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Nonoscillation of First-Order Neutral Impulsive Difference Equations

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Abstract

In this work, we establish sufficient conditions for the existence of nonoscillatory solutions of a class of first-order neutral impulsive difference equations with fixed moments of impulsive effect. The main tools used for the proof of the existence of nonoscillatory solutions are Knaster–Tarski fixed point theorem and Banach's contraction mapping principle.

AMS Subject Classifications: 39A10, 39A12.
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1 Introduction

Consider the class of first-order impulsive neutral difference equations of the form

$$(E)\begin{cases}\Delta[y(n)+p(n)y(n-\tau)]+q(n)G(y(n-\sigma))=0,\ n\neq m_j, j\in\mathbb{N},\\ \underline{\Delta}[y(m_j-1)+p(m_j-1)y(m_j-\tau-1)]+r(m_j-1)G(y(m_j-\sigma-1))=0,\end{cases}$$

where τ, $\sigma > 0$ are integers, p, q, r are real valued functions with discrete arguments such that $q(n) > 0, r > 0, |p(n)| < \infty$ for $n \in \mathbb{N}(n_0) = \{n_0, n_0+1, \cdots\}, G \in C(\mathbb{R},\mathbb{R})$ satisfying the properties $xG(x) > 0$ for $x \neq 0$ and Δ is the forward difference operator defined by $\Delta u(n) = u(n+1) - u(n)$. Let $m_1, m_2, m_3, \cdots$ be the discrete moments of impulsive effect satisfying the property $0 < m_1 < m_2 < \cdots, \lim_{j\to\infty} m_j = +\infty$. Here, $\underline{\Delta}$ is the difference operator defined by $\underline{\Delta} y(m_j-1) = y(m_j) - y(m_j-1)$.

In recent years, much effort has been given to study the oscillation and nonoscillation of impulsive differential/difference equations, since it much more richer that the

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corresponding theory of differential/difference equations and such equation adequate mathematical modelling of real world phenomena observed in population dynamics, industrial robotics, rhythmical beating, discontinuity of solutions, etc.

In this work, our objective is to study the existence of nonoscillatory solutions of the system (E) when $|p(n)| < \infty$. About the impulsive differential/difference equations we refer the monograph [5] and some of the works [6]– [16] to the reader and the references cited there in.

In [11], Peng has established the oscillation criteria for second order impulsive delay difference equations of the form

$$(E_1)\begin{cases}\Delta(a_{n-1}|\Delta x(n-1)|^{\alpha-1}\Delta x(n-1)) + f(n, x(n), x(n-l)) = 0, \ n \neq n_k\\ a_{n_k}|\Delta x(n_k)|^{\alpha-1}\Delta x(n_k) = N_k(a_{n_k-1}|\Delta x(n_k-1)|^{\alpha-1}\Delta x(n_k-1)), \ k \in \mathbb{N},\end{cases}$$

In [9], Lu et al. have established the oscillation criteria for nonlinear third order difference equations with impulse of the form

$$(E_2)\begin{cases}\Delta^3 y(n) + p(n)f(y(n-\tau)) = 0, \ n \neq n_k\\ \Delta^i y(n_k) = g_{i,k}\Delta^i y(n_k-1), \ i = 0, 1, 2, \ k \in \mathbb{N}.\end{cases}$$

Unlike the above method, our impulsive effect satisfies another neutral equation corresponding to its difference equation . Inspired and motivated by the above mention works, an attempt is made to find some sufficient condition for nonoscillation of (E) using Knaster–Tarski fixed point theorem and Banach's contraction mapping principle.

Definition 1.1. By a solution of (E) we mean a real valued function $y(n)$ defined on $\mathbb{N}(n_0 - \rho)$ which satisfy (E) for $n \geq n_0$ with the initial conditions $y(i) = \phi(i)$, $i = n_0 - \rho, \cdots, n_0$, where $\phi(i)$, $i = n_0 - \rho, \cdots, n_0$ are given and $\rho = \max\{\tau, \sigma\}$.

