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Stability of Dynamical Systems on Time Scales

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Abstract

In this study, we develop general theorems that assume the existence of Lyapunov type functions and analyze boundedness and stability of the solutions of nonlinear dynamical systems, including Volterra integro dynamic equations on time scales. We also provide examples to show the application potential of our results.

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1 Introduction

In this paper, we develop new and general theorems to study the existence of solutions, boundedness of solutions, and stability of the zero solution of nonlinear dynamical systems on time scales. Our theorems assume the existence of Lyapunov type functions. The advantage of having general theorems is that they can be applied to a wide class of systems (see, for instance, [10] and Example 3.7 in this paper). For an extensive theory of stability and boundedness of the systems in the continuous case, we refer to [4] and [5].

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We begin this paper by introducing or reminding the readers of the definition of type I Lyapunov function on time scales and it derivatives based on the results of Peterson and Tisdell [11]. Throughout this paper we use the notation

$$[t_0, \infty)_{\mathbb{T}} =: [t_0, \infty) \cap \mathbb{T}.$$

We begin by considering the boundedness and uniqueness of solutions to the first-order dynamic equation

$$x^{\Delta} = f(t, x), \quad t \geq 0, \tag{1.1}$$

subject to the initial condition

$$x(t_0) = x_0, \quad t_0 \geq 0, \quad x_0 \in \mathbb{R}, \tag{1.2}$$

where here $x(t) \in \mathbb{R}^n$, $f : [0, \infty)_{\mathbb{T}} \times D \to \mathbb{R}^n$ where $D \subset \mathbb{R}^n$ and open and f is a continuous, nonlinear function and t is from a so-called time scale $\mathbb{T}$ (which is a nonempty closed subset of $\mathbb{R}$). Throughout the paper we assume $0 \in \mathbb{T}$ (for convenience) and that $f(t, 0) = 0$, for all t in the time scale interval $[0, \infty)_{\mathbb{T}}$, and call the zero function the trivial solution of (1.1). Equation (1.1) subject to (1.2) is known as an initial value problem (IVP) on time scales.

If $\mathbb{T} = \mathbb{R}$ then $x^{\Delta} = x'$ and (1.1), (1.2) become the following IVP for ordinary differential equations

$$x' = f(t, x), \quad t \geq 0, \tag{1.3}$$

$$x(t_0) = x_0, \quad t_0 \geq 0. \tag{1.4}$$

Recently, Raffoul [13] used Lyapunov-type functions to formulate some sufficient conditions that ensure all solutions to (1.3), (1.4) are bounded, while in a more classical setting, Hartman [6, Chapter 3] employed Lyapunov-type functions to prove that solutions to (1.3), (1.4) are unique.

Definition 1.1. We say $V : \mathbb{R}^n \to \mathbb{R}^+$ is a **"type I" Lyapunov function** on $\mathbb{R}^n$ provided

$$V(x) = \sum_{i=1}^{n} V_i(x_i) = V_1(x_1) + \ldots + V_n(x_n),$$

where each $V_i : \mathbb{R}^+ \to \mathbb{R}^+$ is continuously differentiable and $V_i(0) = 0$.

The following chain rule shall be very useful throughout the remainder of this paper. Its proof can be found in Keller [9] and Pötzsche [12]. See also Bohner and Peterson [2, Theorem 1.90].

Theorem 1.2. *Let $V : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose that $x : \mathbb{T} \to \mathbb{R}$ is delta differentiable. Then $V \circ x$ is delta differentiable and*

$$(V \circ x)^{\Delta}(t) = \left\{ \int_0^1 V'\left(x(t) + h\mu(t)x^{\Delta}(t)\right) dh \right\} x^{\Delta}(t).$$

Definition 1.3. $V : \mathbb{R}^n \to \mathbb{R}$ is a "type I" function when

$$V(x) = \sum_{i=1}^{n} V_i(x_i) = V_1(x_1) + \ldots + V_n(x_n),$$

where each $V_i : \mathbb{R} \to \mathbb{R}$ is continuously differentiable.

For illustration purposes, suppose that $V(x) = \sum_{i=1}^{n} a_i x_i^2$, for $x \in \mathbb{R}^n$ and $a_i > 0$, $i = 1, 2, \cdots, n$. For $x \in \mathbb{R}^n$, we define the associated weighted vector by

$$w(x) := \langle a_1 x_1, a_2 x_2, \cdots, a_n x_n \rangle.$$

Then

$$\dot{V}(t,x) = 2w(x) \cdot f(t,x) + \mu(t) w(f(t,x)) \cdot f(t,x).$$

In particular, if $V(x) = \|x\|^2 = \sum_{i=1}^{n} x_i^2$, then

$$\dot{V}(t,x) = 2x \cdot f(t,x) + \mu(t) \|f(t,x)\|^2. \tag{1.5}$$

Now assume that $V : \mathbb{R}^n \to \mathbb{R}$ is a type I function and x is a solution to (1.1). Consider

$$\begin{aligned}
(V \circ x)^{\Delta}(t) &= \sum_{i=1}^{n} (V_i \circ x_i)^{\Delta}(t) \\
&= \sum_{i=1}^{n} \left\{ \int_0^1 V_i'(x_i(t) + h\mu(t) x_i^{\Delta}(t)) dh \right\} x_i^{\Delta}(t) \\
&= \sum_{i=1}^{n} \left\{ \int_0^1 V_i' \left(x_i(t) + h\mu(t) f_i(t, x(t)) \right) dh \right\} f_i(t, x(t)) \\
&= \left\{ \int_0^1 \nabla V(x + h\mu(t) f(t,x)) dh \right\} f(t,x),
\end{aligned}$$

where $\nabla = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$ is the gradient operator. This motivates us to define $\dot{V} : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}$ by either of the following identities

$$\begin{aligned}
\dot{V}(t,x) &= \left\{ \int_0^1 \nabla V(x + h\mu(t) f(t,x)) dh \right\} f(t,x) \\
&= \sum_{i=1}^{n} \left\{ \int_0^1 V_i' \left(x_i + h\mu(t) f_i(t,x) \right) dh \right\} f_i(t,x) \\
&= \begin{cases} \displaystyle\sum_{i=1}^{n} \left\{ V_i(x_i + \mu(t) f_i(t,x)) - V_i(x_i) \right\} / \mu(t), & \text{when } \mu(t) \neq 0 \\ \\ \displaystyle\sum_{i=1}^{n} V_i'(x_i) f_i(t,x), & \text{when } \mu(t) = 0. \end{cases}
\end{aligned} \tag{1.6}$$

