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Homogeneity of Classical and Dynamic Inequalities Compatible on Time Scales

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Abstract

In this paper, we present an extension of dynamic Renyi's inequality on time scales. Furthermore, we find generalizations of classical Rogers–Hölder's inequality and Radon's inequality and their reverse versions on time scales. Our investigations unify and extend some continuous inequalities and their corresponding discrete analogues.

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1 Introduction

We introduce here some well-known classical inequalities.

If $x_i \geq 0$, $y_i \geq 0$ for $i = 1, 2, \ldots, n$ and $\lambda > 0$ with $\lambda \neq 1$, then

$$\frac{1}{\lambda - 1}\left(\sum_{i=1}^{n} x_i\right)^{\lambda}\left(\sum_{i=1}^{n} y_i\right)^{1-\lambda} \leq \frac{1}{\lambda - 1}\sum_{i=1}^{n} x_i^{\lambda} y_i^{1-\lambda}. \tag{1.1}$$

The inequality from (1.1) is called, in literature, Renyi's inequality as given in [13].

If $x_i \geq 0$, $y_i \geq 0$ for $i = 1, 2, \ldots, n$ and $\lambda_1 > 0$, $\lambda_2 > 0$ with $\lambda_1 + \lambda_2 \leq 1$, then

$$\sum_{i=1}^{n} x_i^{\lambda_1} y_i^{\lambda_2} \leq n^{1-\lambda_1-\lambda_2}\left(\sum_{i=1}^{n} x_i\right)^{\lambda_1}\left(\sum_{i=1}^{n} y_i\right)^{\lambda_2}. \tag{1.2}$$

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The inequality from (1.2) is an application of Rogers–Hölder's inequality as given in [13]. The inequality of (1.2) is reversed if $\lambda_1 < 0$ and $\lambda_2 < 0$.

If $x_i \geq 0$, $y_i > 0$ for $i = 1, 2, \ldots, n$ and $\lambda > 0$, then

$$\frac{\left(\sum\limits_{i=1}^{n} x_i\right)^{\lambda+1}}{\left(\sum\limits_{i=1}^{n} y_i\right)^{\lambda}} \leq \sum_{i=1}^{n} \frac{x_i^{\lambda+1}}{y_i^{\lambda}}. \tag{1.3}$$

The inequality from (1.3) is called, in literature, Radon's inequality as given in [15].

We will unify and extend these results on time scales. The calculus of time scales was initiated by Stefan Hilger as given in [11]. A time scale is an arbitrary nonempty closed subset of the real numbers. The theory of time scales is applied to combine results in one comprehensive form. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus, i.e., when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$ where $q > 1$. The time scales calculus is studied as delta calculus, nabla calculus and diamond-α calculus. This hybrid theory is also widely applied on dynamic inequalities. The basic work on dynamic inequalities is done by Ravi Agarwal, George Anastassiou, Martin Bohner, Allan Peterson, Donal O'Regan, Samir Saker and many other authors.

In this paper, it is assumed that all considerable integrals exist and are finite and $\mathbb{T}$ is a time scale, $a, b \in \mathbb{T}$ with $a < b$ and an interval $[a, b]_{\mathbb{T}}$ means the intersection of a real interval with the given time scale.

2 Preliminaries

We need here basic concepts of delta calculus. The results of delta calculus are adapted from monographs [7, 8].

For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}.$$

The mapping $\mu : \mathbb{T} \to \mathbb{R}_0^+ = [0, +\infty)$ such that $\mu(t) := \sigma(t) - t$ is called the forward graininess function. The backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

The mapping $\nu : \mathbb{T} \to \mathbb{R}_0^+ = [0, +\infty)$ such that $\nu(t) := t - \rho(t)$ is called the backward graininess function. If $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$, we say that t is left-scattered. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. If $\mathbb{T}$ has a left-scattered maximum M, then $\mathbb{T}^k = \mathbb{T} - \{M\}$, otherwise $\mathbb{T}^k = \mathbb{T}$.

For a function $f : \mathbb{T} \to \mathbb{R}$, the delta derivative f^{Δ} is defined as follows:

Let $t \in \mathbb{T}^k$. If there exists $f^{\Delta}(t) \in \mathbb{R}$ such that for all $\epsilon > 0$, there is a neighborhood U of t, such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s|,$$

for all $s \in U$, then f is said to be delta differentiable at t, and $f^{\Delta}(t)$ is called the delta derivative of f at t.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be right-dense continuous (rd-continuous), if it is continuous at each right-dense point and there exists a finite left-sided limit at every left-dense point. The set of all rd-continuous functions is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$.

