\frac{1}{3}\right)^{\frac{1}{\alpha}} < g(\gamma)^{\frac{1}{\alpha}} = \left(\frac{\gamma}{2 - 3\gamma}\right)^{\frac{1}{\alpha}} < 1 \implies \left(\frac{1}{3}\right)^{\frac{1}{\alpha}} c_2 < \left(\frac{\gamma}{2 - 3\gamma}\right)^{\frac{1}{\alpha}} c_2 = y^\sigma < c_2.$$

Finally, note that $(1/3)^{1/\alpha} c_2 = (1/3)^{1/\alpha}(-3/(f\mu))^{1/\alpha} = (-1/(f\mu))^{1/\alpha} = c_1$ as desired. □

Remark 5.19. Note that in the theorems for the other cases (Theorems 5.7, 5.12, 5.14, 5.15, and 5.17), the asymptotic behavior of the solution for a given initial condition could be determined after a finite number of applications of the theorem. For example, in Theorem 5.7, every initial condition would eventually lead to applying exactly one of Parts 1 or 2 or 6 or 8. And, once one of these parts was reached for the first time while analyzing a particular initial condition, the asymptotic behavior was completely determined for that solution.

However, in Theorem 5.18, solutions that have an initial condition that satisfies the hypotheses of Parts 3-5 will eventually get to Part 6, and Part 6 guarantees that y^σ will satisfy the hypotheses of Parts 3-5 again, or even possible Parts 1 or 7 or 8! In this situation, the asymptotic behavior is quite complicated. We explore this in more detail in the next section.

Now with Theorem 5.17 and 5.18 (along with the trivial case of Lemma 4.8), we are able to give the bifurcation diagram for the case $\alpha = \dfrac{m}{n}$ with m odd, n even in Figure 5.5.

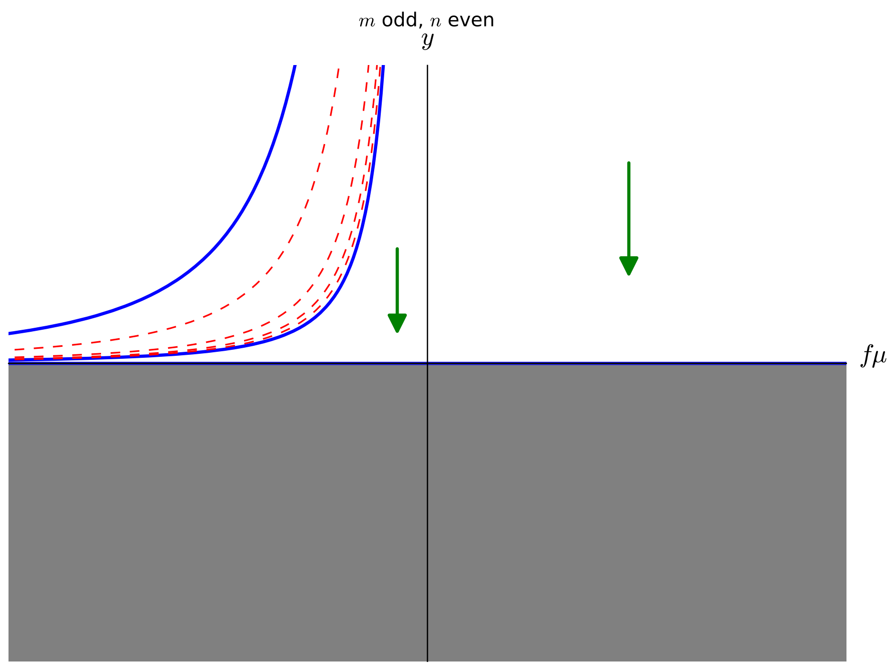


Figure 5.5: Bifurcation diagram for the m odd, n even case. The solid curves are the critical points, and the dashed curves are the first several T_k's. Arrows indicate monotonic solutions approaching 0. Shaded regions are where y^σ is undefined.

6 Limit Cycle Example

While repeatedly applying Theorem 5.18 gives some insight into the solution behavior, it does not give any clear asymptotic results because of the chaotic nature of a such a solution. We will show in this section, using a specific example, that even in the midst of these chaotic solutions, certain well-behaved solutions can be found.