Definition 1.2. A nontrivial solution $y(n)$ of (E) is said to be nonoscillatory, if it is either eventually positive or eventually negative. Otherwise, the solution is said to be oscillatory.

The following fixed point theorem will be used in the proof of main results.

Theorem 1.3 (Knaster–Tarski Fixed Point Theorem, see [5])**.** *Let X be a partially ordered Banach space with ordering $\leq$. Let S be a subset of X with the following properties: the infimum of S belongs to S and every nonempty subset of S has a supremum which belongs to S. Let $T : S \to S$ be a increasing mapping, i.e., $x \leq y$ implies that $Tx \leq Ty$. Then T has a fixed point in S.*

Theorem 1.4 (Banach‘s Contraction Mapping Principle, see [5])**.** *A contraction mapping on a complete metric space has exactly one fixed point.*

2 Main Results

Theorem 2.1. *Assume that*

(H_1) $\sum_{n=N}^{\infty} q(n) + \sum_{j=1}^{\infty} r(m_j - 1) < \infty, \ N > 0$

holds. If one of the following conditions holds:

$$(A)\ 0 \leq p_1 \leq p(n) \leq p_2 < 1, \qquad (B)\ 1 < p_3 \leq p(n) \leq p_4 < \infty,$$
$$(C)\ -1 < p_5 \leq p(n) \leq p_6 \leq 0, \quad (D)\ -\infty < p_7 \leq p(n) \leq p_8 < -1,$$

then (E) has a bounded nonoscillatory solution.

Proof. **Case-A.** $0 \leq p_1 \leq p(n) \leq p_2 < 1$

Let $X = l_\infty^{n_0}$ be the Banach space of real valued bounded functions $y(n)$ for $n \geq n_0$ with sup norm defined by $\|y\| = \sup\{|y(n)| : n \geq n_0\}$. Set

$$\mu = \{y \in X : \alpha_1 \leq y(n) \leq \alpha_2, n \geq n_0\},$$

where α_1 and α_2 are two positive constants such that

$$\alpha_1 < (1 - p_2)\alpha_2 \quad \text{and} \quad \alpha_1 + p_2\alpha_2 < \beta < \alpha_2.$$

Clearly, μ is a closed subset of X implies that μ is a complete metric space. From (H_1), it is possible to choose $n_1 > n_0$ such that

$$\sum_{s=n}^{\infty} q(s) + \sum_{n_1 \leq m_j - 1 \leq n} r(m_j - 1) < M, \ n \geq n_1, \tag{2.1}$$

where

$$M = \min\left\{\frac{\alpha_2 - \beta}{L}, \frac{\beta - (\alpha_1 + p_2\alpha_2)}{L}, \frac{1 - (p_1 + p_2)}{L}\right\},$$

and $L = \max\{L_1, G(\alpha_2)\}$, L_1 is the Lipschitz constant of G on $[\alpha_1, \alpha_2]$. Define a map $\mathcal{T} : \mu \to \mu$ by

$$\mathcal{T}y(n) = \begin{cases} \mathcal{T}y(n_1 + \rho), \ n_1 \leq n \leq n_1 + \rho, \\ \beta - p(n)y(n - \tau) + \sum_{s=n}^{\infty} q(s)G(y(s - \sigma)) \\ + \sum_{n_1 \leq m_j - 1 \leq n} r(m_j - 1)G(y(m_j - \sigma - 1)), \ n \geq n_1 + \rho, \end{cases}$$

where $\rho = \max\{\tau, \sigma\}$. For $y \in \mu$ and using (2.1), we have

$$\mathcal{T}y(n) \leq \beta + \sum_{s=n}^{\infty} q(s)G(y(s - \sigma)) + \sum_{n_1 \leq m_j - 1 \leq n} r(m_j - 1)G(y(m_{-}\sigma - 1))$$