The next definition is given in [7, 8].

Definition 2.1. A function $F : \mathbb{T} \to \mathbb{R}$ is called a delta antiderivative of $f : \mathbb{T} \to \mathbb{R}$, provided that $F^{\Delta}(t) = f(t)$ holds for all $t \in \mathbb{T}^k$. Then the delta integral of f is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a).$$

The following results of nabla calculus are taken from [3, 7, 8].

If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T} - \{m\}$, otherwise $\mathbb{T}_k = \mathbb{T}$. A function $f : \mathbb{T}_k \to \mathbb{R}$ is called nabla differentiable at $t \in \mathbb{T}_k$, with nabla derivative $f^{\nabla}(t)$, if there exists $f^{\nabla}(t) \in \mathbb{R}$ such that given any $\epsilon > 0$, there is a neighborhood V of t, such that

$$|f(\rho(t)) - f(s) - f^{\nabla}(t)(\rho(t) - s)| \leq \epsilon|\rho(t) - s|,$$

for all $s \in V$.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be left-dense continuous (ld-continuous), provided it is continuous at all left-dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$. The set of all ld-continuous functions is denoted by $C_{ld}(\mathbb{T}, \mathbb{R})$.

The next definition is given in [3, 7, 8].

Definition 2.2. A function $G : \mathbb{T} \to \mathbb{R}$ is called a nabla antiderivative of $g : \mathbb{T} \to \mathbb{R}$, provided that $G^{\nabla}(t) = g(t)$ holds for all $t \in \mathbb{T}_k$. Then the nabla integral of g is defined by

$$\int_a^b g(t)\nabla t = G(b) - G(a).$$

Now we present short introduction of the diamond-α derivative as given in [2, 19].

Definition 2.3. Let $\mathbb{T}$ be a time scale and $f(t)$ be differentiable on $\mathbb{T}$ in the Δ and ∇ senses. For $t \in \mathbb{T}_k^k$, where $\mathbb{T}_k^k = \mathbb{T}^k \cap \mathbb{T}_k$, the diamond-$\alpha$ dynamic derivative $f^{\diamond_\alpha}(t)$ is defined by

$$f^{\diamond_\alpha}(t) = \alpha f^{\Delta}(t) + (1-\alpha) f^{\nabla}(t), \quad 0 \le \alpha \le 1.$$

Thus f is diamond-α differentiable if and only if f is Δ and ∇ differentiable.

The diamond-α derivative reduces to the standard Δ-derivative for $\alpha = 1$, or the standard ∇-derivative for $\alpha = 0$. It represents a weighted dynamic derivative for $\alpha \in (0,1)$.

Theorem 2.4 (See [19]). *Let $f, g : \mathbb{T} \to \mathbb{R}$ be diamond-α differentiable at $t \in \mathbb{T}$ and we write $f^{\sigma}(t) = f(\sigma(t))$, $g^{\sigma}(t) = g(\sigma(t))$, $f^{\rho}(t) = f(\rho(t))$ and $g^{\rho}(t) = g(\rho(t))$. Then*

(i) $f \pm g : \mathbb{T} \to \mathbb{R}$ is diamond-α differentiable at $t \in \mathbb{T}$, with

$$(f \pm g)^{\diamond_\alpha}(t) = f^{\diamond_\alpha}(t) \pm g^{\diamond_\alpha}(t).$$

(ii) $fg : \mathbb{T} \to \mathbb{R}$ is diamond-α differentiable at $t \in \mathbb{T}$, with

$$(fg)^{\diamond_\alpha}(t) = f^{\diamond_\alpha}(t) g(t) + \alpha f^{\sigma}(t) g^{\Delta}(t) + (1-\alpha) f^{\rho}(t) g^{\nabla}(t).$$

(iii) For $g(t)g^{\sigma}(t)g^{\rho}(t) \neq 0$, $\dfrac{f}{g} : \mathbb{T} \to \mathbb{R}$ is diamond-α differentiable at $t \in \mathbb{T}$, with

$$\left(\frac{f}{g}\right)^{\diamond_\alpha}(t) = \frac{f^{\diamond_\alpha}(t) g^{\sigma}(t) g^{\rho}(t) - \alpha f^{\sigma}(t) g^{\rho}(t) g^{\Delta}(t) - (1-\alpha) f^{\rho}(t) g^{\sigma}(t) g^{\nabla}(t)}{g(t) g^{\sigma}(t) g^{\rho}(t)}.$$

Definition 2.5 (See [19]). Let $a, t \in \mathbb{T}$ and $h : \mathbb{T} \to \mathbb{R}$. Then the diamond-$\alpha$ integral from a to t of h is defined by

$$\int_a^t h(s) \diamond_\alpha s = \alpha \int_a^t h(s) \Delta s + (1-\alpha) \int_a^t h(s) \nabla s, \quad 0 \le \alpha \le 1,$$

provided that there exist delta and nabla integrals of h on $\mathbb{T}$.