Consider $\mu = 1$, $\alpha = 1/2$ (i.e., $m = 1$, $n = 2$), and $f = -1/2$. Note that the recurrence relation (5.1) simplifies to

$$y^{\sigma} = \frac{y}{\left(2 - \frac{1}{2}\sqrt{y}\right)^2} = \frac{4y}{(4 - \sqrt{y})^2}. \tag{6.1}$$

Further, with these specific values of the parameters, the hypotheses of Theorem 5.18 are satisfied. The three critical points are 0, $c_1 = 4$, and $c_2 = 36$, and the first few values of the sequence $(T_k)_1^{\infty}$ defined in Lemma 4.5 are approximately:

$$(T_k) \approx (16, 7.11111, 5.22448, 4.55111, 4.26222, 4.12799, 4.06324, \cdots).$$

In this case, an interesting solution behavior is observed that is not present in any of the other cases we have analyzed. For the initial condition $y_0 = 9$, we note that Theorem 5.18.4 guarantees that $y_1 > T_1 = 16$. This is certainly true, but applying the recurrence relation (5.1) to $y_0 = 9$ yields $y_1 = 36$. We have found an initial condition that "lands on the equilibrium" $c_2 = 36$.

The natural next question is, what value of y_0 yields $y_1 = 9$? A quick computation using (6.1) produces $y_0 = 144$. Thus, $(y_k) = (144, 9, 36, 36, \cdots)$ is a solution. Repeating this process, we find that there are *two* values of y that yield $y^{\sigma} = 144$. In particular, $y = 576/49$ and $y = 576/25$. Thus, two more solutions of (6.1) are of the form

$$y_0 = \frac{576}{49} \implies (y_k) = \left(\frac{576}{49}, 144, 9, 36, \cdots\right)$$
$$y_0 = \frac{576}{25} \implies (y_k) = \left(\frac{576}{25}, 144, 9, 36, \cdots\right)$$

Repeating again, we continue to find more initial conditions that will ultimately lead to the solution taking on the value 36. Below are the solutions that are found at the next step of this process:

$$(y_k) = \left(\frac{2304}{361}, \frac{576}{49}, 144, 9, 36, \cdots\right) \qquad (y_k) = \left(\frac{2304}{289}, \frac{576}{25}, 144, 9, 36, \cdots\right)$$
$$(y_k) = \left(\frac{2304}{25}, \frac{576}{49}, 144, 9, 36, \cdots\right) \qquad (y_k) = \left(\frac{2304}{49}, \frac{576}{25}, 144, 9, 36, \cdots\right)$$

The above demonstrates the existence of a limit cycle with period one. But, that is not all we can find here! The following theorem gives a complete characterization of all initial conditions that will tend toward limit cycles of *any period* for this particular example. (Remark 6.3 illustrates how the theorem is used to accomplish this.)

We do need one well-known property of the greatest common divisor:

Lemma 6.1. *If* $m \in \mathbb{Z}$*, then* $\gcd(a, b + ma) = \gcd(a, b)$.

Theorem 6.2. *Assume* $p, q \in \mathbb{N}$ *are odd such that* $\gcd(p, q) = 1$*. Let* $y = \left(\frac{2^k p}{q}\right)^2$ *where* $k \in \mathbb{N}_0$*. Then, the next value,* y^{σ}*, of the solution to* (6.1) *satisfies:*

1. If $k \geq 3$*, then* $y^\sigma = \left(\frac{2^{k-1}p}{q - 2^{k-2}p}\right)^2$.

2. If $k = 2$*, then* $y^\sigma = \left(\frac{p}{\frac{1}{2}(q-p)}\right)^2$.

3. If $k = 0$*, then* $y^\sigma = \left(\frac{2p}{4q-p}\right)^2$.

4. If $k = 1$*, then* $y^\sigma = \left(\frac{2p}{2q-p}\right)^2$.

In each case, the fraction in the expression for y^σ *is in lowest terms.*

Proof. Suppose that $p, q \in \mathbb{N}$ are odd such that $\gcd(p, q) = 1$.

Part 1: Fix $k \in \mathbb{N}$ with $k \geq 3$, and assume $y = \left(\frac{2^k p}{q}\right)^2$. Calculating y^σ using (6.1) yields:

$$y^\sigma = \left(\frac{4\left(\frac{2^k p}{q}\right)^2}{\left(4 - \frac{2^k p}{q}\right)^2}\right)^2 = \left(\frac{2^{k+1}p}{4q - 2^k p}\right)^2 = \left(\frac{4 \cdot 2^{k-1}p}{4(q - 2^{k-2}p)}\right)^2 = \left(\frac{2^{k-1}p}{q - 2^{k-2}p}\right)^2.$$

Note that $\gcd(p, q - 2^{k-2}p) = 1$, which follows immediately from Lemma 6.1:

$$\gcd(p, q - 2^{k-2}p) = \gcd(p, q) = 1.$$

Further, $q - 2^{k-2}p$ is odd since it is an even number ($2^{k-2}p$) subtracted from an odd number (q). So, $\gcd(2^{k-1}p, q - 2^{k-2}) = 1$ and the fraction is in lowest terms as desired.