2 Boundedness

We begin this section by proving a new lemma that we need for our general theorems. The next lemma is new and necessary for the upcoming theorem.

Lemma 2.1. *Let the function* $p : [0,\infty)_{\mathbb{T}} \to \mathbb{R}$ *be regressive* ($1+\mu(t)p(t) \neq 0$)*. Then*

$$e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(\tau)} p(\tau)\Delta\tau\right) \leq \exp\left(\int_s^t |p(\tau)|\Delta\tau\right). \tag{2.1}$$

Proof. Assume $1+\mu(u)p(u) > 0$ and define $f : (-1,\infty) \to \mathbb{R}$ by

$$f(x) = x - \log(1+x).$$

Then we have $f(x) \geq 0$ for all $x > -1$. By definition, we have

$$\xi_{\mu(u)}\left(p(u)\right) = \begin{cases} p(u) & \text{if } \mu(u) = 0 \\ \dfrac{\log\left(1+\mu(u)p(u)\right)}{\mu(u)} & \text{if } \mu(u) > 0. \end{cases}$$

Thus for $\mu > 0$ and $1+\mu(u)p(u) > 0$ we have that

$$\begin{aligned} \xi_{\mu(u)}\left(p(u)\right) &= \frac{\log\left(1+\mu(u)p(u)\right)}{\mu(u)} \\ &= p(u) - \frac{f\left(\mu(u)p(u)\right)}{\mu(u)} \\ &\leq p(u). \end{aligned}$$

Thus

$$e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(\tau)} p(\tau)\Delta\tau\right) \leq \exp\left(\int_s^t |p(\tau)|\Delta\tau\right).$$

Next we assume $1+\mu(u)p(u) < 0$ and define $f : (-\infty,-1) \to \mathbb{R}$ by

$$f(x) = |x| - \log|1+x|.$$

Then we have $f(x) \geq 0$ for all $x < -1$. By [2, Theorem 2.44], we know that if $1+\mu(u)p(u) < 0$ on $\mathbb{T}^k$, then $e_p(t,s) = \alpha(t,t_0)(-1)^{n_t}$ for all $t \in [0,\infty)_{\mathbb{T}}$ where

$$\alpha(t,t_0) = \exp\left(\int_{t_0}^t \frac{\log\left(1+\mu(s)p(s)\right)}{\mu(s)}\Delta s\right) > 0$$

and

$$n_t = \begin{cases} |[t_0,t)| & \text{if } t \geq t_0 \\ |[t,t_0)| & \text{if } t < t_0. \end{cases}$$

That is

$$e_p(t, t_0) = \alpha(t, t_0).$$

Since

$$\begin{aligned}\frac{\log(1 + \mu(s)p(s))}{\mu(s)} &= |p(s)| - \frac{f(\mu(s)p(s))}{\mu(s)}\\ &\le |p(s)|,\end{aligned}$$

we have

$$\alpha(t, t_0) \le \exp\left(\int_{t_0}^{t} |p(s)|\Delta s\right).$$

That is

$$e_p(t, t_0) = \exp\left(\int_{t_0}^{t} |p(s)|\Delta s\right).$$

This completes the proof. □

Theorem 2.2. *Let the function* $h : [0, \infty)_{\mathbb{T}} \to [0, \infty)$ *be regressive* $(1 + \mu(t)h(t) \neq 0)$*. Given* $x_0 \neq 0$ *suppose there exists a "type I" function* $V(t, x)$ *and such that for* $t \in [t_0, \infty)_{\mathbb{T}}$

$$W_1(||x||) \le V(t, x)$$

and

$$\dot{V}(t, x) \le h(t)V(t, x) + g(t), \tag{2.2}$$

where $\dot{V}(t, x)$ *is given by* (1.6)*. Then all solutions of* (1.1) *are bounded provided that* $\int_0^{\infty} |h(t)|\Delta t \le M_1$ *and* $\int_0^{\infty} |g(t)|\Delta t \le M_2$ *for positive constants* M_1 *and* M_2*.*

Proof. Using Lemma 2.1 and an application of the variation of parameters formula on (2.2) give

$$\begin{aligned}V(t, x) &\le V(t_0, x_0)e_h(t, t_0) + \int_{t_0}^{t} e_h(t, \sigma(\tau))g(\tau)\Delta\tau\\ &\le \exp(M_1)\left[V(t_0, x_0) + \int_{t_0}^{t} |g(t)|\Delta t\right]\\ &\le \exp(M_1)\left[V(t_0, x_0) + M_2\right],\end{aligned}$$

since $e_h(t, \sigma(s)) \le \exp\left(\int_{\sigma(s)}^{t} |h(s)|\Delta s\right)$. Thus from the above inequality we arrive at

$$||x|| \le \left[W_1^{-1} \exp(M_1)\left(V(t_0, x_0) + M_2\right)\right]^{1/2}. \tag{2.3}$$

This completes the proof. □

Remark 2.3. It is clear from (2.3) that if $V(t_0, x_0) \leq C(|x_0|)$ where C is a positive constant independent of t_0, then solutions of (1.1) are uniformly bounded.