Theorem 2.6 (See [19]). *Let $a, b, t \in \mathbb{T}$, $c \in \mathbb{R}$. Assume that $f(s)$ and $g(s)$ are $\diamond_\alpha$-integrable functions on $[a,b]_{\mathbb{T}}$. Then*

(i) $\displaystyle\int_a^t [f(s) \pm g(s)] \diamond_\alpha s = \int_a^t f(s) \diamond_\alpha s \pm \int_a^t g(s) \diamond_\alpha s;$

(ii) $\displaystyle\int_a^t c f(s) \diamond_\alpha s = c \int_a^t f(s) \diamond_\alpha s;$

(iii) $\displaystyle\int_a^t f(s) \diamond_\alpha s = -\int_t^a f(s) \diamond_\alpha s;$

(iv) $\displaystyle\int_a^t f(s)\diamond_\alpha s = \int_a^b f(s)\diamond_\alpha s + \int_b^t f(s)\diamond_\alpha s$;

(v) $\displaystyle\int_a^a f(s)\diamond_\alpha s = 0$.

We need the following results.

Theorem 2.7 (See [2]). *Let $a, b \in \mathbb{T}$ with $a < b$ and $w, f, g \in C([a,b]_{\mathbb{T}}, \mathbb{R})$ be $\diamond_\alpha$-integrable functions with $\int_a^b |w(x)||g(x)|^q \diamond_\alpha x > 0$. If $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$, then*

$$\int_a^b |w(x)||f(x)g(x)| \diamond_\alpha x \leq \left(\int_a^b |w(x)||f(x)|^p \diamond_\alpha x\right)^{\frac{1}{p}} \left(\int_a^b |w(x)||g(x)|^q \diamond_\alpha x\right)^{\frac{1}{q}}. \quad (2.1)$$

Inequality (2.1) is weighted form of Rogers–Hölder's inequality on time scales.

Theorem 2.8 (See [9, 10]). *Let $a, b \in \mathbb{T}$ with $a < b$ and $w, f_k \in C([a,b]_{\mathbb{T}}, \mathbb{R})$ for $k = 1, 2, \ldots, m$ be $\diamond_\alpha$-integrable functions.*

(i) If $\lambda_k > 1$ such that $\sum_{k=1}^m \frac{1}{\lambda_k} = 1$, then

$$\int_a^b |w(x)| \prod_{k=1}^m |f_k(x)| \diamond_\alpha x \leq \prod_{k=1}^m \left(\int_a^b |w(x)||f_k(x)|^{\lambda_k} \diamond_\alpha x\right)^{\frac{1}{\lambda_k}}. \quad (2.2)$$

(ii) If $0 < \lambda_1 < 1$, $\lambda_k < 0$, $k = 2, 3, \ldots, m$ such that $\sum_{k=1}^m \frac{1}{\lambda_k} = 1$, then

$$\int_a^b |w(x)| \prod_{k=1}^m |f_k(x)| \diamond_\alpha x \geq \prod_{k=1}^m \left(\int_a^b |w(x)||f_k(x)|^{\lambda_k} \diamond_\alpha x\right)^{\frac{1}{\lambda_k}}. \quad (2.3)$$

Theorem 2.9 (See [2]). *Let $a, b \in \mathbb{T}$ and $c, d \in \mathbb{R}$. Suppose that $g \in C([a,b]_{\mathbb{T}}, (c,d))$ and $w \in C([a,b]_{\mathbb{T}}, \mathbb{R})$ with $\int_a^b |w(s)| \diamond_\alpha s > 0$. If $\Phi \in C((c,d), \mathbb{R})$ is convex, then generalized Jensen's inequality is*

$$\Phi\left(\frac{\int_a^b |w(s)|g(s)\diamond_\alpha s}{\int_a^b |w(s)|\diamond_\alpha s}\right) \leq \frac{\int_a^b |w(s)|\Phi(g(s))\diamond_\alpha s}{\int_a^b |w(s)|\diamond_\alpha s}. \quad (2.4)$$

If Φ is strictly convex, then the inequality sign "$\leq$" in the above inequality (2.4) *can be replaced by "$<$".*

3 Renyi's Inequality

In order to present our main results, first we give a simple proof for an extension of dynamic Renyi's inequality on time scales.

