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The Discrete Diamond-Alpha Imaginary Ellipse and Hyers–Ulam Stability

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Abstract

We introduce the imaginary diamond-alpha ellipse, which unifies and extends the left Hilger imaginary circle (forward, Delta case) and the right Hilger imaginary circle (backward, nabla case), for the discrete diamond-alpha derivative with constant step size. We then establish the Hyers–Ulam stability (HUS) of the first-order linear complex constant coefficient discrete diamond-alpha derivative equation, proving that the imaginary diamond-alpha ellipse fails to have HUS, while inside and outside the ellipse the equation has HUS. In particular, for each parameter value not on the diamond-alpha ellipse, we determine explicitly the best (minimum) HUS constant in terms of the elliptical real part of the coefficient.

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1 Hilger Circles and the Imaginary Diamond-α Ellipse

Given the complex number $\lambda \in \mathbb{C}$ and fixed step size $h > 0$, key quantities in the Hilger [6] complex plane are $|1+\lambda h|$ and $|1-\lambda h|$. Let $\mathbb{I}_h$ be the (left) Hilger imaginary circle; see also Bohner and Peterson [4, pages 51–53]. If $\lambda \in \mathbb{I}_h$, that is to say if there exist $\alpha, \beta \in \mathbb{R}$ with

$$\lambda = \alpha + i\beta, \qquad \left(\alpha + \frac{1}{h}\right)^2 + \beta^2 = \frac{1}{h^2},$$

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then $|1+\lambda h|=1$. If $0<|1+\lambda h|<1$ then λ is inside the left Hilger circle; if $|1+\lambda h|=1$ then $\lambda\in\mathbb{I}_h$ and λ is on the circle; if $|1+\lambda h|>1$ then λ is outside the left Hilger imaginary circle. Now for $\lambda\in\mathbb{C}\setminus\left\{\frac{-1}{h}\right\}$, the Hilger real part of λ is defined ([6] or [4, Definition 2.3]) by

$$\mathrm{Re}_h(\lambda):=\frac{|1+\lambda h|-1}{h}.$$

It follows that $\mathrm{Re}_h(\lambda)=0$ if and only if $\lambda\in\mathbb{I}_h$, with $\mathrm{Re}_h(\lambda)<0$ for those $\lambda\in\mathbb{C}\setminus\left\{\frac{-1}{h}\right\}$ inside the left Hilger imaginary circle, and $\mathrm{Re}_h(\lambda)>0$ for those $\lambda\in\mathbb{C}\setminus\left\{\frac{-1}{h}\right\}$ outside the left Hilger imaginary circle.

In an analogous way, let $\widehat{\mathbb{I}}_h$ be the right Hilger imaginary circle; this was presented by Ortigueira, Coito, and Trujillo [12] to facilitate a derivative-based discrete-time signal processing. If $\lambda\in\widehat{\mathbb{I}}_h$, that is to say if there exist $\alpha,\beta\in\mathbb{R}$ with

$$\lambda=\alpha+i\beta,\qquad\left(\alpha-\frac{1}{h}\right)^2+\beta^2=\frac{1}{h^2},$$

then $|1-\lambda h|=1$. If $0<|1-\lambda h|<1$ then λ is inside the right Hilger circle; if $|1-\lambda h|=1$ then $\lambda\in\widehat{\mathbb{I}}_h$ and λ is on the circle; if $|1-\lambda h|>1$ then λ is outside the right Hilger imaginary circle. Now for $\lambda\in\mathbb{C}\setminus\left\{\frac{1}{h}\right\}$, the right Hilger real part of λ is defined here by

$$\widehat{\mathrm{Re}}_h(\lambda):=\frac{1-|1-\lambda h|}{h}.$$

It follows that $\widehat{\mathrm{Re}}_h(\lambda)=0$ if and only if $\lambda\in\widehat{\mathbb{I}}_h$, with $\widehat{\mathrm{Re}}_h(\lambda)<0$ for those $\lambda\in\mathbb{C}\setminus\left\{\frac{1}{h}\right\}$ outside the right Hilger imaginary circle, and $\widehat{\mathrm{Re}}_h(\lambda)>0$ for those $\lambda\in\mathbb{C}\setminus\left\{\frac{1}{h}\right\}$ inside the right Hilger imaginary circle. Notice that in either case,

$$\lim_{h\to0^+}\mathrm{Re}_h(\lambda)=\mathrm{Re}(\lambda)=\lim_{h\to0^+}\widehat{\mathrm{Re}}_h(\lambda),$$

where $\mathrm{Re}(\lambda)$ is the standard real part of $\lambda\in\mathbb{C}$.

Combining these two notions, we introduce the imaginary $\Diamond_\alpha$ ellipse $\mathcal{E}_{(h,\alpha)}$, where $\lambda\in\mathcal{E}_{(h,\alpha)}$ means $\lambda\in\mathbb{C}$ takes the form

$$\lambda=\frac{(1-2\alpha)(1-\cos\theta)+i\sin\theta}{h}\in\mathcal{E}_{(h,\alpha)}\quad\text{for any}\quad\theta\in[0,2\pi].\tag{1.1}$$

Note that $\mathcal{E}_{(h,1)}=\mathbb{I}_h$, the (left) Hilger circle, and $\mathcal{E}_{(h,0)}=\widehat{\mathbb{I}}_h$, the (right) Hilger circle, respectively. For $\lambda\notin\mathcal{E}_{(h,\alpha)}$, we represent λ via

$$\lambda=\frac{R(1-2\alpha)+[\alpha(R^2+1)-1]\cos\theta}{hR}+i\frac{[1+\alpha(R^2-1)]\sin\theta}{hR}\in\mathbb{C}\tag{1.2}$$

for $\theta \in [0, 2\pi]$, where $R > 0$ is a real parameter with $R \neq 1$ and $R \neq \dfrac{1-\alpha}{\alpha}$. These are basically shifted elliptical coordinates for $\lambda \in \mathbb{C}$, while they technically describe a vertical line segment in the complex plane if $R = \sqrt{\dfrac{1-\alpha}{\alpha}}$ for $\alpha \in (0, 1)$.