We have the following application.

Example 2.4. Consider the scalar dynamical equation

$$x^{\Delta}(t) = a(t)x(t) + f(t), \;\; x(t_0) = x_0, \; t \in [t_0, \infty)_{\mathbb{T}} \text{ with } t_0 \geq 0. \tag{2.4}$$

For the rest of the example we write a and f; we suppress t in both functions. Let $V(t, x) = x^2$. By (1.5) we have along the solutions of (2.4) that

$$\dot{V}(t, x) = (2a + \mu(t)a)\, x^2 + (1 + \mu(t)a)\, 2xf + \mu(t)f^2. \tag{2.5}$$

We notice that

$$\begin{aligned}(1 + \mu(t)a)\, 2xf &\leq |\, (1 + \mu(t)a)\, |2|x||f|^{1/2}|f|^{1/2} \\ &\leq |1 + \mu(t)a| \left(x^2|f| + |f|\right).\end{aligned}$$

Thus, (2.5) reduces to

$$\dot{V}(t, x) = (2a + \mu(t)a + |1 + \mu(t)a||f|)\, x^2 + |1 + \mu(t)a||f| + \mu(t)f^2. \tag{2.6}$$

Let

$$h(t) := 2a + \mu(t)a + |1 + \mu(t)a||f| \;\text{ and }\; g(t) := |1 + \mu(t)a||f| + \mu(t)f^2,$$

then (2.6) is equivalent to

$$\dot{V}(t, x) \leq h(t)V(t, x) + g(t).$$

If we let

$$a(t) = f(t) = \frac{1}{(t + 1)^3},$$

then we always have

$$h(t) = \frac{3}{(t + 1)^3} + \frac{2\mu(t)}{(t + 1)^3} + \frac{\mu(t)}{(t + 1)^6} \text{ and } g(t) = \frac{1}{(t + 1)^3} + \frac{2\mu(t)}{(t + 1)^6}. \tag{2.7}$$

1. If $\mathbb{T} = \mathbb{R}$, we have $\mu(t) = 0$ and hence $h(t) = \dfrac{3}{(t + 1)^2}$ and $g(t) = \dfrac{1}{(t + 1)^2}$. Moreover, $\displaystyle\int_0^\infty |h(t)|\Delta t = 3$ and $\displaystyle\int_0^\infty |g(t)|\Delta t = 1$ and by Theorem 2.2 all solutions of (2.4) are bounded.

2. If $\mathbb{T} = \mathbb{Z}$, we have $\mu(t) = 1$ and hence

$$\int_0^\infty |h(t)|\Delta t = \sum_{k=0}^\infty \left[\frac{5}{(k+1)^3} + \frac{1}{(k+1)^6}\right] = 5\zeta(3) + \frac{\pi^6}{945} < \infty,$$

and

$$\int_0^\infty |g(t)|\Delta t = \sum_{k=0}^\infty \left[\frac{1}{(k+1)^3} + \frac{2}{(k+1)^6}\right] = \zeta(3) + \frac{2\pi^6}{945} < \infty,$$

where $\zeta(s)$ is Riemann zeta function. By Theorem 2.2 all solutions of (2.4) are bounded.

3. If $\mathbb{T} = \mathbb{P}_{1,1} = \bigcup_{k\in\mathbb{Z}}[2k, 2k+1]$, then

$$\sigma(t) = \begin{cases} t+1 & \text{if } t \in \bigcup_{k\in\mathbb{Z}}\{2k+1\} \\ t & \text{if } t \in \bigcup_{k\in\mathbb{Z}}[2k, 2k+1) \end{cases}$$

and hence

$$\mu(t) = \begin{cases} 1 & \text{if } t \in \bigcup_{k\in\mathbb{Z}}\{2k+1\} \\ 0 & \text{if } t \in \bigcup_{k\in\mathbb{Z}}[2k, 2k+1). \end{cases}$$

For any $\xi : \mathbb{P}_{1,1} \to \mathbb{R}$ we have

$$\int_t^{\sigma(t)} \xi(t)\Delta t = \begin{cases} \xi(t) & \text{if } t \in \bigcup_{k\in\mathbb{Z}}\{2k+1\} \\ 0 & \text{if } t \in \bigcup_{k\in\mathbb{Z}}[2k, 2k+1). \end{cases}$$

By (2.7),

$$h(t) = \frac{3}{(t+1)^3} + \frac{2\mu(t)}{(t+1)^3} + \frac{\mu(t)}{(t+1)^6} \text{ and } g(t) = \frac{1}{(t+1)^3} + \frac{2\mu(t)}{(t+1)^6}.$$

Then

$$\int_0^\infty |h(t)|\,\Delta t = \sum_{k=0}^\infty \int_{2k}^{2k+1} \frac{3}{(t+1)^3}\Delta t + \sum_{k=0}^\infty \int_{2k+1}^{2k+2} \frac{3}{(t+1)^3}\Delta t$$
$$+ \sum_{k=0}^\infty \int_{2k+1}^{2k+2} \left[\frac{2}{(t+1)^3} + \frac{1}{(t+1)^6}\right]\Delta t$$

$$= \frac{3}{4}\sum_{k=0}^{\infty}\left[\frac{1}{(2k+1)^4} - \frac{1}{(2k+2)^4}\right] + \sum_{k=0}^{\infty}\frac{5}{(2k+2)^3}\Delta t$$
$$+ \sum_{k=0}^{\infty}\frac{1}{(2k+2)^6}$$
$$= \frac{7\pi^4}{960} + \frac{5\zeta(3)}{8} + \frac{\pi^6}{60480} < \infty$$