Theorem 3.1. *Let* $w, f, g \in C([a,b]_{\mathbb{T}}, \mathbb{R})$ *be* $\diamond_\alpha$*-integrable functions. If* $\lambda \in \mathbb{R}_0^+ - \{0,1\}$*, then*

$$\frac{1}{\lambda-1}\left(\int_a^b |w(x)||f(x)| \diamond_\alpha x\right)^{\lambda}\left(\int_a^b |w(x)||g(x)| \diamond_\alpha x\right)^{1-\lambda} \leq \frac{1}{\lambda-1}\int_a^b |w(x)||f(x)|^{\lambda}|g(x)|^{1-\lambda} \diamond_\alpha x. \quad (3.1)$$

Proof. Case (i). If $\lambda > 1$, then dynamic Rogers–Hölder's inequality (2.1) becomes

$$\int_a^b |w(x)||f(x)g(x)| \diamond_\alpha x \leq \left(\int_a^b |w(x)||f(x)|^{\lambda} \diamond_\alpha x\right)^{\frac{1}{\lambda}}\left(\int_a^b |w(x)||g(x)|^{\frac{\lambda}{\lambda-1}} \diamond_\alpha x\right)^{\frac{\lambda-1}{\lambda}}. \quad (3.2)$$

Replacing $|g(x)|$ by $|g(x)|^{\frac{\lambda-1}{\lambda}}$ in inequality (3.2), we obtain

$$\int_a^b |w(x)||f(x)||g(x)|^{\frac{\lambda-1}{\lambda}} \diamond_\alpha x \leq \left(\int_a^b |w(x)||f(x)|^{\lambda} \diamond_\alpha x\right)^{\frac{1}{\lambda}}\left(\int_a^b |w(x)||g(x)| \diamond_\alpha x\right)^{\frac{\lambda-1}{\lambda}}. \quad (3.3)$$

Taking power $\lambda > 1$ on both sides of inequality (3.3), we get

$$\left(\int_a^b |w(x)||f(x)||g(x)|^{\frac{\lambda-1}{\lambda}} \diamond_\alpha x\right)^{\lambda}\left(\int_a^b |w(x)||g(x)| \diamond_\alpha x\right)^{1-\lambda} \leq \int_a^b |w(x)||f(x)|^{\lambda} \diamond_\alpha x. \quad (3.4)$$

Replacing $|f(x)|$ by $|f(x)||g(x)|^{\frac{1-\lambda}{\lambda}}$ in inequality (3.4) and multiplying both sides by $\frac{1}{\lambda-1}$, we get the desired claim.

Case (ii). If $\lambda \in (0,1)$, then by using dynamic Rogers–Hölder's inequality (2.1) for $p = \frac{1}{\lambda} > 1$ and $q = \frac{1}{1-\lambda} > 1$, we obtain

$$\int_a^b |w(x)||f(x)|^{\lambda}|g(x)|^{1-\lambda} \diamond_\alpha x$$
$$\leq \left(\int_a^b |w(x)||f(x)| \diamond_\alpha x\right)^{\lambda} \left(\int_a^b |w(x)||g(x)| \diamond_\alpha x\right)^{1-\lambda}. \quad (3.5)$$

Multiplying both sides of inequality (3.5) by $\frac{1}{\lambda - 1}$, we get the desired claim. Thus, the proof of Theorem 3.1 is completed. □

Remark 3.2. Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $b = n + 1$, $w \equiv 1$, $f(i) = x_i \in [0, +\infty)$ and $g(i) = y_i \in [0, +\infty)$ for $i = 1, 2, \ldots, n$. Then inequality (3.1) reduces to inequality (1.1).

4 Rogers–Hölder's Inequality

Next, we give the following extension of Rogers–Hölder's inequality and its reversed version on time scales. Discrete version of Rogers–Hölder's inequality is given in [12].