Part 2: Assume $k = 2$; then, $y = \left(\frac{2^2 p}{q}\right)^2$. Applying (6.1) as in Part 1, we have $y^\sigma = \left(\frac{2p}{q-p}\right)^2$. Note that $q - p$ is even. Again, straight from Lemma 6.1,

$$\gcd(p, q - p) = \gcd(p, q) = 1.$$

Hence, $\gcd(2p, q - p) = 2$, and it follows that $y^\sigma = \left(\frac{p}{\frac{1}{2}(q-p)}\right)^2$ is in lowest terms.

Parts 3 and 4 are nearly identical: directly calculate y^σ from (6.1), and then use properties of the greatest common divisor to establish that the fraction is written in lowest terms. □

Remark 6.3. The key to seeing Theorem 6.2's relationship to limit cycles is to note the importance of each fraction being expressed in lowest terms, and to focus on the pattern found in the numerators. Let p be an odd number. Then, for example, take $k = 5$ and consider $y_0 = \left(\frac{2^5 p}{q}\right)^2$. Using the theorem to calculate y^σ, we examine the numerators in each of the fractions which are written in lowest terms. Repeating the process gives the following pattern for the numerators:

	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	
	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	
numerator:	2^5p	2^4p	2^3p	2^2p	p	$2p$	$2p$	$2p$	$\cdots$

Once the numerator $2p$ is achieved, we have entered the limit cycle. The period of the limit cycle depends on the magnitude of p. Additional limit cycles of the same period are found by varying the q, but making sure $(2p/q)^2 > 4$. This restriction stems from Theorem 5.18, which states that if $y \leq 4 = c_1$, then the solution is monotonic and approaches 0.

For example, take $p = 3$ and $q = 1$, and we obtain the one period limit cycle we already knew about: $y = 36$. But, Theorem 6.2 shows exactly which initial conditions will eventually enter this limit cycle. For example, with $p = 3$, and $q = 1$, and taking $k = 4$, $y = (2^4p/q)^2 = (48)^2 = 2304$, and the theorem guarantees that we will enter the limit cycle at y_4. Using Theorem 6.2, we can also compute the solution explicitly:

$$\left(\left(2^4 \cdot 3\right)^2, \left(\frac{2^3 \cdot 3}{11}\right)^2, \left(\frac{2^2 \cdot 3}{5}\right)^2, 3^2, 2 \cdot 3^2, \cdots \right) = \left(2304, \frac{576}{121}, \frac{144}{25}, 9, 36, 36, \cdots \right).$$

What if we wanted to generate another solution that ends in this limit cycle? Use a different odd number q! For example, $q = 5$ with the same $p = 3$, $k = 4$, will again enter the limit cycle at y_4:

$$\left(\left(\frac{2^4 \cdot 3}{5}\right)^2, \left(\frac{2^3 \cdot 3}{7}\right)^2, \left(2^2 \cdot 3\right)^2, 3^2, 2 \cdot 3^2, \cdots \right) = \left(2304, \frac{576}{49}, 144, 9, 36, 36, \cdots \right).$$

Finally, we generate limit cycles of other periods using this procedure. All we need to do is change p to different odd numbers, and take the initial condition to be $y_0 = (2 \cdot p)^2$. Of course, we could choose q to be odd numbers other than 1 (as long as $\gcd(p, q) = 1$), which would generate addition limit cycles of a desired period, but here we will use $q = 1$ consistently.

In Figure 6.1 we plot several limit cycles of higher period. In particular, we have plotted the following cases: $p = 5$ generates a two cycle; $p = 7$ generates a three cycle; $p = 11$ generates a five cycle; and $p = 29$ generates a 14 cycle.