and

$$\int_0^{\infty}|g(t)|\,\Delta t = \sum_{k=0}^{\infty}\int_{2k}^{2k+1}\frac{1}{(t+1)^3}\Delta t + \sum_{k=0}^{\infty}\int_{2k+1}^{2k+2}\frac{1}{(t+1)^3}\Delta t$$
$$+ \sum_{k=0}^{\infty}\int_{2k+1}^{2k+2}\frac{2}{(t+1)^6}\Delta t$$
$$= \frac{1}{4}\sum_{k=0}^{\infty}\left[\frac{1}{(2k+1)^4} - \frac{1}{(2k+2)^4}\right] + \sum_{k=0}^{\infty}\frac{1}{(2k+2)^3}$$
$$+ \sum_{k=0}^{\infty}\frac{2}{(2k+2)^6}$$
$$= \frac{7\pi^4}{960} + \frac{\zeta(3)}{8} + \frac{\pi^6}{30240} < \infty$$

where $\zeta(s)$ is Riemann zeta function. By Theorem 2.2 all solutions of (2.4) are bounded.

4. If $\mathbb{T} = q^{\mathbb{Z}} = \{q^n : n \in \mathbb{Z}$ and $q > 1$ is constant$\}$, then $\mu(t) = (q-1)t$. By (2.7) we have

$$\int_0^{\infty}|h(t)|\,\Delta t = \sum_{k=0}^{\infty}(q-1)q^k\left[\frac{3}{(q^k+1)^3} + \frac{2(q-1)q^k}{(q^k+1)^3} + \frac{(q-1)q^k}{(q^k+1)^6}\right] < \infty$$

and

$$\int_0^{\infty}|g(t)|\,\Delta t = \sum_{k=0}^{\infty}(q-1)q^k\left[\frac{1}{(q^k+1)^3} + \frac{2(q-1)q^k}{(q^k+1)^6}\right] < \infty.$$

By Theorem 2.2 all solutions of (2.4) are bounded.

We let $x(t) = x(t, t_0, x_0)$ denote a solution of the initial value problem (IVP) (1.1), with

$$x(t_0) = x_0, \quad t_0 \geq 0, \quad x_0 \in \mathbb{R}.$$

Next, we give a formula for the Δ derivative of $|x|^{\Delta}$, that we use throughout this paper. Details can be found in [1]. One can easily find $\frac{d}{dt}|x(t)| = \frac{x(t)}{|x(t)|}x'(t)$ by using the equation $x^2(t) = |x(t)|^2$ and the product rule in real case. However, on time scales, using the product rule $(fg)^\Delta = f^\Delta g^\sigma + fg^\Delta$ with $f(t) = g(t) = |x(t)|$, we have

$$|x|^\Delta = \frac{x + x^\sigma}{|x| + |x^\sigma|} x^\Delta \text{ for } x \neq 0. \tag{2.8}$$

That is, the coefficient of x^Δ in (2.8) depends not only on the sign of $x(t)$ but also on that of $x^\sigma(t)$. Therefore, the equality $|x|^\Delta = \frac{x}{|x|} x^\Delta$ holds only if $xx^\sigma \geq 0$ and $x \neq 0$. Let us keep this case distinct from the case $xx^\sigma < 0$ by separating the time scale $\mathbb{T}$ into two parts as follows (see also ([1]))

$$\mathbb{T}_- : = \{s \in \mathbb{T} : x(s)\, x^\sigma(s) < 0\},$$
$$\mathbb{T}_+ : = \{s \in \mathbb{T} : x(s)\, x^\sigma(s) \geq 0\}.$$

Note that the set $\mathbb{T}_-$ consists only of right scattered points of $\mathbb{T}$. To see the relation between $|x|^\Delta$ and $\frac{x}{|x|} x^\Delta$ we prove the next result.

Lemma 2.5 (See [1, Lemma 5]). *Let $x \neq 0$ be Δ-differentiable. Then*

$$|x(t)|^\Delta = \begin{cases} \dfrac{x(t)}{|x(t)|} x^\Delta(t) & \text{if } t \in \mathbb{T}_+ \\ -\dfrac{2}{\mu(t)} |x(t)| - \dfrac{x(t)}{|x(t)|} x^\Delta(t) & \text{for } t \in \mathbb{T}_- \end{cases}.$$

In particular,

$$\frac{x}{|x|} x^\Delta \leq |x|^\Delta \leq -\frac{x}{|x|} x^\Delta(t) \text{ for all } t \in \mathbb{T}_-. \tag{2.9}$$

For $x \in \mathbb{R}^n$, $||x||$ denotes the Euclidean norm of x. For any $n \times n$ matrix A, define the norm of A by $|A| = \sup\{|Ax| : ||x|| \leq 1\}$.

Definition 2.6. A solution x of (1.1) is said to be bounded if for any $t_0 \in [0, \infty)$ and number r there exists a number $\alpha(t_0, r)$ depending on t_0 and r such that $||x(t, t_0, x_0)|| \leq \alpha(t_0, r)$ for all $t \geq t_0$ and x_0, $|x_0| < r$. It is uniformly bounded if α is independent of the initial time t_0.