Remark 1.1 (Elliptical Real Part). Let $\alpha \in [0, 1]$ and $h > 0$ be given. For any $\lambda \in \mathbb{C}$, λ can be expressed in terms of the imaginary ellipse (1.2). Define the elliptical real part of λ in (1.2) to be

$$\mathrm{Re}_{(h,\alpha)}(\lambda) := \begin{cases} \dfrac{(R-1)(1-\alpha-R\alpha)}{hR} & : \alpha \in \left(0, \dfrac{1}{2}\right], \\[2ex] \dfrac{(R-1)(\alpha R+\alpha-1)}{hR} & : \alpha \in \left(\dfrac{1}{2}, 1\right). \end{cases} \tag{1.3}$$

For the convenience of the reader, we note the following cases.

(i) If $R = 1$ or $R = \dfrac{1-\alpha}{\alpha}$, then $\mathrm{Re}_{(h,\alpha)}(\lambda) = 0$, placing λ on the imaginary $\Diamond_\alpha$ ellipse, that is $\lambda \in \mathcal{E}_{(h,\alpha)}$.

(ii) If $\alpha \in \left(0, \dfrac{1}{2}\right)$ and $0 < R < 1$ or $1 < \dfrac{1-\alpha}{\alpha} < R$, then $\mathrm{Re}_{(h,\alpha)}(\lambda) < 0$ and λ is outside the ellipse.

(iii) If $\alpha \in \left(0, \dfrac{1}{2}\right)$ and $1 < R < \dfrac{1-\alpha}{\alpha}$, then $\mathrm{Re}_{(h,\alpha)}(\lambda) > 0$ and λ is inside the ellipse.

(iv) If $\alpha = \dfrac{1}{2}$ there is a discontinuity, and the imaginary $\Diamond_\alpha$ ellipse $\mathcal{E}_{\left(h,\frac{1}{2}\right)}$ degenerates to the straight line segment $\dfrac{i\sin\theta}{h}$ on the imaginary axis.

(v) If $\alpha \in \left(\dfrac{1}{2}, 1\right)$ and $0 < R < \dfrac{1-\alpha}{\alpha} < 1$ or $R > 1$, then $\mathrm{Re}_{(h,\alpha)}(\lambda) > 0$ and λ is outside the ellipse.

(vi) If $\alpha \in \left(\dfrac{1}{2}, 1\right)$ and $\dfrac{1-\alpha}{\alpha} < R < 1$, then $\mathrm{Re}_{(h,\alpha)}(\lambda) < 0$ and λ is inside the ellipse.

For the two extreme cases, note that

$$\mathrm{Re}_{(h,0)}(\lambda) = \frac{R-1}{hR} = \frac{1-|1-\lambda h|}{h} = \widehat{\mathrm{Re}}_h(\lambda), \qquad R = |1-\lambda h|^{-1}$$

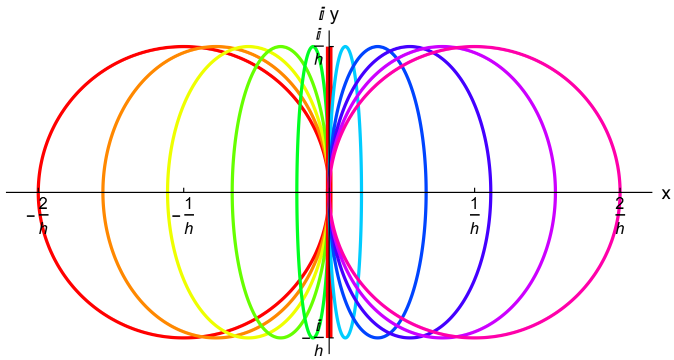

Figure 1.1: Various imaginary $\Diamond_\alpha$ ellipses for $\alpha \in [0,1]$, with $\alpha = 1$ generating the left Hilger circle, $\alpha = \dfrac{1}{2}$ the line segment, and $\alpha = 0$ the right circle.

and

$$\mathrm{Re}_{(h,1)}(\lambda) = \frac{R-1}{h} = \frac{|1+\lambda h|-1}{h} = \mathrm{Re}_h(\lambda), \qquad R = |1+\lambda h|,$$

which agree with the right Hilger real part and the left Hilger real part, respectively; see (1.2). ♡