$$\begin{aligned}
&\leq \beta + G(\alpha_2)\Big[\sum_{s=n}^{\infty} q(s) + \sum_{n_1 \leq m_j - 1 \leq n} r(m_j - 1)\Big] \\
&\leq \beta + G(\alpha_2)M \\
&= \alpha_2
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{T}y(n) &\geq \beta - p(n)y(n-\tau) \\
&\geq \alpha_1 + p_2\alpha_2 - p_2\alpha_2 \\
&= \alpha_1
\end{aligned}$$

implies that $\mathcal{T}y \in \mu$ for every $n \geq n_1$. In order to apply the Banach's contraction principle we have to show that $\mathcal{T}$ is a contraction mapping on μ. For $y_1, y_2 \in \mu$, we have

$$\begin{aligned}
|\mathcal{T}y_1(n) - \mathcal{T}y_2(n)| \leq\; & |P(n)|\, |y_1(n-\tau) - y_2(n-\tau)| \\
& + L_1 \sum_{s=n}^{\infty} q(s)|y_1(s-\sigma) - y_2(s-\sigma)| \\
& + L_1 \sum_{n_1 \leq m_j - 1 \leq n} r(m_j - 1)|y_1(m_j - \sigma - 1) - y_2(m_j - \sigma - 1)|,
\end{aligned}$$

that is,

$$\begin{aligned}
|\mathcal{T}y_1(n) - \mathcal{T}y_2(n)| &\leq p_2\|y_1 - y_2\| + L_1\|y_1 - y_2\|\big[\sum_{s=n}^{\infty} q(s) + \sum_{n_1 \leq m_j - 1 \leq n} r(m_j - 1)\big] \\
&\leq [p_2 + 1 - (p_1 + p_2)]\|y_1 - y_2\|
\end{aligned}$$

implies that

$$\|\mathcal{T}y_{-}\mathcal{T}y_2\| \leq \lambda\|y_1 - y_2\|.$$

Therefore, $\mathcal{T}$ is a contraction with $\lambda = 1 - p_1 < 1$. Hence, by Banach's fixed point theorem, $\mathcal{T}$ has a unique point $y \in \mu$ such that $\mathcal{T}y = y$. Therefore,

$$y(n) = \begin{cases} y(n_1 + \rho), \ n_1 \leq n \leq n_1 + \rho, \\ \beta - p(n)y(n-\tau) + \sum_{s=n}^{\infty} q(s)G(y(s-\sigma)) \\ + \sum_{j=1}^{\infty} r(m_j - 1)G(y(m_j - \sigma - 1)), n \geq n_1 + \rho \end{cases}$$

and it is easy to see that $y(n)$ is a nonoscillatory solution of (E).

Case-B. $1 < p_3 \leq p(n) \leq p_4 < \infty$

Let $X = l_\infty^{n_0}$ be the Banach space of real valued bounded functions $y(n)$ for $n \geq n_0$ with sup norm defined by $\|y\| = \sup\{|y(n)| : n \geq n_0\}$. Set

$$\mu = \{y \in X : \alpha_3 \leq y(n) \leq \alpha_4, n \geq n_0\},$$

where α_3 and α_4 are two positive constants such that

$$p_4\alpha_3 < (p_3 - 1)\alpha_4 \quad \text{and} \quad \alpha_4 + p_4\alpha_3 < \beta < p_3\alpha_4.$$

Since μ is a closed subset of X then μ is a complete metric space. From (H_1), it is possible to choose $n_1 > n_0$ such that

$$\sum_{s=n}^{\infty} q(s) + \sum_{n_1 \leq m_j - 1 \leq n} r(m_j - 1) < M, \; n \geq n_1, \tag{2.2}$$

where

$$M = \min\Big\{\frac{p_3\alpha_4 - \beta}{L}, \frac{\beta - (\alpha_4 + p_4\alpha_3)}{L}, \frac{(p_3 - 1)}{L}\Big\},$$

and $L = \max\{L_1, G(\alpha_4)\}$, L_1 is the Lipschitz constant of G on $[\alpha_3, \alpha_4]$. Define a map $\mathcal{T} : \mu \to \mu$ by

$$\mathcal{T}y(n) = \begin{cases} \mathcal{T}y(n_1 + \rho), \; n_1 \leq n \leq n_1 + \rho, \\ \dfrac{\beta}{p(n+\tau)} - \dfrac{y(n+\tau)}{p(n+\tau)} + \dfrac{1}{p(n+\tau)}\big[\displaystyle\sum_{s=n}^{\infty} q(s)G(y(s-\sigma)) \\ + \displaystyle\sum_{n_1 \leq m_j - 1 \leq n} r(m_j - 1)G(y(m_j - \sigma - 1))\big], \; n \geq n_1 + \rho. \end{cases}$$