Definition 2.7. Let x be solution of (1.1) with respect to initial condition x_0 and y be solution of (1.1) with respect to initial condition y_0. The solution x is then said to be stable, if, whenever $\epsilon > 0$ is given, there exists $\delta(\epsilon) > 0$ for which

$$||x(t) - y(t)|| < \epsilon, \quad \text{whenever} \quad ||x_0 - y_0|| < \delta.$$

Let the time scale be the set of nonnegative integers and consider the linear difference equation

$$x^\Delta = 1, \; x(t_0) = x_0. \tag{2.10}$$

It is easy to check that $x(t) = x_0 + (t - t_0)$ is the solution of (2.10). If y is another solution with $y(t_0) = y_0$, then we have $y(t) = y_0 + (t - t_0)$. For any $\epsilon > 0$, let $\delta = \epsilon$. Then

$$||x(t) - y(t)|| = ||x_0 + (t - t_0) - y_0 - (t - t_0)|| = ||x_0 - y_0)|| < \epsilon,$$

whenever $||x_0 - y_0|| < \delta$. Hence, the solution x is stable, but unbounded. This simple example shows that the properties of boundedness of all solutions and stability of a solution do not coincide.

Hereafter, we denote wedges by W_i, $i = 1, 2, 3, ...$, such that $W_i[0, \infty)_{\mathbb{T}} \to [0, \infty)$ be continuous with $W_i(0) = 0,\ W_i(r)$ strictly increasing, and $W_i(r) \to \infty$ as $r \to \infty$.

Theorem 2.8. *Suppose there exists a "type I" function $V(t, x)$ and W_1 such that*

$$W_1(||x||) \le V(t, x),\ t \ge t_0 \tag{2.11}$$

$$\dot{V}(t, x) \le 0, \tag{2.12}$$

where $\dot{V}(t, x)$ is given by (1.6) *and*

$$W_1(||x||) \to \infty,\ \text{as } ||x|| \to \infty. \tag{2.13}$$

Assume for any initial time t_0 with $x(t_0) = x_0$, and $V(t_0, x_0)$ is bounded, then solutions of (1.1) *are bounded.*

Proof. Let r_0 be any positive constant such that $||x_0|| \le r_0$. By (2.13) and since $V(t_0, x_0)$ is bounded, there exists a function $\alpha(t_0, r_0)$ such that

$$V(t_0, x_0) \le W_1(\alpha(t_0, r_0)).$$

Utilizing conditions (2.11) and (2.12) we have

$$W_1(||x||) \le V(t, x) \le V(t_0, x_0) \le W_1(\alpha(t_0, r_0)). \tag{2.14}$$

Taking the inverse in (2.14) we arrive at $||x(t, t_0, x_0)|| \le \alpha(t_0, r_0)$. This completes the proof. □

For more on such results, we refer to [7].

Example 2.9. Consider the scalar nonlinear Volterra integro-dynamic equation

$$x^{\Delta}(t) = a(t)x(t) + b(t)\frac{x(t)}{1 + \int_0^t x^2(s)\Delta s},\ t \in [0, \infty)_{\mathbb{T}}. \tag{2.15}$$

Let the function $\alpha : \mathbb{T} \to \mathbb{R}$ be defined by

$$\alpha(t) := \begin{cases} \left(-\frac{2}{\mu(t)} - a(t)\right) & \text{for } t \in [0, \infty)_{\mathbb{T}_-} \\ a(t) & \text{for } t \in [0, \infty)_{\mathbb{T}_+} \end{cases}. \tag{2.16}$$

If there exists a $\beta > 0$ such that

$$\alpha(t) + |b(t)| \leq -\beta \text{ for all } t \in [0, \infty)_{\mathbb{T}},$$

then all solutions of (2.15) are bounded. To see this, consider the Lyapunov function

$$V(t, x) = |x(t)|.$$

Then along the solutions of (2.15) we have by using Lemma 2.5

$$\begin{aligned}
\dot{V}(t, x) &= \frac{x}{|x|} x^{\Delta}(t) \\
&= \frac{x}{|x|} \left(a(t)x(t) + b(t) \frac{x(t)}{1 + \int_0^t x^2(s)\Delta s} \right) \\
&= \frac{x^2}{|x|} \left(a(t) + b(t) \frac{1}{1 + \int_0^t x^2(s)\Delta s} \right) \\
&\leq (a(t) + |b(t)|)\, |x(t)| \\
&= (\alpha(t) + |b(t)|)\, |x(t)| \\
&\leq -\beta |x(t)|
\end{aligned}$$

for $t \in [0, \infty)_{\mathbb{T}_+}$. On the other hand, for $t \in [0, \infty)_{\mathbb{T}_-}$, we have

$$\begin{aligned}
\dot{V}(t, x) &= -\frac{2}{\mu(t)} |x(t)| - \frac{x(t)}{|x(t)|} x^{\Delta}(t) \\
&= -\frac{2}{\mu(t)} |x(t)| - \frac{x}{|x|} \left(a(t)x(t) + b(t) \frac{x(t)}{1 + \int_0^t x^2(s)\Delta s} \right) \\
&= -\frac{2}{\mu(t)} |x(t)| - \frac{x^2}{|x|} \left(a(t) + b(t) \frac{1}{1 + \int_0^t x^2(s)\Delta s} \right) \\
&= |x(t)| \left\{ \left(-\frac{2}{\mu(t)} - a(t) \right) - b(t) \frac{1}{1 + \int_0^t x^2(s)\Delta s} \right\} \\
&\leq \left\{ \left(-\frac{2}{\mu(t)} - a(t) \right) + |b(t)| \right\} |x(t)| \\
&= (\alpha(t) + |b(t)|)\, |x(t)| \\
&\leq -\beta |x(t)|.
\end{aligned}$$

Clearly,

$$W_1(||x||) = ||x|| \to \infty, \text{ as } ||x|| \to \infty.$$

Then boundedness of the solutions of (2.15) follow from Theorem 2.8.

3 Stability

Definition 3.1. Let the function $V : [0,\infty)_{\mathbb{T}} \times D \to \mathbb{R}$, be right-dense continuous in x and $\dot{V}(t,x)$ with respect to (1.1) is given by (1.6).