2 Hyers–Ulam Stability for a Diamond-α Equation

Given $h\mathbb{Z} := \{hn : n \in \mathbb{Z}\}$, for any nonempty closed interval $I \subseteq \mathbb{R}$, let $\mathbb{T} := h\mathbb{Z} \cap I$. Define

$$\mathbb{T}_\kappa := \begin{cases} \mathbb{T}\backslash\{\min \mathbb{T}\} & : \min \mathbb{T} \text{ exists}, \\ \mathbb{T} & : \text{otherwise}, \end{cases} \qquad \mathbb{T}^\kappa := \begin{cases} \mathbb{T}\backslash\{\max \mathbb{T}\} & : \max \mathbb{T} \text{ exists}, \\ \mathbb{T} & : \text{otherwise}, \end{cases}$$

and set $\mathbb{T}_\kappa^\kappa = \mathbb{T}_\kappa \cap \mathbb{T}^\kappa$. In this paper we consider on $\mathbb{T}$ the Hyers–Ulam stability of the first-order linear homogeneous discrete diamond-alpha derivative equation with complex constant coefficient given by

$$\Diamond_\alpha x(t) - \lambda x(t) = 0, \quad \Diamond_\alpha x(t) := \alpha \Delta_h x(t) + (1-\alpha)\nabla_h x(t), \quad \alpha \in [0,1], \tag{2.1}$$

where $\lambda \in \mathbb{C}$ and $t \in \mathbb{T}_\kappa^\kappa$; here we use the forward difference operator $\Delta_h x(t) := \dfrac{x(t+h)-x(t)}{h}$ and the backward difference operator $\nabla_h x(t) := \dfrac{x(t)-x(t-h)}{h}$ for all $t \in \mathbb{T}_\kappa^\kappa$. Note that if a function x exists on $\mathbb{T}$, then $\Delta_h x$ exists on $\mathbb{T}^\kappa$ and $\nabla_h x$ exists on $\mathbb{T}_\kappa$. Thus, for the remainder of the paper, we assume that $\mathbb{T}$ and $\mathbb{T}_\kappa^\kappa$ are nonempty sets in $\mathbb{R}$. For more on the diamond-α derivative, see Sheng, Fadag, Henderson, and Davis [15] or Rogers Jr. and Sheng [14] and the references therein.

Definition 2.1. We say that (2.1) has Hyers–Ulam stability (HUS) on $\mathbb{T}$ if and only if there exists a constant $K > 0$ with the following property. For arbitrary $\varepsilon > 0$, if a function $\phi : \mathbb{T} \to \mathbb{C}$ satisfies

$$|\Diamond_\alpha \phi(t) - \lambda\phi(t)| \leq \varepsilon \quad \text{for all} \quad t \in \mathbb{T}^\kappa_\kappa,$$

then there exists a solution $x : \mathbb{T} \to \mathbb{C}$ of (2.1) such that $|\phi(t) - x(t)| \leq K\varepsilon$ for all $t \in \mathbb{T}$. Such a constant K is called an HUS constant for (2.1) on $\mathbb{T}$.

We start with values for which (2.1) does not have HUS, namely when $\lambda \in \mathcal{E}_{(h,\alpha)}$ is on the $\Diamond_\alpha$ imaginary ellipse introduced in (1.1).

Theorem 2.2. *For any $\alpha \in [0,1]$, if λ is on the diamond-α imaginary ellipse* (1.1)*, that is $\lambda \in \mathcal{E}_{(h,\alpha)}$, then* (2.1) *does not have Hyers–Ulam stability on $h\mathbb{Z}$.*

Proof. Let λ be as in (1.1). Given $\varepsilon > 0$, let

$$\phi(t) := \frac{\varepsilon t e^{\frac{i\theta t}{h}}}{1 + \alpha(-1 + e^{2i\theta})}.$$

Then $|\Diamond_\alpha \phi(t) - \lambda\phi(t)| = \left|(e^{i\theta})^{\frac{t-h}{h}} \varepsilon\right| = \varepsilon$ for all $t \in h\mathbb{Z}$. As

$$x(t) = c_1 \left(\frac{\alpha - 1}{\alpha e^{i\theta}}\right)^{\frac{t}{h}} + c_2 e^{\frac{i\theta t}{h}}$$

is the general solution to $\Diamond_\alpha x(t) - \lambda x(t) = 0$, we see that $|\phi(t) - x(t)| \to \infty$ as $t \to \pm\infty$ for any choice of the constants $c_1, c_2 \in \mathbb{R}$, so that (2.1) does not have HUS on $h\mathbb{Z}$. □

In light of Theorem 2.2 above, we now consider cases where (2.1) does have HUS, given $\alpha \in [0,1]$ and $\lambda \in \mathbb{C} \setminus \mathcal{E}_{(h,\alpha)}$. We begin with the special cases of $\alpha = 1$ and $\alpha = 0$, respectively, in the next few theorems. The $\alpha = 1$ case is known from Anderson and Onitsuka [3], while the $\alpha = 0$ is new to the literature.

Theorem 2.3. *[3, Theorem 2.6] Let $\alpha = 1$, so that $\Diamond_\alpha = \Delta_h$ in* (2.1)*. If $\lambda \in \mathbb{C}\backslash\{-1/h\}$ with $|1 + \lambda h| \neq 1$, then* (2.1) *has Hyers–Ulam stability with minimum HUS constant $\dfrac{h}{|1 - |1 + \lambda h||} = \dfrac{1}{|\operatorname{Re}_h(\lambda)|}$ on $h\mathbb{Z}$.*

Theorem 2.4. *Let $\alpha = 0$, so that $\Diamond_\alpha = \nabla_h$ in* (2.1)*. If $\lambda \in \mathbb{C}\backslash\{1/h\}$ with $|1 - \lambda h| \neq 1$, then* (2.1) *has Hyers–Ulam stability with minimum HUS constant*

$$\frac{h}{|1 - |1 - \lambda h||} =: \frac{1}{\left|\widehat{\operatorname{Re}}_h(\lambda)\right|}$$

on $h\mathbb{Z}$.