For $y \in \mu$ and using (2.2), we have

$$\begin{aligned} \mathcal{T}y(n) &\leq \frac{\beta}{p(n+\tau)} + \frac{G(\alpha_4)}{p(n+\tau)}\Big[\sum_{s=n}^{\infty} q(s) + \sum_{n_1 \leq m_j - 1 \leq n} r(m_j - 1)\Big] \\ &\leq \frac{1}{p_3}\big[\beta + G(\alpha_4)M\big] \\ &= \alpha_4 \end{aligned}$$

and

$$\begin{aligned} \mathcal{T}y(n) &\geq \frac{\beta}{p(n+\tau)} - \frac{y(n+\tau)}{p(n+\tau)} \\ &\geq \frac{1}{p_4}\big[p_4\alpha_3 + \alpha_4 - \alpha_4\big] \\ &= \alpha_3 \end{aligned}$$

implies that $\mathcal{T}y \in \mu$ for every $n \geq n_1$. For $y_1, y_2 \in \mu$, we have

$$\begin{aligned}|\mathcal{T}y_1(n)-&\mathcal{T}y_2(n)| \\ &\leq \frac{1}{|P(n+\tau)|}\,|y_1(n+\tau)-y_2(n+\tau)| \\ &+\frac{L_1}{|P(n+\tau)|}\sum_{s=n}^{\infty} q(s)|y_1(s-\sigma)-y_2(s-\sigma)| \\ &+\frac{L_1}{|P(n+\tau)|}\sum_{n_1\leq m_j-1\leq n} r(m_j-1)|y_1(m_j-\sigma-1)-y_2(m_j-\sigma-1)|,\end{aligned}$$

that is,

$$\begin{aligned}|\mathcal{T}y_1(n)-\mathcal{T}y_2(n)| &\leq \frac{1}{p_3}\|y_1-y_2\|+\frac{L_1}{p_3}\|y_1-y_2\|\Big[\sum_{s=n}^{\infty} q(s)+\sum_{n_1\leq m_j-1\leq n} r(m_j-1)\Big] \\ &\leq \frac{1}{p_3}\|y_1-y_2\|+\frac{L_1}{p_3}M\|y_1-y_2\| \\ &\leq \frac{1}{p_3}[1+\frac{p_3-1}{2}]\|y_1-y_2\|\end{aligned}$$

implies that

$$\|\mathcal{T}y_{-}\mathcal{T}y_2\| \leq \lambda\|y_1-y_2\|.$$

Therefore, $\mathcal{T}$ is a contraction with $\lambda = \dfrac{1+p_3}{2p_3} < 1$. Hence, by Banach's fixed point theorem, $\mathcal{T}$ has a unique point $y \in \mu$ such that $\mathcal{T}y = y$. Therefore,

$$y(n) = \begin{cases} y(n_1+\rho), \; n_1 \leq n \leq n_1+\rho, \\ \dfrac{\beta}{p(n+\tau)} - \dfrac{y(n+\tau)}{p(n+\tau)} + \dfrac{1}{p(n+\tau)}\Big[\displaystyle\sum_{s=n}^{\infty} q(s)G(y(s-\sigma)) \\ +\displaystyle\sum_{n_1\leq m_j-1\leq n} r(m_j-1)G(y(m_j-\sigma-1))\Big], \; n \geq n_1+\rho \end{cases}$$

and it is easy to see that $y(n)$ is a nonoscillatory solution of (E).