1. The function V is said to be a Lyapunov function/functional on a subset H of $\mathbb{R}^n$ if

   i) $V(0) = 0$, and $V(x) > 0$, for $x \neq 0$ and

   ii) $\dot{V}(t,x) \leq 0$, whenever x and $f(x)$ belong to the set D.

2. The function V is said to be a strict Lyapunov function/functional on a subset D of $\mathbb{R}^n$ if $\dot{V}(t,x) < 0$.

Definition 3.2. A function $U : [0,\infty)_{\mathbb{T}} \times D \to [0,\infty)$ is called

1. positive definite if $U(t,0) = 0$ and if there is a wedge W_1 with $U(t,x) \geq W_1(|x|)$,
2. decrescent if there is a wedge W_2 with $U(t,x) \leq W_2(|x|)$,
3. negative definite if $-U(t,x)$ is positive definite,
4. radially unbounded if $D = \mathbb{R}^n$ and there is a wedge $W_3(|x|) \leq U(t,x)$ and $W_3(r) \to \infty$ as $r \to \infty$.

Next, we state some definitions regarding the stability of the zero solution of (1.1). For more on such theorems, we refer to [8].

Definition 3.3. The zero solution of (1.1) is stable **(S)** if for each $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$ such that $|x(t_0)| < \delta$ implies $|x(t,t_0,x_0)| < \varepsilon$. It is uniformly stable **(US)** if δ is independent of t_0.

Definition 3.4. The zero solution of (1.1) is asymptotically stable **(AS)** if its (S) and $|x(t,t_0,x_0)| \to 0$, as $t \to \infty$.

Definition 3.5. The zero solution of (1.1) is uniformly asymptotically stable **(UAS)** if it is (US) and there exists a $\gamma > 0$ with the property that for each $\mu > 0$ there exists $T > 0$ such that $\left[t_0 \geq 0, \text{ with } |x_0| < \gamma, t \geq t_0 + T\right]$ implies $|x(t,t_0,\phi)| < \mu$.

Theorem 3.6. *Assume $f(t,0) = 0$ and there is a Lyapunov function V for* (1.1) *(see Definition 3.1.)*

1. *If V is positive definite, then $x = 0$ is stable.*
2. *If V is positive definite and decrescent, then $x = 0$ is uniformly stable.*

3. *If V is positive definite and decrescent, and $\dot{V}(t,x)$ is negative definite, then $x = 0$ is uniformly asymptotically stable .*

4. *If $D = \mathbb{R}^k$ and if V is radially unbounded, then all solutions of* (1.1) *are bounded.*

Proof. 1. We have $\dot{V}(t,x) \leq 0$, V is right-dense continuous in x, $V(t,0) = 0$, and $W_1(|x|) \leq V(t,x)$. Let $\epsilon > 0$ and $t_0 \geq 0$ be given. We must find δ such that $|x_0| < \delta$ and $t \geq t_0$ imply $|x(t,t_0,x_0)| < \epsilon$. (Throughout these proofs we assume ϵ is small enough so that $|x(t,t_0,x_0)| < \epsilon$ implies that $x \in D$.) As V is right-dense continuous in x and $V(t,0) = 0$ there is a $\delta > 0$ such that $|x_0| < \delta$ implies $V(t_0,x_0) < W_1(\epsilon)$. Thus, if $t \geq t_0$ and $|x_0| < \delta$ and $x = x(t,t_0,x_0)$, we have

$$W_1(|x(t)|) \leq V(t,x) \leq V(t_0,x_0) < W_1(\epsilon),$$

or $|x(t)| < \epsilon$ as required.

2. For a given ϵ we select a $\delta > 0$ such that $W_2(\delta) < W_1(\epsilon)$ where $W_1(|x|) \leq V(t,x) \leq W_2(|x|)$. If $t_0 \geq 0$, we have

$$\begin{aligned} W_1(|x(t)|) \leq V(t,x) &\leq V(t_0,x_0) \\ &\leq W_2(|x_0|) < W_2(\delta) < W_1(\epsilon), \end{aligned}$$

or $|x(t)| < \epsilon$ as required.

3. Let $\epsilon = 1$, and find δ of uniform stability and call it η. Let γ be given. We must find $T > 0$ such that

$$|x_0| < \eta, \quad n_0 \geq 0, \quad \text{and } t \geq t_0 + T$$

imply $|x(t,t_0,x_0)| < \gamma$. Pick $\mu > 0$ with $W_2(\mu) < W_1(\gamma)$, so that there is $t_1 \geq n_0$ with $|x(t_1)| < \mu$, then, for $n \geq t_1$, we have

$$\begin{aligned} W_1(|x(t)|) \leq V(t,x) &\leq V(t_1,x_1) \\ &\leq W_2(|x_1|) < W_2(\delta) < W_1(\gamma), \end{aligned}$$

or $|x(t_1)| < \gamma$. Since $\dot{V}(t,x) \leq -W_3(|x|)$, so as long as $|x(t)| > \mu$, then $\triangle V(t,x) \leq -W_3(\mu)$; thus

$$\begin{aligned} V(t,x(t)) &\leq V(t_0,x_0) - \int_{t_0}^{t} W_3(|x(s)|)\Delta s \\ &\leq W_2(|x_0|) - W_3(\mu)(t-t_0) \\ &\leq W_2(\eta) - W_3(\mu)(t-t_0), \end{aligned}$$

which vanishes at

$$t = t_0 + \frac{W_2(\eta)}{W_3(\mu)} \geq t_0 + T,$$

where $T \geq \frac{W_2(\eta)}{W_3(\mu)}$. Hence, if $T > \frac{W_2(\eta)}{W_3(\mu)}$, then $|x(t)| > \mu$ fails, and we have $|x(t)| < \gamma$ for all $t \geq t_0 + T$. This proves (UAS).