Proof. From $\nabla_h\phi(t) = \Delta_h\phi(t-h) = \dfrac{\phi(t)-\phi(t-h)}{h}$, we have

$$\begin{aligned}\nabla_h\phi(t) - \lambda\phi(t) &= \Delta_h\phi(t-h) - \lambda\left(h\Delta_h\phi(t-h) + \phi(t-h)\right)\\ &= (1-\lambda h)\Delta_h\phi(t-h) - \lambda\phi(t-h)\\ &= (1-\lambda h)\left(\Delta_h\phi(t-h) - \frac{\lambda}{1-\lambda h}\phi(t-h)\right) \qquad (2.2)\end{aligned}$$

for all $t \in \mathbb{T}_\kappa$. Note that $1-\lambda h \neq 0$ since $\lambda \neq 1/h$. Therefore, using the assumption $|\nabla_h\phi(t) - \lambda\phi(t)| \leq \varepsilon$ for all $t \in \mathbb{T}_\kappa$, we get

$$\left|\Delta_h\phi(t-h) - \frac{\lambda}{1-\lambda h}\phi(t-h)\right| \leq \frac{\varepsilon}{|1-\lambda h|}, \qquad t \in \mathbb{T}_\kappa.$$

That is,

$$\left|\Delta\phi(t) - \frac{\lambda}{1-\lambda h}\phi(t)\right| \leq \frac{\varepsilon}{|1-\lambda h|}, \qquad t \in \mathbb{T}^\kappa; \qquad (2.3)$$

note that $\dfrac{\lambda}{1-\lambda h} \neq \dfrac{-1}{h}$ is satisfied, so the conclusion of Theorem 2.3 holds. That is, there exists a solution x of (2.1), with λ replaced by $\dfrac{\lambda}{1-\lambda h}$ and ε replaced by $\dfrac{\varepsilon}{|1-\lambda h|}$, such that

$$|\phi(t) - x(t)| \leq \frac{h}{\left|1 - \left|\frac{1}{|1-\lambda h|}\right|\right|}\frac{\varepsilon}{|1-\lambda h|} = \frac{h\varepsilon}{||1-\lambda h| - 1|}.$$

It is straightforward to check that

$$\Delta_h x(t) - \frac{\lambda}{1-\lambda h}x(t) = 0$$

is equivalent to

$$\nabla_h x(t) - \lambda x(t) = 0.$$

Consequently, by Theorem 2.3 with $\alpha = 0$ in (2.1), since $\lambda \in \mathbb{C}\backslash\{1/h\}$ with $|1-\lambda h| \neq 1$, (2.1) has HUS with best HUS constant $\dfrac{h}{||1-\lambda h|-1|}$. This ends the proof. $\square$

By using Theorems 2.2, 2.3, and 2.4, we can establish the following result.

Theorem 2.5. *Let $\alpha = 0$, and assume from* (1.1) *that $\lambda \neq \dfrac{(1-\cos\theta) + i\sin\theta}{h}$ for all $\theta \in [0, 2\pi]$, namely $|1-\lambda h| \neq 1$, and assume $\lambda \neq \dfrac{1}{h}$. Let $\varepsilon > 0$ be a given arbitrary constant, and let $\phi : \mathbb{T} \to \mathbb{C}$ satisfy*

$$|\nabla_h\phi(t) - \lambda\phi(t) - f(t)| \leq \varepsilon, \qquad t \in \mathbb{T}_\kappa,$$

where f is a complex-valued function on $\mathbb{T}$. Then there exists a solution $x : \mathbb{T} \to \mathbb{C}$ of

$$\nabla_h x(t) - \lambda x(t) - f(t) = 0 \tag{2.4}$$

such that

$$|\phi(t) - x(t)| \leq \frac{h\varepsilon}{|1 - |1 - \lambda h||} = \frac{\varepsilon}{\left|\widehat{\mathrm{Re}}_h(\lambda)\right|}$$

for all $t \in \mathbb{T}$.

Proof. We assume that

$$|\nabla_h \phi(t) - \lambda\phi(t) - f(t)| \leq \varepsilon$$

for all $t \in \mathbb{T}_\kappa$. Let $u(t) = (1 - \lambda h)^{\frac{-t}{h}}\ \nabla_h^{-1}\left[f(t)(1 - \lambda h)^{\frac{t-h}{h}}\right]$ on $\mathbb{T}$, where ∇_h^{-1} is an anti-backward difference operator. We will check that u is a solution of (2.4). Note that

$$\begin{aligned} f(t)(1-\lambda h)^{\frac{t-h}{h}} &= \nabla_h\left[u(t)(1-\lambda h)^{\frac{t}{h}}\right] \\ &= \frac{1}{h}\left[u(t)(1-\lambda h)^{\frac{t}{h}} - u(t-h)(1-\lambda h)^{\frac{t-h}{h}}\right] \\ &= \frac{1}{h}[(1-\lambda h)u(t) - u(t-h)](1-\lambda h)^{\frac{t-h}{h}} \\ &= [\nabla_h u(t) - \lambda u(t)](1-\lambda h)^{\frac{t-h}{h}} \end{aligned}$$