Theorem 4.1. *Let* $w, f_k \in C([a, b]_{\mathbb{T}}, \mathbb{R})$ *for* $k = 1, 2, \ldots, m$ *with* $\int_a^b |w(x)| \diamond_\alpha x > 0$ *be* $\diamond_\alpha$*-integrable functions.*

(i) If $0 < \lambda_1, \lambda_2, \ldots, \lambda_m < 1$ *with* $\sum_{k=1}^m \lambda_k \leq 1$*, then*

$$\int_a^b |w(x)| \prod_{k=1}^m |f_k(x)|^{\lambda_k} \diamond_\alpha x$$
$$\leq \left(\int_a^b |w(x)| \diamond_\alpha x\right)^{1-\sum_{k=1}^m \lambda_k} \prod_{k=1}^m \left(\int_a^b |w(x)||f_k(x)| \diamond_\alpha x\right)^{\lambda_k}. \quad (4.1)$$

(ii) If $\lambda_1, \lambda_2, \ldots, \lambda_m < 0$*, then*

$$\int_a^b |w(x)| \prod_{k=1}^m |f_k(x)|^{\lambda_k} \diamond_\alpha x$$
$$\geq \left(\int_a^b |w(x)| \diamond_\alpha x\right)^{1-\sum_{k=1}^m \lambda_k} \prod_{k=1}^m \left(\int_a^b |w(x)||f_k(x)| \diamond_\alpha x\right)^{\lambda_k}. \quad (4.2)$$

Proof. Case (i). We note that

$$\int_a^b |w(x)| \prod_{k=1}^m |f_k(x)|^{\lambda_k} \diamond_\alpha x = \int_a^b |w(x)| \prod_{k=1}^m \left(|f_k(x)|^{\sum_{k=1}^m \lambda_k}\right)^{\frac{\lambda_k}{\sum_{k=1}^m \lambda_k}} \diamond_\alpha x. \quad (4.3)$$

Applying generalized Rogers–Hölder's inequality (2.2), we get

$$\int_a^b |w(x)| \prod_{k=1}^m |f_k(x)|^{\lambda_k} \diamond_\alpha x \leq \prod_{k=1}^m \left(\int_a^b |w(x)| |f_k(x)|^{\sum\limits_{k=1}^m \lambda_k} \diamond_\alpha x \right)^{\frac{\lambda_k}{\sum\limits_{k=1}^m \lambda_k}}. \tag{4.4}$$

We consider a function $\Phi : \mathbb{R}_0^+ := [0, +\infty) \to \mathbb{R}_0^+$ defined by $\Phi(x) = x^{\sum\limits_{k=1}^m \lambda_k}$, $x \in [0, +\infty)$, which is concave for $\sum\limits_{k=1}^m \lambda_k < 1$. Then applying generalized Jensen's inequality given in Theorem 2.9, we get

$$\frac{\int_a^b |w(x)| |g(x)|^{\sum\limits_{k=1}^m \lambda_k} \diamond_\alpha x}{\int_a^b |w(x)| \diamond_\alpha x} \leq \left(\frac{\int_a^b |w(x)| |g(x)| \diamond_\alpha x}{\int_a^b |w(x)| \diamond_\alpha x} \right)^{\sum\limits_{k=1}^m \lambda_k}. \tag{4.5}$$

From inequality (4.5), we have that

$$\left(\int_a^b |w(x)| |f_1(x)|^{\sum\limits_{k=1}^m \lambda_k} \diamond_\alpha x \right)^{\frac{\lambda_1}{\sum\limits_{k=1}^m \lambda_k}} \leq \left(\int_a^b |w(x)| \diamond_\alpha x \right)^{\frac{\lambda_1}{\sum\limits_{k=1}^m \lambda_k} - \lambda_1} \left(\int_a^b |w(x)| |f_1(x)| \diamond_\alpha x \right)^{\lambda_1}. \tag{4.6}$$

Continuing this process, we get

$$\left(\int_a^b |w(x)| |f_m(x)|^{\sum\limits_{k=1}^m \lambda_k} \diamond_\alpha x \right)^{\frac{\lambda_m}{\sum\limits_{k=1}^m \lambda_k}} \leq \left(\int_a^b |w(x)| \diamond_\alpha x \right)^{\frac{\lambda_m}{\sum\limits_{k=1}^m \lambda_k} - \lambda_m} \left(\int_a^b |w(x)| |f_m(x)| \diamond_\alpha x \right)^{\lambda_m}. \tag{4.7}$$

Multiplying these results from 1 to m, we obtain

$$\prod_{k=1}^m \left(\int_a^b |w(x)| |f_k(x)|^{\sum\limits_{k=1}^m \lambda_k} \diamond_\alpha x \right)^{\frac{\lambda_k}{\sum\limits_{k=1}^m \lambda_k}} \leq \left(\int_a^b |w(x)| \diamond_\alpha x \right)^{1 - \sum\limits_{k=1}^m \lambda_k} \prod_{k=1}^m \left(\int_a^b |w(x)| |f_k(x)| \diamond_\alpha x \right)^{\lambda_k}. \tag{4.8}$$

This directly yields inequality (4.1).