4. Since V is radially unbounded, we have $V(t,x) \geq W_1(|x|) \to \infty$ as $|x| \to \infty$. Thus, given $t_0 \geq 0$, and x_0, there is an $r > 0$ with $W_1(r) > V(n_0, x_0)$. Hence, if $t \geq t_0$ and $x(t) = x(t, t_0, x_0)$, then

$$W_1(|x(t)|) \leq V(t, x(t)) \leq V(t_0, x_0) < W_1(r),$$

or $|x(n)| < r$. The proof of Theorem 3.6 is complete.

□

Example 3.7. According to Theorem 3.6, all solutions of (2.15) are bounded and its zero solution is (UAS). However, UAS of the zero solution of (2.15) cannot be obtained from [10, Theorem 3.3] since (2.15) does not satisfy the condition (4) in [10, Theorem 3.3].

In the next example, we show that $\dot{V}(t,x) \leq 0$ is not enough to drive solutions to zero. The continuous version of the next example can be found in [5, page 230]. However, obtaining a time scale version that is valid on all time scales is quite challenging. In preparation for the next example, we prove the following lemma.

Lemma 3.8. *Let $\mathbb{T}$ be any time scale and $g : \mathbb{T} \to \mathbb{R}$ be Δ-differentiable with $g(t) \neq 0$ for all $t \in \mathbb{T}$ and the function $p : [0,\infty)_{\mathbb{T}} \to \mathbb{R}$, defined by*

$$p(t) := 2\frac{g^{\Delta}(t)}{g(t)} + \mu(t)\left(\frac{g^{\Delta}(t)}{g(t)}\right)^2, \tag{3.1}$$

be regressive. Then

$$e_p(t, t_0) = \frac{g^2(t)}{g^2(t)}.$$

Proof. Let $y(t) = g^2(t)/g^2(t_0)$. Then by using $g^{\sigma}(t) = g(t) + \mu(t)g^{\Delta}(t)$, we have

$$\begin{aligned} y^{\Delta}(t) &= \frac{1}{g^2(t_0)}\left[g(t) + g^{\sigma}(t)\right] g^{\Delta}(t) \\ &= \frac{1}{g^2(t_0)}\left[g(t) + g(t) + \mu(t)g^{\Delta}(t)\right] g^{\Delta}(t) \\ &= \left[2\frac{g^{\Delta}(t)}{g(t)} + \mu(t)\left(\frac{g^{\Delta}(t)}{g(t)}\right)^2\right] \frac{g^2(t)}{g^2(t_0)} \end{aligned}$$

$$= p(t)y(t).$$

The proof follows from the uniqueness of the solution $e_p(t,t_0)$ of the following initial value problem

$$y^{\Delta} = p(t)y, \;\; y(t_0) = 1$$

(see [2, Theorem 2.33]). □

Example 3.9. Let $g : [0,\infty)_{\mathbb{T}} \to (0,1]$ be a Δ-differentiable function with $g(0) = 1$ and $\int_0^{\infty} g(s)\Delta s < \infty$. Suppose that the function $p(t)$ defined by (3.1) is regressive. We wish to construct a function $V(t,x) = a(t)x^2$ with $a(t) > 0$ and with the derivative of V along any solutions of

$$x^{\Delta} = \left[\frac{g^{\Delta}(t)}{g(t)}\right] x \tag{3.2}$$

satisfying

$$\dot{V}(t,x) = -x^2.$$

We shall, thereby, see that $V \geq 0$ and $\dot{V}$ negative definite do not imply that solutions tend to zero, because $x(t) = g(t)$ is a solution of (3.2). To this end, we compute

$$\begin{aligned}\dot{V}(t,x) &= a^{\sigma}(t)\left[x(t) + x^{\sigma}(t)\right] x^{\Delta}(t) + a^{\Delta}(t)x^2(t) \\ &= \left[a^{\sigma}(t)\left(2\frac{g^{\Delta}(t)}{g(t)} + \mu(t)\left(\frac{g^{\Delta}(t)}{g(t)}\right)^2\right) + a^{\Delta}(t)\right] x^2(t)\end{aligned}$$

where we used (3.2) and $x^{\sigma}(t) = x(t) + \mu(t)x^{\Delta}(t)$. Setting

$$\dot{V}(t,x) = -x^2(t)$$

we have

$$a^{\Delta}(t) = -a^{\sigma}(t)\left(2\frac{g^{\Delta}(t)}{g(t)} + \mu(t)\left(\frac{g^{\Delta}(t)}{g(t)}\right)^2\right) - 1.$$

Using the variation of parameters formula and Lemma 3.8, the equality

$$e_{\ominus p}(t,s) = \left(e_p(t,s)\right)^{-1},$$

we get

$$\begin{aligned}a(t) &= e_{\ominus p}(t,0)a(0) - \int_0^t e_{\ominus p}(t,s)\Delta s \\ &= \frac{a(0)}{g^2(t)} - \int_0^t \frac{g^2(s)}{g^2(t)}\Delta s,\end{aligned}$$

where $p(t)$ is given by (3.1). Since $0 < g(t) \leq 1$ and g is in $L^1[0,\infty)_{\mathbb{T}}$, we have $\int_0^t g^2(s)\,\Delta s < \infty$ (see [3, Theorem 5.52]). That means, we may choose $a(0)$ large enough so that $a(t) > 1$ on $[0,\infty)_{\mathbb{T}}$. Thus we have shown that $V \geq 0$ and $\dot{V}(t,x)$ is negative definite do not imply that solutions tend to zero. Notice that V is not decrescent. That is, there is no wedge W_2 with $V(t,x) \leq W_2(|x|)$. In the next theorem we obtain necessary conditions for the instability of the zero solution of (1.1).