holds for all $t \in \mathbb{T}_\kappa$. From this, we see that

$$\begin{aligned} \nabla_h(\phi(t) - u(t)) - \lambda(\phi(t) - u(t)) &= \nabla_h\phi(t) - \lambda\phi(t) - (\nabla_h u(t) - \lambda u(t)) \\ &= \nabla_h\phi(t) - \lambda\phi(t) - f(t) \end{aligned}$$

for all $t \in \mathbb{T}_\kappa$, and thus,

$$|\nabla_h(\phi(t) - u(t)) - \lambda(\phi(t) - u(t))| = |\nabla_h\phi(t) - \lambda\phi(t) - f(t)| \leq \varepsilon$$

for all $t \in \mathbb{T}_\kappa$, by assumption. Using Theorem 2.4, we can find a solution $v : \mathbb{T} \to \mathbb{C}$ of (2.1) with $\alpha = 0$ such that $|(\phi(t) - u(t)) - v(t)| \leq \dfrac{h\varepsilon}{|1 - |1 - \lambda h||}$ for all $t \in \mathbb{T}$. Let $x(t) = u(t) + v(t)$ for all $t \in \mathbb{T}$. Then we see that

$$\nabla_h x(t) = \nabla_h u(t) + \nabla_h v(t) = \lambda u(t) + f(t) + \lambda v(t) = \lambda x(t) + f(t)$$

holds on $\mathbb{T}_\kappa$. This means that x is a solution of (2.4) on $\mathbb{T}$. This completes the proof. □

3 HUS for the General Discrete Diamond-α Case

The Ulam type of stability was initiated by Ulam [17], joined almost immediately by Hyers [7], and generalized by Rassias [13]. For the study of this stability for ordinary differential equations of first order, see Miura, Miyajima, and Takahasi [9] and Jung [8]; more recently, see Onitsuka and Shoji [11]. For difference equations and the influence of the step size on Hyers–Ulam stability, see Onitsuka [10] and [3]. On general time scales, see András and Mészáros [1] and Shen [16]. Also recently, Brzdęk, Popa, Raşa and Xu [5] wrote on the Ulam stability of operators.

In the remainder of the paper, we will explore HUS and HUS constants for (2.1) in the case that $\alpha \in (0,1)$, and by Theorem 2.2, $\lambda \in \mathbb{C} \setminus \mathcal{E}_{(h,\alpha)}$. That is, λ is either inside or outside the imaginary $\Diamond_\alpha$ ellipse $\mathcal{E}_{(h,\alpha)}$, but not on it. With these considerations in mind, let $\lambda \in \mathbb{C} \setminus \mathcal{E}_{(h,\alpha)}$ be denoted as in (1.2). Note that upon expansion, (2.1) is equivalent to the second-order linear difference equation

$$\alpha x(t+h) + (1-2\alpha-\lambda h)x(t) + (\alpha-1)x(t-h) = 0. \tag{3.1}$$

If we use λ in (1.2) for $R > 0$ with $R \neq 1$ and $R \neq \dfrac{1-\alpha}{\alpha}$, we may denote the characteristic zeros of this equation as

$$\Lambda_R := Re^{i\theta} \quad \text{and} \quad \Lambda_\alpha := \frac{\alpha-1}{\alpha Re^{i\theta}}; \tag{3.2}$$

then the general solution to (2.1) is

$$x(t) = c_1(\Lambda_R)^{\frac{t}{h}} + c_2(\Lambda_\alpha)^{\frac{t}{h}}, \qquad t \in \mathbb{T}, \tag{3.3}$$

for arbitrary constants $c_1, c_2 \in \mathbb{C}$.

Remark 3.1. We can think of the main equation (2.1) as the eigenvalue problem for the operator $\Diamond_\alpha$,

$$\Diamond_\alpha x(t) = \lambda x(t),$$

with eigenpairs (λ, x). By introducing the elliptical form of the eigenvalue λ in (1.2) ahead of time, we have arrived at a nice radial form for the eigenfunctions via (3.2) and (3.3). Thus, we avoid the branch cuts in the complex plane that would normally arise by using the quadratic formula to find the zeros of the characteristic equation (3.1) and needing to use a square root. Not realizing this earlier hampered us in [2]. ♡

Theorem 3.2. *Let $\alpha \in (0,1)$, $\lambda \in \mathbb{C}$ be given by* (1.2) *with $R \in (0,1) \cup (1,\infty)$ but $R \neq \dfrac{1-\alpha}{\alpha}$, and let Λ_R and Λ_α be given as in* (3.2)*. Let $\varepsilon > 0$ be a given arbitrary constant, and let $\phi : \mathbb{T} \to \mathbb{C}$ satisfy*

$$|\Diamond_\alpha \phi(t) - \lambda\phi(t)| \leq \varepsilon \quad \textit{for all} \quad t \in \mathbb{T}_\kappa^\kappa.$$

Then one of the following holds.