Case (ii). The left-hand side of inequality (4.2) can be written as

$$\int_a^b |w(x)||f_1(x)|^{\lambda_1}\left(\prod_{k=2}^m |f_k(x)|^{\lambda_k}\right)\diamond_\alpha x$$
$$= \int_a^b |w(x)|\left(|f_1(x)|^{\frac{\lambda_1}{1-\sum_{k=2}^m \lambda_k}}\right)^{1-\sum_{k=2}^m \lambda_k}\left(\prod_{k=2}^m |f_k(x)|^{\lambda_k}\right)\diamond_\alpha x. \quad (4.9)$$

Applying reverse Rogers–Hölder's inequality (2.3), we get

$$\int_a^b |w(x)||f_1(x)|^{\lambda_1}\left(\prod_{k=2}^m |f_k(x)|^{\lambda_k}\right)\diamond_\alpha x$$
$$\geq \left(\int_a^b |w(x)||f_1(x)|^{\frac{\lambda_1}{1-\sum_{k=2}^m \lambda_k}}\diamond_\alpha x\right)^{1-\sum_{k=2}^m \lambda_k}\prod_{k=2}^m\left(\int_a^b |w(x)||f_k(x)|\diamond_\alpha x\right)^{\lambda_k}. \quad (4.10)$$

We consider a function $\Phi : \mathbb{R}_0^+ := [0,+\infty) \to \mathbb{R}_0^+$ defined by $\Phi(x) = x^{\frac{\lambda_1}{1-\sum_{k=2}^m \lambda_k}}$, $x \in [0,+\infty)$, which is a convex function for $\lambda_1 < 0$, $1 - \sum_{k=2}^m \lambda_k > 0$. Then applying generalized Jensen's inequality given in Theorem 2.9, we get

$$\left(\frac{\int_a^b |w(x)||f_1(x)|\diamond_\alpha x}{\int_a^b |w(x)|\diamond_\alpha x}\right)^{\frac{\lambda_1}{1-\sum_{k=2}^m \lambda_k}} \leq \frac{\int_a^b |w(x)||f_1(x)|^{\frac{\lambda_1}{1-\sum_{k=2}^m \lambda_k}}\diamond_\alpha x}{\int_a^b |w(x)|\diamond_\alpha x}. \quad (4.11)$$

From inequality (4.11), we have that

$$\left(\int_a^b |w(x)|\diamond_\alpha x\right)^{1-\sum_{k=1}^m \lambda_k}\left(\int_a^b |w(x)||f_1(x)|\diamond_\alpha x\right)^{\lambda_1}$$
$$\leq \left(\int_a^b |w(x)||f_1(x)|^{\frac{\lambda_1}{1-\sum_{k=2}^m \lambda_k}}\diamond_\alpha x\right)^{1-\sum_{k=2}^m \lambda_k}. \quad (4.12)$$

From inequalities (4.10) and (4.12), we get the desired claim. Thus, the proof of Theorem 4.1 is completed. □

Remark 4.2. Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $b = n + 1$, $w \equiv 1$, $k = 1, 2$, $f_1(i) = x_i \in [0,+\infty)$ and $f_2(i) = y_i \in [0,+\infty)$ for $i = 1, 2, \ldots, n$. Then inequality (4.1) reduces to (1.2) and inequality (4.2) reduces to the reverse version of (1.2).

5 Radon's Inequality

In order to conclude our main results, we give a proof for an extension of dynamic Radon's inequality on time scales.