Case-C. $-1 < p_5 \leq p(n) \leq p_6 \leq 0$

Let $X = l_\infty^{n_0}$ be the Banach space of real valued bounded functions $y(n)$ for $n \geq n_0$ with sup norm defined by $\|y\| = \sup\{|y(n)| : n \geq n_0\}$.

Let $K = \{y(n) \in X : y(n) \geq 0 \text{ for } n \geq n_0\}$. For $y_1, y_2 \in X$ we define $y_1 \leq y_2$ if and only if $y_2 - y_1 \in K$. Thus, X is a partially ordered Banach space. Set

$$\mu = \{y \in X : \alpha_5 \leq y(n) \leq \alpha_6, n \geq n_0\},$$

where α_5 and α_6 are two positive constants such that

$$\alpha_5 < \beta < (1+p_5)\alpha_6.$$

From (H_1), it is possible to choose $n_1 > n_0$ such that

$$\sum_{n=n_1}^{\infty} q(n) + \sum_{n_1 \le m_j - 1 \le n} r(m_j - 1) < \frac{(1+p_5)\alpha_6 - \beta}{G(\alpha_6)},\ n \ge n_1. \tag{2.3}$$

Let $x_0(n) = \alpha_5$ for $n \ge n_0$. Then $x_0(n) \in \mu$ and $x_0(n) = \inf \mu$. In addition, if $\phi \subset \mu^* \subset \mu$, then

$$\mu^* = \{y \in X : l_1 \le y(n) \le l_2, \alpha_5 \le l_1, l_2 \le \alpha_6, n \ge n_0\}.$$

Let $x_1(n) = l_2' = \sup\{l_2 : \alpha_5 \le l_2 \le \alpha_6\}$. Then $x_1(n) \in \mu$ and $x_1(n) = \sup \mu^*$. Since μ is a closed subset of X, μ is a complete metric space. Define a map $\mathcal{T} : \mu \to \mu$ by

$$\mathcal{T}y(n) = \begin{cases} \mathcal{T}y(n_1+\rho),\ n_1 \le n \le n_1+\rho, \\ \beta - p(n)y(n-\tau) + \sum_{s=n}^{\infty} q(s)G(y(s-\sigma)) \\ + \sum_{n_1 \le m_j - 1 \le n} r(m_j-1)G(y(m_j-\sigma-1)),\ n \ge n_1+\rho. \end{cases}$$

For $y \in \mu$ and using (2.3), we have

$$\begin{aligned} \mathcal{T}y(n) &\le \beta - p(n)y(n-\tau) + G(\alpha_6)\Big[\sum_{s=n}^{\infty} q(s) + \sum_{n_1 \le m_j - 1 \le n} r(m_j-1)\Big] \\ &\le \beta - p_5\alpha_6 + G(\alpha_6)\Big[\frac{(1+p_5)\alpha_6 - \beta}{G(\alpha_6)}\Big] \\ &= \alpha_6 \end{aligned}$$

and

$$\mathcal{T}y(n) \ge \beta \ge \alpha_5$$

implies that $\mathcal{T}y \in \mu$ for every $n \ge n_1$. Let $y_1, y_2 \in \mu$ such that $y_1 \le y_2$. It is easy to verify that $\mathcal{T}y_1 \le \mathcal{T}y_2$. Hence, by Knaster–Tarski fixed point theorem, $\mathcal{T}$ has a unique $y \in \mu$ such that $\mathcal{T}y = y$. Therefore,

$$y(n) = \begin{cases} y(n_1+\rho),\ n_1 \le n \le n_1+\rho, \\ \beta - p(n)y(n-\tau) + \sum_{s=n}^{\infty} q(s)G(y(s-\sigma)) \\ + \sum_{j=1}^{\infty} r(m_j-1)G(y(m_j-\sigma-1)), n \ge n_1+\rho. \end{cases}$$

Clearly, $y(n)$ is a nonoscillatory solution of (E).
Case-D. $-\infty < p_7 \le p(n) \le p_8 < -1$