(i) If $\alpha \in \left(0, \frac{1}{2}\right)$ *and* $1 < R < \frac{1-\alpha}{\alpha}$*, then* (2.1) *has Hyers–Ulam stability with an HUS constant* $\frac{hR}{(R-1)(1-\alpha-R\alpha)}$ *on* $\mathbb{T}$*;*

(ii) If $\alpha \in \left(0, \frac{1}{2}\right]$ *and* $1 \leq \frac{1-\alpha}{\alpha} < R$*, then* (2.1) *has Hyers–Ulam stability with an HUS constant* $\frac{hR}{(R-1)(R\alpha+\alpha-1)}$ *on* $\mathbb{T}$*;*

(iii) If $\alpha \in \left[\frac{1}{2}, 1\right)$ *and* $0 < R < \frac{1-\alpha}{\alpha} \leq 1$*, then* (2.1) *has Hyers–Ulam stability with an HUS constant* $\frac{hR}{(1-R)(1-\alpha-R\alpha)}$ *on* $\mathbb{T}$*;*

(iv) If $\alpha \in \left(\frac{1}{2}, 1\right)$ *and* $0 < \frac{1-\alpha}{\alpha} < R < 1$*, then* (2.1) *has Hyers–Ulam stability with an HUS constant* $\frac{hR}{(1-R)(R\alpha+\alpha-1)}$ *on* $\mathbb{T}$.

Proof. First, we see

$$\Lambda_\alpha + \Lambda_R = \frac{\lambda h + 2\alpha - 1}{\alpha} \quad \text{and} \quad \Lambda_\alpha \Lambda_R = \frac{\alpha - 1}{\alpha} \tag{3.4}$$

holds. Now, for arbitrary $\varepsilon > 0$, we assume that a function $\phi : \mathbb{T} \to \mathbb{C}$ satisfies

$$|\Diamond_\alpha \phi(t) - \lambda\phi(t)| \leq \varepsilon$$

for all $t \in \mathbb{T}^\kappa_\kappa$. Let $\psi(t) = (\alpha h \Lambda_\alpha)\nabla_h\phi(t) - \alpha(\Lambda_\alpha - 1)\phi(t)$ for all $t \in \mathbb{T}_\kappa$. Using (3.4), we have

$$\begin{aligned}
\Delta_h\psi(t) - \left(\frac{\Lambda_R - 1}{h}\right)\psi(t) &= (\alpha h \Lambda_\alpha)\Delta_h(\nabla_h\phi(t)) - \alpha(\Lambda_\alpha - 1)\Delta_h\phi(t) \\
&\quad -\alpha h \Lambda_\alpha \left(\frac{\Lambda_R - 1}{h}\right)\nabla_h\phi(t) \\
&\quad +\alpha(\Lambda_\alpha - 1)\left(\frac{\Lambda_R - 1}{h}\right)\phi(t) \\
&= \alpha\Lambda_\alpha\Delta_h(\phi(t) - \phi(t-h)) - \alpha(\Lambda_\alpha - 1)\Delta_h\phi(t) \\
&\quad -\alpha(\Lambda_\alpha\Lambda_R - \Lambda_\alpha)\nabla_h\phi(t) \\
&\quad +\frac{\alpha}{h}(\Lambda_\alpha\Lambda_R - \Lambda_\alpha - \Lambda_R + 1)\phi(t) \\
&= \alpha\Lambda_\alpha\Delta_h\phi(t) - \alpha\Lambda_\alpha\nabla_h\phi(t) - \alpha(\Lambda_\alpha - 1)\Delta_h\phi(t) \\
&\quad -\alpha(\Lambda_\alpha\Lambda_R - \Lambda_\alpha)\nabla_h\phi(t) - \lambda\phi(t)
\end{aligned}$$

$$\begin{aligned} &= \alpha\Delta_h\phi(t) - \alpha\Lambda_\alpha\Lambda_R\nabla_h\phi(t) - \lambda\phi(t) \\ &= \Diamond_\alpha\phi(t) - \lambda\phi(t) \end{aligned} \tag{3.5}$$

for all $t \in \mathbb{T}_\kappa^\kappa$. In summary, we have

$$\left|\Delta_h\psi(t) - \left(\frac{\Lambda_R - 1}{h}\right)\psi(t)\right| = |\Diamond_\alpha\phi(t) - \lambda\phi(t)| \le \varepsilon \tag{3.6}$$

for all $t \in \mathbb{T}_\kappa^\kappa$. The proof can be divided into four cases: (i) $\alpha \in \left(0, \frac{1}{2}\right)$ and $1 < R < \frac{1-\alpha}{\alpha}$; (ii) $\alpha \in \left(0, \frac{1}{2}\right]$ and $1 \le \frac{1-\alpha}{\alpha} < R$; (iii) $\alpha \in \left[\frac{1}{2}, 1\right)$ and $0 < R < \frac{1-\alpha}{\alpha} \le 1$; (iv) $\alpha \in \left(\frac{1}{2}, 1\right)$ and $0 < \frac{1-\alpha}{\alpha} < R < 1$.