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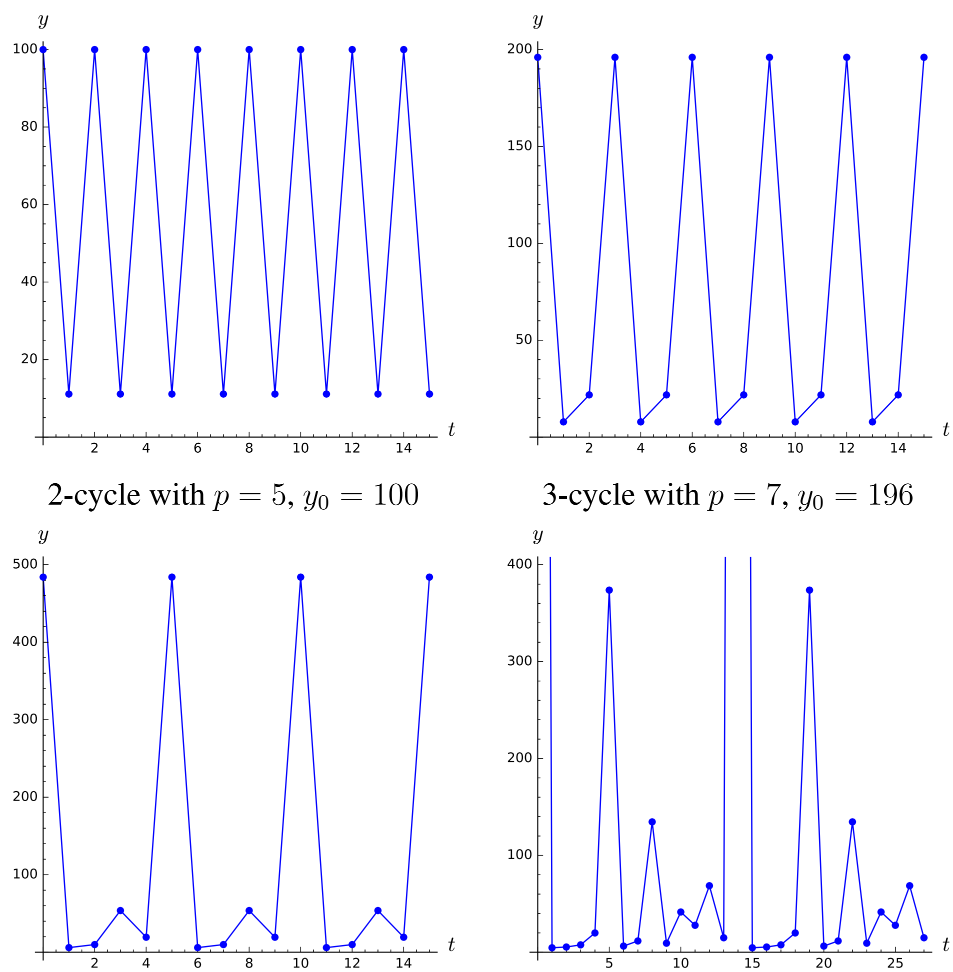
References

[1] R. Agarwal, M. Bohner, D. O'Regan, and A. Peterson, Dynamic equations on time scales: a survey, *J. Comput. Appl. Math.*, 141:1-2 (2002), 1–26.

[2] C. Ahrendt, A. Wells, and T. Williams, A comparison of the behavior of solutions of the two logistic dynamic equations on time scales, *Commun. Appl. Anal.*, 18 (2014),



2-cycle with $p = 5$, $y_0 = 100$

3-cycle with $p = 7$, $y_0 = 196$

5-cycle with $p = 11$, $y_0 = 484$

14-cycle with $p = 29$, $y_0 = 3364$ (two periods shown)

Figure 6.1: Limit cycles generated using Theorem 6.2.

19–40.

[3] E. Akin-Bohner and M. Bohner, Miscellaneous Dynamic Equations, *Methods Appl. Anal.* 10:1 (2003), 11–30.

[4] M. Bohner and A. Peterson, *Dynamic equations on time scales: An introduction with applications*, Birkhäuser, Boston, Massachusetts, 2001.

[5] M. Bohner and A. Peterson (eds.), *Advances in dynamic equations on time scales*, Birkhäuser, Boston, Massachusetts, 2003.

[6] W. Kelley and A. Peterson, *Difference Equations: An Introduction with Applications*, 2nd ed., Academic Press, New York, 2001.

[7] R. J. Marks II, I. A. Gravagne, J. M. Davis, and J. J. DaCunha, Nonregressivity in switched linear circuits and mechanical systems, *Math. Comput. Model.*, 43 (2006), 1383–1392.

[8] I. S. Sokolnikoff, and T. M. Redheffer, *Mathematics of Physics and Modern Engineering*, McGraw-Hill, New York, 1958.