Definition 3.10. The zero solution of (1.1) is unstable if there is an $\varepsilon > 0$, and $t_0 \geq 0$, such that for any $\delta > 0$ there is an x_0 with $|x_0| < \delta$ and there is an $t_1 > t_0$ such that $|x(t_1,t_0,x_0)| \geq \varepsilon$.

Theorem 3.11. *Let the function* $V : [0,\infty)_{\mathbb{T}} \times D \to \mathbb{R}$, *be right-dense continuous in* x *and* $\dot{V}(t,x)$ *with respect to* (1.1) *is given by* (1.6) *which is locally Lipschitz in* x *such that*

$$W_1(|x|) \leq V(t,x) \leq W_2(|x|), \tag{3.3}$$

and along the solutions of (1.1) *we have*

$$\dot{V}(t,x) \geq W_3(|x|). \tag{3.4}$$

Then the zero solution of (1.1) *is unstable.*

Proof. Suppose not, then for $\epsilon = \min\{1, d(0,\partial D)\}$ we can find a $\delta > 0$ such that $|x_0| < \delta$ and $t \geq 0$ imply that $|x(t,0,x_0)| < \epsilon$. We may pick x_0 in such a way so that $|x_0| = \delta/2$ and find $\gamma > 0$ with $W_2(\gamma) = W_1(\delta/2)$. Then for $x(t) = x(t,0,x_0)$ we have $\dot{V}(t,x) \geq 0$ so that

$$W_2(|x(t)|) \geq V(t,x(t)) \geq V(0,x_0) \geq W_1(\delta/2) = W_2(\gamma)$$

from which we conclude that $\gamma \leq |x(t)|$ for $t \geq 0$. Thus

$$\dot{V}(t,x) \geq W_3(|x(t)|) \geq W_3(\gamma).$$

Thus,

$$W_2(|x(t)|) \geq V(t,x(t)) \geq V(0,x_0) + tW_3(\gamma),$$

from which we conclude that $|x(t)| \to \infty$, which is a contradiction. This completes the proof. □

In the next example, we show that stability or instability is inherently dependent on the chosen time scale.

Example 3.12. Let $\mathbb{T}$ be any time scale that includes 0 and consider the autonomous system

$$x^{\Delta} = -x - xy$$

$$y^{\Delta} = -y + x^2 \tag{3.5}$$

for $t \in [0, \infty)_{\mathbb{T}}$. Let

$$V(x, y) = x^2 + y^2.$$

Then we have along the solutions of (3.5) that

$$\begin{aligned} \dot{V}(t, x) &= 2x \cdot f(t, x) + \mu(t)\|f(t, x)\|^2 \\ &= 2x(-x - xy) + 2y(-y + x^2) \\ &+ \mu(t)\left(-x - xy\right)^2 + \mu(t)\left(-y + x^2\right)^2 \\ &= -2(x^2 + y^2) + \mu(t)\left(x^2 + y^2 + x^2y^2 + x^4\right). \end{aligned} \tag{3.6}$$

Thus, for $\mathbb{T} = \mathbb{R}$ we have $\mu(t) = 0$ and from the above inequality we see that the zero solution of (3.5) is (UAS) by Theorem 3.6. On the other hand, If $\mathbb{T} = \mathbb{Z}$ then $\mu(t) = 1$ and

$$\begin{aligned} \dot{V}(t, x) &= -(x^2 + y^2) + x^2(x^2 + y^2) \\ &\geq (x^2 + y^2)(x^2 - 1) > 0 \end{aligned}$$

on the set

$$D = \{(x, y) \in \mathbb{R}^2 : |x| > 1\}.$$

Thus, the zero solution is unstable by Theorem 3.11.

Next we let $\mathbb{T} = h\mathbb{N}_0 = \{0, 2h, \ldots\}$, then $\mu(t) = h$ and in this case system (3.5) takes the form

$$\begin{aligned} \frac{\Delta x(k)}{h} &= -x(k) - x(k)y(k) \\ \frac{\Delta y(k)}{h} &= -y(k) + x^2(k), \end{aligned}$$

$k = 0, h, 2h, \cdots$. Then for $h \geq 2$, we have from (3.6) that

$$\begin{aligned} \dot{V}(t, x) &= -2(x^2 + y^2) + h(x^2 + y^2 + x^2y^2 + x^4) \\ &\geq x^2y^2 + x^4 \geq 0 \end{aligned}$$

and hence the zero solution is unstable by Theorem 3.11.

Theorem 3.13. *Assume the function $a(t)$ is right-dense continuous and that $a(t) > 0$ for all $t \in [0, \infty)_{\mathbb{T}}$ and consider the two dimensional system*

$$\begin{aligned} x^{\Delta} &= -a(t)x + a(t)y \\ y^{\Delta} &= -a(t)x - a(t)y. \end{aligned} \tag{3.7}$$

Let

$$1 - \mu(t)a(t) > 0, \tag{3.8}$$

then $(0, 0)$ of (3.7) is (UAS).

Proof. Let

$$V(x,y) = x^2 + y^2.$$

Then we have along the solutions of (3.7) that

$$\begin{aligned}\dot{V}(t,x) &= 2x \cdot f(t,x) + \mu(t)\|f(t,x)\|^2 \\ &= 2x(-a(t)x + a(t)y) + 2y(-a(t)x - a(t)y) \\ &+ \mu(t)\big(-a(t)x + a(t)y\big)^2 + \mu(t)\big(-a(t)x - a(t)y\big)^2 \\ &= -2a(t)(x^2+y^2) + 2\mu(t)\big(a^2(t)x^2 + a^2(t)y^2\big) \\ &= -2a(t)(x^2+y^2)\big[1 - \mu(t)a(t)\big].\end{aligned}$$

Thus, by Theorem 3.6, $(0,0)$ of (3.7) is (UAS). □

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