First, we consider case (i) $\alpha \in \left(0, \frac{1}{2}\right)$ and $1 < R < \frac{1-\alpha}{\alpha}$. Using Theorem 2.3 with (3.6), we conclude that there exists a solution $y : \mathbb{T}_\kappa \to \mathbb{C}$ of

$$\Delta_h y(t) - \left(\frac{\Lambda_R - 1}{h}\right)y(t) = 0 \tag{3.7}$$

such that $|\psi(t) - y(t)| \le \dfrac{h\varepsilon}{|1 - |\Lambda_R||}$ for all $t \in \mathbb{T}_\kappa$. This inequality implies that

$$\left|\nabla_h\phi(t) - \frac{\Lambda_\alpha - 1}{h\Lambda_\alpha}\phi(t) - \frac{y(t)}{h\alpha\Lambda_\alpha}\right| \le \frac{R\varepsilon}{(1-\alpha)(|\Lambda_R| - 1)} = \frac{R\varepsilon}{(1-\alpha)(R-1)} \tag{3.8}$$

for all $t \in \mathbb{T}_\kappa$. Using Theorem 2.5 with (3.8), we can find a solution $x : \mathbb{T} \to \mathbb{C}$ of

$$\nabla_h x(t) - \frac{\Lambda_\alpha - 1}{h\Lambda_\alpha}x(t) - \frac{y(t)}{h\alpha\Lambda_\alpha} = 0 \tag{3.9}$$

such that

$$|\phi(t) - x(t)| \le \frac{\frac{hR\varepsilon}{(1-\alpha)(R-1)}}{\left|1 - \left|\frac{1}{|\Lambda_\alpha|}\right|\right|} = \frac{hR\varepsilon}{(R-1)(1-\alpha-R\alpha)}$$

for all $t \in \mathbb{T}$. Now, we will show that this x is a solution of (2.1) on $\mathbb{T}$. Recalling (3.5), we have

$$\begin{aligned} \Diamond_\alpha x(t) - \lambda x(t) &= \Delta_h\left[h\alpha\Lambda_\alpha(\nabla_h x(t)) - \alpha(\Lambda_\alpha - 1)x(t)\right] \\ &\quad - \left(\frac{\Lambda_R - 1}{h}\right)\left[h\alpha\Lambda_\alpha\nabla_h x(t) - \alpha(\Lambda_\alpha - 1)x(t)\right] \end{aligned}$$

for all $t \in \mathbb{T}_\kappa^\kappa$. Using (3.7) and (3.9), we get

$$\Diamond_\alpha x(t) - \lambda x(t) = \Delta y(t) - \left(\frac{\Lambda_R - 1}{h}\right)y(t) = 0$$

for all $t \in \mathbb{T}_\kappa^\kappa$. Thus, this x is a solution of (2.1) on $\mathbb{T}$. Consequently, (2.1) has Hyers–Ulam stability with an HUS constant $\dfrac{hR}{(R-1)(1-\alpha-R\alpha)}$ on $\mathbb{T}$.

Next, we consider case (ii) $\alpha \in \left(0, \dfrac{1}{2}\right]$ and $1 \leq \dfrac{1-\alpha}{\alpha} < R$. Repeating the same argument as in the proof of case (i), we see that there exists a solution $y : \mathbb{T}_\kappa \to \mathbb{C}$ of (3.7) such that $|\psi(t) - y(t)| \leq \dfrac{h\varepsilon}{R-1}$ for all $t \in \mathbb{T}_\kappa$. That is, (3.8) holds for all $t \in \mathbb{T}_\kappa$. Using Theorem 2.5 with (3.8), we can find a solution $x : \mathbb{T} \to \mathbb{C}$ of (3.9) such that

$$|\phi(t) - x(t)| \leq \frac{\frac{hR\varepsilon}{(1-\alpha)(R-1)}}{\frac{R\alpha}{1-\alpha} - 1} = \frac{hR\varepsilon}{(R-1)(R\alpha+\alpha-1)}$$

for all $t \in \mathbb{T}$. Repeating the same argument as in the proof of case (i), x is a solution of (2.1) on $\mathbb{T}$. Consequently, (2.1) has Hyers–Ulam stability with an HUS constant $\dfrac{hR}{(R-1)(R\alpha+\alpha-1)}$ on $\mathbb{T}$.

Now, we consider case (iii) $\alpha \in \left[\dfrac{1}{2}, 1\right)$ and $0 < R < \dfrac{1-\alpha}{\alpha} \leq 1$. Using Theorem 2.3 with (3.6), we conclude that there exists a solution $y : \mathbb{T}_\kappa \to \mathbb{C}$ of (3.7) such that $|\psi(t) - y(t)| \leq \dfrac{\varepsilon}{1-R}$ for all $t \in \mathbb{T}_\kappa$. That is,

$$\left|\nabla\phi(t) - \frac{\Lambda_\alpha - 1}{h\Lambda_\alpha}\phi(t) - \frac{y(t)}{h\alpha\Lambda_\alpha}\right| \leq \frac{R\varepsilon}{(1-\alpha)(1-R)}$$

holds for all $t \in \mathbb{T}_\kappa$. Using Theorem 2.5, we can find a solution $x : \mathbb{T} \to \mathbb{C}$ of (3.9) such that

$$|\phi(t) - x(t)| \leq \frac{\frac{hR\varepsilon}{(1-\alpha)(1-R)}}{\frac{(1-\alpha)-R\alpha}{(1-\alpha)}} = \frac{hR\varepsilon}{(1-R)(1-\alpha-R\alpha)}$$

for all $t \in \mathbb{T}$. Again, x is a solution of (2.1) on $\mathbb{T}$. Consequently, (2.1) has Hyers–Ulam stability with an HUS constant $\dfrac{hR}{(1-R)(1-\alpha-R\alpha)}$ on $\mathbb{T}$.