Let $X = l_\infty^{n_0}$ be the Banach space of real valued bounded functions $y(n)$ for $n \geq n_0$ with sup norm defined by $\|y\| = \sup\{|y(n)| : n \geq n_0\}$. Set

$$\mu = \{y \in X : \alpha_7 \leq y(n) \leq \alpha_8, n \geq n_0\},$$

where α_7 and α_8 are two positive constants such that

$$-p_7\alpha_7 < \beta < (-1 - p_8)\alpha_8.$$

Since μ is a closed subset of X, μ is a complete metric space. From (H_1), it is possible to choose $n_1 > n_0$ such that

$$\sum_{s=n}^{\infty} q(s) + \sum_{n_1 \leq m_j - 1 \leq n} r(m_j - 1) < M, \; n \geq n_1. \tag{2.4}$$

where

$$M = \min\Big\{\frac{p_7\alpha_7 + \beta}{L}, \frac{-(\beta + (1 + p_8)\alpha_8)}{L}, \frac{(-1 - p_8)}{L}\Big\},$$

and $L = \max\{L_1, G(\alpha_8)\}$, L_1 is the Lipschitz constant of G on $[\alpha_7, \alpha_8]$. For $y \in \mu$ define a map

$$\mathcal{T}y(n) = \begin{cases} \mathcal{T}y(n_1 + \rho), \; n_1 \leq n \leq n_1 + \rho, \\ -\dfrac{\beta}{p(n+\tau)} - \dfrac{y(n+\tau)}{p(n+\tau)} + \dfrac{1}{p(n+\tau)}\big[\displaystyle\sum_{s=n}^{\infty} q(s) G(y(s-\sigma)) \\ + \displaystyle\sum_{n_1 \leq m_j - 1 \leq n} r(m_j - 1) G(y(m_j - \sigma - 1))\big], \; n \geq n_1 + \rho. \end{cases}$$

For $y \in \mu$ and using (2.4), we have

$$\begin{aligned} \mathcal{T}y(n) &\leq -\frac{\beta}{p(n+\tau)} - \frac{y(n+\tau)}{p(n+\tau)} \\ &\leq -\frac{1}{p_8}\big[(-1 - p_8)\alpha_8 + \alpha_8\big] \\ &= \alpha_8 \end{aligned}$$

and

$$\begin{aligned} \mathcal{T}y(n) &\geq -\frac{\beta}{p(n+\tau)} + \frac{G(\alpha_8)}{p(n+\tau)} \sum_{s=n}^{\infty} q(s) + \sum_{n_1 \leq m_j - 1 \leq n} r(m_j - 1) \\ &\geq -\frac{1}{p(n+\tau)}\Big[\beta - G(\alpha_8)\big[\sum_{s=n}^{\infty} q(s) + \sum_{n_1 \leq m_j - 1 \leq n} r(m_j - 1)\big]\Big] \\ &\geq -\frac{1}{p_7}\big[\beta + G(\alpha_8) M\big] \end{aligned}$$

$$\geq -\frac{1}{p_7}\big[\beta - \beta - p_7\alpha_7\big]$$
$$= \alpha_7$$

implies that $\mathcal{T}y \in \mu$ for every $n \geq n_1$. The remaining part of the proof is similar to that of Case-B, hence it is omitted. This completes the proof of the theorem. □

3 Discussion and an Example

In [14], Tripathy and Chhatria have studied the oscillatory behaviour of solutions of the system (E) under the sufficient condition

$$(H_2) \quad \sum_{n=N}^{\infty} q(n) + \sum_{j=1}^{\infty} r(m_j - 1) = \infty,\ N > 0.$$

Because of [14, Theorem 2.1], (H_2) may be a necessary and sufficient condition for oscillation of the solutions of the system (E) for all ranges of the neutral coefficient $p(n)$.

We conclude this section with the following example to illustrates our main results.