Theorem 5.1. *Let $w, f, g \in C([a,b]_{\mathbb{T}}, \mathbb{R})$ be $\diamond_\alpha$-integrable functions, neither $w \equiv 0$ nor $g \equiv 0$.*

(i) If $\lambda_1, \lambda_2, \lambda_3 > 0$ with $\lambda_1 - \lambda_2 = \lambda_3$, then

$$\frac{\left(\int_a^b |w(x)||f(x)|^{\lambda_3} \diamond_\alpha x\right)^{\lambda_1}}{\left(\int_a^b |w(x)||g(x)|^{\lambda_3} \diamond_\alpha x\right)^{\lambda_2}} \leq \left(\int_a^b \frac{|w(x)||f(x)|^{\lambda_1}}{|g(x)|^{\lambda_2}} \diamond_\alpha x\right)^{\lambda_3}. \tag{5.1}$$

(ii) If $\lambda_1, \lambda_3 > 0$, $\lambda_2 < 0$ with $\lambda_1 - \lambda_2 = \lambda_3$, then

$$\frac{\left(\int_a^b |w(x)||f(x)|^{\lambda_3} \diamond_\alpha x\right)^{\lambda_1}}{\left(\int_a^b |w(x)||g(x)|^{\lambda_3} \diamond_\alpha x\right)^{\lambda_2}} \geq \left(\int_a^b \frac{|w(x)||f(x)|^{\lambda_1}}{|g(x)|^{\lambda_2}} \diamond_\alpha x\right)^{\lambda_3}. \tag{5.2}$$

Proof. Case (i). Let $\gamma = \dfrac{\lambda_2}{\lambda_1}$ and $\delta = \dfrac{\lambda_3}{\lambda_1}$. Then $\gamma + \delta = 1$ with $\gamma < 1$ and $\delta < 1$. Applying Rogers–Hölder's inequality (2.1) for $p = \dfrac{1}{\gamma} > 1$, $q = \dfrac{1}{\delta} > 1$, we get

$$\int_a^b |w(x)||f(x)|^{\gamma}|g(x)|^{\delta} \diamond_\alpha x \leq \left(\int_a^b |w(x)||f(x)| \diamond_\alpha x\right)^{\gamma} \left(\int_a^b |w(x)||g(x)| \diamond_\alpha x\right)^{\delta}. \tag{5.3}$$

Letting $|f(x)|$ and $|g(x)|$ be replaced by $|g(x)|^{\lambda_3}$ and $|f(x)|^{\lambda_1}|g(x)|^{-\lambda_2}$ in the above inequality (5.3), respectively, we get

$$\int_a^b |w(x)||f(x)|^{\lambda_3} \diamond_\alpha x \leq \left(\int_a^b |w(x)||g(x)|^{\lambda_3} \diamond_\alpha x\right)^{\frac{\lambda_2}{\lambda_1}} \left(\int_a^b \frac{|w(x)||f(x)|^{\lambda_1}}{|g(x)|^{\lambda_2}} \diamond_\alpha x\right)^{\frac{\lambda_3}{\lambda_1}}. \tag{5.4}$$

Taking power $\lambda_1 > 0$ on both sides of inequality (5.4), we get the inequality (5.1).

Case (ii). Let $\gamma = \dfrac{\lambda_2}{\lambda_1}$ and $\delta = \dfrac{\lambda_3}{\lambda_1}$. Then $\gamma + \delta = 1$ with $\gamma < 0$ and $\delta > 0$. Applying reverse Rogers–Hölder's inequality (2.3) for $w, f, g \in C([a,b]_{\mathbb{T}}, \mathbb{R})$, we get

$$\int_a^b |w(x)||f(x)|^\gamma |g(x)|^\delta \diamond_\alpha x \\ \geq \left(\int_a^b |w(x)||f(x)| \diamond_\alpha x\right)^\gamma \left(\int_a^b |w(x)||g(x)| \diamond_\alpha x\right)^\delta. \quad (5.5)$$

Letting $|f(x)|$ and $|g(x)|$ be replaced by $|g(x)|^{\lambda_3}$ and $|f(x)|^{\lambda_1}|g(x)|^{-\lambda_2}$ in the above inequality (5.5), respectively, we get

$$\int_a^b |w(x)||f(x)|^{\lambda_3} \diamond_\alpha x \\ \geq \left(\int_a^b |w(x)||g(x)|^{\lambda_3} \diamond_\alpha x\right)^{\frac{\lambda_2}{\lambda_1}} \left(\int_a^b \frac{|w(x)||f(x)|^{\lambda_1}}{|g(x)|^{\lambda_2}} \diamond_\alpha x\right)^{\frac{\lambda_3}{\lambda_1}}. \quad (5.6)$$

Taking power $\lambda_1 > 0$ on both sides of inequality (5.6), we get the inequality (5.2). □