Finally, we consider case (iv) $\alpha \in \left(\dfrac{1}{2}, 1\right)$ and $0 < \dfrac{1-\alpha}{\alpha} < R < 1$. Using these facts and the same arguments as above, we see that (2.1) has Hyers–Ulam stability with an HUS constant $\dfrac{hR}{(1-R)(R\alpha+\alpha-1)}$ on $\mathbb{T}$. This completes the proof. □

From Theorems 2.2, 2.3, 2.4 and 3.2, we obtain the following result.

Theorem 3.3. *For any* $\alpha \in [0, 1]$ *and* $\theta \in [0, 2\pi]$, *if*

$$\lambda \neq \frac{(1-2\alpha)(1-\cos\theta) + i\sin\theta}{h} \in \mathbb{C},$$

then (2.1) *has Hyers–Ulam stability on* $\mathbb{T}$*, with an HUS constant*

$$\frac{hR}{|R-1||\alpha R+\alpha-1|}=\frac{1}{\left|\mathrm{Re}_{(h,\alpha)}(\lambda)\right|}$$

where $\lambda\in\mathbb{C}\setminus\mathcal{E}_{(h,\alpha)}$ *is given by* (1.2) *for* $R>0$ *with* $R\neq 1$ *and* $R\neq\dfrac{1-\alpha}{\alpha}$*, and* $\mathrm{Re}_{(h,\alpha)}(\lambda)$ *is the elliptical real part of* λ *given by* (1.3).

Proof. Let $\alpha\in[0,1]$, $\lambda\in\mathbb{C}$ be given by (1.2) with $R\in(0,1)\cup(1,\infty)$. Consider the case $\alpha=1$. From Theorem 2.3, (2.1) has HUS on $\mathbb{T}$. Moreover, an HUS constant for (2.1) is $\dfrac{h}{|1-|1+\lambda h||}$. When $\alpha=1$, from (1.2) we have $1+\lambda h=Re^{i\theta}$ and

$$\frac{hR}{|R-1||R\alpha+\alpha-1|}=\frac{h}{|R-1|}=\frac{h}{||1+\lambda h|-1|}.$$

Thus, the assertion is true when $\alpha=1$.

Next, we consider the case $\alpha=0$. From Theorems 2.4 and 2.5, (2.1) has HUS on $\mathbb{T}$. Moreover, an HUS constant for (2.1) is $\dfrac{h}{|1-|1-\lambda h||}$. When $\alpha=0$, from (1.2) we have $1-\lambda h=(Re^{i\theta})^{-1}$ and

$$\frac{hR}{|R-1||R\alpha+\alpha-1|}=\frac{hR}{|1-R|}=\frac{h}{|1-|1-\lambda h||}.$$

Thus, the assertion is true when $\alpha=0$. By Theorem 3.2, we can conclude that the case $\alpha\in(0,1)$ is true as well, completing the proof. □

Remark 3.4. Consider Theorem 3.3. If $\theta=\dfrac{\pi}{2}$ and $h>0$, then λ in (1.2) becomes

$$\lambda=\frac{1-2\alpha}{h}+\frac{i\left(1+\alpha\left(R^2-1\right)\right)}{hR}$$

for $R>0$ with $R\neq 1$ and $R\neq\dfrac{1-\alpha}{\alpha}$. Since $\lambda\in\mathbb{C}\setminus\mathcal{E}_{(h,\alpha)}$, by Theorem 3.3 equation (2.1) has Hyers–Ulam stability with an HUS constant

$$\frac{hR}{|R-1||\alpha R+\alpha-1|}=\frac{1}{\left|\mathrm{Re}_{(h,\alpha)}(\lambda)\right|}$$

on $\mathbb{T}$. Consider $\phi:\mathbb{T}\to\mathbb{C}$ given by

$$\phi(t):=\frac{-hR\varepsilon(i)^{\frac{t-h}{h}}}{(R-1)(\alpha R+\alpha-1)}+c_1\left(\frac{i(\alpha-1)}{-\alpha R}\right)^{\frac{t}{h}}+c_2(iR)^{\frac{t}{h}},$$

which satisfies

$$|\Diamond_\alpha \phi(t) - \lambda\phi(t)| = \left|(i)^{\frac{t}{h}}\varepsilon\right| = \varepsilon \quad \text{for all} \quad t \in \mathbb{T}^\kappa_\kappa = h\mathbb{Z}$$

for any given arbitrary constant $\varepsilon > 0$ and for arbitrary constants $c_1, c_2 \in \mathbb{C}$. Clearly

$$x(t) = c_1 \left(\frac{i(\alpha-1)}{-\alpha R}\right)^{\frac{t}{h}} + c_2(iR)^{\frac{t}{h}}$$

is a solution of (2.1) for this value of θ, yielding

$$|\phi(t) - x(t)| = \left|\frac{-hR\varepsilon(i)^{\frac{t-h}{h}}}{(R-1)(\alpha R+\alpha-1)}\right| = \frac{hR\varepsilon}{|R-1||\alpha R+\alpha-1|}, \quad t \in \mathbb{T}.$$

Thus

$$\frac{hR}{|R-1||\alpha R+\alpha-1|} = \frac{1}{\left|\mathrm{Re}_{(h,\alpha)}(\lambda)\right|}$$

is the best (minimum) HUS constant in this case. ♡

Remark 3.5. This improves and extends the results in [2], where only $\lambda \in \mathbb{R}$ were considered, and the best HUS constant could not be found in all cases [2, Remark 1], since as seen above a complex-valued λ is required. ♡

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