Example 3.1. Consider the impulsive difference equation of the form

$$(E_3)\begin{cases}\Delta[y(n) + \frac{1}{e^n}y(n-3)] + \big(\frac{e-1}{e}\big)e^{-n}y(n-2) = 0,\ n \neq m_j,\ n > 3,\\ \underline{\Delta}[y(m_j-1) + \frac{1}{e^{m_j-1}}y(m_j-4)] + \big(\frac{e-1}{e}\big)e^{-m_j}y(m_j-3) = 0,\ j \in \mathbb{N},\end{cases}$$

where $\tau = 3$, $\sigma = 2$, $p(n) = \frac{1}{e^n}$, $q(n) = (\frac{e-1}{e})e^{-n}$, $r(m_j - 1) = (\frac{e-1}{e})e^{-m_j}$, $G(u) = u$, $m_j = 3j$ for $j \in \mathbb{N}$. Here

$$\sum_{s=3}^{n} q(s) + \sum_{3\leq m_j-1\leq n} r(m_j-1), n \geq 3$$
$$= \sum_{s=2}^{n} \big(\frac{e-1}{e}\big)e^{-n} + \sum_{2\leq m_j-1\leq n} \big(\frac{e-1}{e}\big)e^{-m_j}$$
$$< \infty,\ \text{as } n \to \infty.$$

Indeed, all conditions of Theorem 2.1 holds true. Clearly, $y(n) = e$ is a nonoscillatory solution of the first equation of (E_3). It is easy to see that e is a nonoscillatory solution of the second equation of (E_3).

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References

[1] R. P. Agarwal, Difference Equations and Inequalities, Marcel Dekker, New York, 2000.

[2] R. P. Agarwal, M. Bohner S. R. Grace and D. O'Regan, Discrete Oscillation Theory, Hindawi Publishing Corporation, New York, 2005.

[3] G. E. Chatzarakis, P. Gokulraj, T. Kalaimani and V. Sadhasivam, Oscillatory solutions of nonlinear fractional difference equations, Int. J. Difference Equ., 13: 19–31, 2018.

[4] G. E. Chatzarakis, P. Gokulraj and T. Kalaimani, Oscillation tests for fractional difference equations, Tatra Mt. Math. Publ., 71: 53–64, 2018.

[5] V. Lakshmikantham, D. D. Bainov and P. S. Simeonov, Oscillation Theory of Impulsive Differential Equations, World Scientific, Singapore, 1989.

[6] J. Li and J. Shen, Positive solutions for first order difference equations with impulses, Int. J. Difference Equ., 1: 225–239, 2006.

[7] X. Li and Q. Xi, Oscillatory and asymptotic properties of impulsive difference equations with time- varying delays, Int. J. Difference Equ., 4: 201–209, 2009.

[8] Q. L. Li, Z. G. Zhang, F. Gou and H. Y. Liang, Oscillation criteria for third-order difference equations with impulses, J. Comput. Appl. Math., 225: 80–86, 2009.

[9] W. Lu, W. G. Ge and Z. H. Zhao, Oscillatory criteria for third-order nonlinear difference equations with impulses, J. Comput. Appl. Math., 234: 3366–3372, 2010.

[10] M. Peng, Oscllation theorems for second- order nolnlinear neutral dealy difference equations with impulses, Comput. Math. Appl., 44: 741–749, 2002.

[11] M. Peng, Oscillation criteria for second-order impulsive delay difference equations, Appl. Math. Comput., 146: 227–235, 2003.

[12] A. M. Samoilenko and N. A. Perestynk, Differential Equations with Impulse Effect, Visca Skola, Kiev, 1987.

[13] A. K. Tripathy and G. N. Chhatria, Oscillation criteria for forced first order nonlinear neutral impulsive difference system, Tatra Mt. Math. Publ., 71: 175–193, 2018.

[14] A. K. Tripathy and G. N. Chhatria, On oscillatory first order neutral impulsive difference equations, Math. Bohem., DOI: 10.21136/MB.2019.0002-18 (online first).

[15] G. P. Wei, The persistance of nonoscillatory solutions of difference equation under impulsive perturbations, Comput. Math. Appl., 50: 1579–1586, 2005.

[16] H. Zhang and L. Chen, Oscillations criteria for a class of second-order impulsive delay difference equations, Adv. Complex Syst., 9: 69–76, 2006.