Remark 5.2. Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $b = n+1$, $w \equiv 1$, $f(i) = x_i \in [0, +\infty)$ and $g(i) = y_i \in (0, +\infty)$ for $i = 1, 2, \ldots, n$. Then inequality (5.1) reduces to

$$\frac{\left(\sum_{i=1}^n x_i^{\lambda_3}\right)^{\lambda_1}}{\left(\sum_{i=1}^n y_i^{\lambda_3}\right)^{\lambda_2}} \leq \left(\sum_{i=1}^n \frac{x_i^{\lambda_1}}{y_i^{\lambda_2}}\right)^{\lambda_3} \quad (5.7)$$

and inequality (5.2) reduces to

$$\frac{\left(\sum_{i=1}^n x_i^{\lambda_3}\right)^{\lambda_1}}{\left(\sum_{i=1}^n y_i^{\lambda_3}\right)^{\lambda_2}} \geq \left(\sum_{i=1}^n \frac{x_i^{\lambda_1}}{y_i^{\lambda_2}}\right)^{\lambda_3}. \quad (5.8)$$

Similar inequalities related to (5.7) and (5.8) may be found in [13].

Next, we give the following extension of Radon's inequality [15] on time scales for diamond-α integral and its generalized forms are proved in [16–18].

Corollary 5.3. *Let* $w, f, g \in C([a,b]_{\mathbb{T}}, \mathbb{R})$ *be* $\diamond_\alpha$*-integrable functions, neither* $w \equiv 0$ *nor* $g \equiv 0$*. If* $\lambda > 0$*, then*

$$\frac{\left(\int_a^b |w(x)||f(x)| \diamond_\alpha x\right)^{\lambda+1}}{\left(\int_a^b |w(x)||g(x)| \diamond_\alpha x\right)^{\lambda}} \leq \int_a^b \frac{|w(x)||f(x)|^{\lambda+1}}{|g(x)|^\lambda} \diamond_\alpha x. \quad (5.9)$$

Proof. If $\lambda_2 = \lambda$ and $\lambda_3 = 1$, then $\lambda_1 - \lambda = 1$. The inequality (5.9) follows from inequality (5.1). □

Remark 5.4. Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $b = n + 1$ and $w \equiv 1$. If $n \in \mathbb{N}$, $f(i) = x_i \geq 0$ and $g(i) = y_i > 0$ for $i = 1, 2, \ldots, n$, then the inequality (1.3) follows from inequality (5.9).

Remark 5.5. Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $b = n + 1$ and $w \equiv 1$. If $n \in \mathbb{N}$, $f(i) = x_i \in \mathbb{R}$, $g(i) = y_i > 0$ for $i = 1, 2, \ldots, n$, $\lambda_1 = 2$ and $\lambda_2 = \lambda_3 = 1$, then inequality (5.1) reduces to

$$\frac{\left(\sum_{i=1}^{n} x_i\right)^2}{\sum_{i=1}^{n} y_i} \leq \sum_{i=1}^{n} \frac{x_i^2}{y_i}, \tag{5.10}$$

with equality if and only if $\dfrac{x_1}{y_1} = \dfrac{x_2}{y_2} = \ldots = \dfrac{x_n}{y_n}$.

The inequality from (5.10) is called, in literature, Bergström's inequality as given in [4–6, 14].

6 Conclusion and Future Work

There have been recent developments of the theory and applications of dynamic inequalities on time scales, see [1, 2]. In this research article, we have presented some dynamic inequalities for the diamond-α integral, which is the linear combination of the delta and nabla integrals. As is well-known that Rogers–Hölder's inequality and Radon's inequality and their various extensions play a very important role in mathematical analysis. Some generalizations of Renyi's inequality, Rogers–Hölder's inequality and Radon's inequality on time scales are investigated. If we set $\alpha = 1$, then we get delta versions and if we set $\alpha = 0$, then we get nabla versions of diamond-α integral operator inequalities presented in this article. Also, if we set $\mathbb{T} = \mathbb{Z}$, then we get discrete versions and if we set $\mathbb{T} = \mathbb{R}$, then we get continuous versions of diamond-α integral operator inequalities presented in this article.

In the future research, we will continue to generalize more dynamic inequalities by using Specht's ratio, Kantorovich's ratio, functional generalization, n-tuple diamond-α integral, fractional Riemann–Liouville integral and fractional derivatives. It will be interesting to find the inequalities in quantum calculus and α,β-symmetric quantum calculus.

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