International Journal of Difference Equations
ISSN 0973-6069, Volume 13, Number 2, pp. 157–171 (2018)
http://campus.mst.edu/ijde

Asymptotic Behavior of the Solution of a System of Difference Equations

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Abstract

In this paper, we study the boundedness character and persistence, local and global behavior, and rate of convergence of positive solutions of following system of rational difference equations

$$x_{n+1} = \frac{\alpha_1 + \beta_1 e^{-y_n}}{a_1 + b_1 x_n},\ y_{n+1} = \frac{\alpha_2 + \beta_2 e^{-x_n}}{a_2 + b_2 y_n},$$

where the parameters $\alpha_i,\ \beta_i,\ a_i,\ b_i$ for $i \in \{1, 2\}$ and the initial conditions x_0, y_0 are positive real numbers. Some numerical example are given to verify our theoretical results.

AMS Subject Classifications: 39A10, 40A05.
Keywords: System of difference equations, boundedness, persistence, asymptotic behavior, rate of convergence.

1 Introduction

The study of asymptotic stability of positive solutions in difference equations is extremely useful in the behavior analysis of mathematical models in various biological systems and other applications. In recent years, the global asymptotic behavior of the difference equations of exponential form has been one of the main topics in the theory of difference equations [3,4,10–13]. In particular, in [5] El-Metwally et al. investigated boundedness character, asymptotic behavior, periodicity nature of the positive solutions, and stability of equilibrium point of the following population model:

$$x_{n+1} = \alpha + \beta x_{n-1} e^{-x_n}.$$

Received September 8, 2018; Accepted November 2, 2018
Communicated by Martin Bohner

where the parameters α and β are positive numbers and the initial conditions x_{-1} and x_0 are arbitrary non-negative numbers.

Aboutaleb et al. [1] investigated the global asymptotic stability of the recursive sequence

$$x_{n+1} = \frac{\alpha - \beta x_n}{\gamma + x_{n-1}}$$

where α, β, γ are non-negative and the initial conditions x_{-1}, x_0 are arbitrary.

Ozturk et al. [9] have investigated the boundedness, asymptotic behavior, periodicity, and stability of the positive solutions of the following difference equation:

$$y_{n+1} = \frac{\alpha + \beta e^{-y_n}}{\gamma + y_{n-1}},$$

where the parameters α, β, γ are positive numbers and the initial conditions are arbitrary non-negative numbers.

Motivated by the aforementioned study, our goal in this paper is to investigate the qualitative behavior of positive solutions following system of exponential difference equations

$$x_{n+1} = \frac{\alpha_1 + \beta_1 e^{-y_n}}{a_1 + b_1 x_n},\ y_{n+1} = \frac{\alpha_2 + \beta_2 e^{-x_n}}{a_2 + b_2 y_n}, \tag{1.1}$$

where the parameters $\alpha_i,\ \beta_i,\ a_i,\ b_i$ for $i \in \{1, 2\}$ and the initial conditions x_0, y_0 are positive real numbers.

More precisely, we investigate the boundedness character, persistence, local asymptotic stability and global behavior of unique positive equilibrium point, and rate of convergence of positive solutions of system (1.1) which converge to its unique positive equilibrium point. For applications and basic theory of difference equations we refer to [2, 6–8, 15].

2 Preliminaries

Let us consider second-dimensional discrete dynamical system of the following form:

$$x_{n+1} = f(x_n, y_n), \quad y_{n+1} = g(x_n, y_n),\ n = 0, 1, ... \tag{2.1}$$

where $f : I \times J \to I$ and $g : I \times J \to J$ are continuously differentiable functions and I, J are some intervals of real numbers. Furthermore, a solution $\{x_n, y_n\}_{n=0}^{\infty}$ of system (2.1) is uniquely determined by initial conditions $(x_0, y_0) \in I \times J$. Along with system (2.1), we consider the corresponding vector map $F = (f, g)$. An equilibrium point of (2.1) is a point $(\overline{x}, \overline{y})$ that satisfies

$$\overline{x} = f(\overline{x}, \overline{y}), \quad \overline{y} = g(\overline{x}, \overline{y}).$$

The point $(\overline{x}, \overline{y})$ is also called a fixed point of the vector map F.

Definition 2.1. Let $(\overline{x}, \overline{y})$ be an equilibrium point of system (2.1).

i) An equilibrium point $(\overline{x}, \overline{y})$ is said to be stable if for any $\varepsilon > 0$ there is $\delta > 0$ such that for every initial conditions (x_0, y_0), if $||(x_0, y_0) - (\overline{x}, \overline{y})|| < \delta$ implies that $||(x_n, y_n) - (\overline{x}, \overline{y})|| < \varepsilon$ for all $n > 0$, where $||.||$ is usual Euclidean norm in $\mathbb{R}^2$.

ii) An equilibrium point $(\overline{x}, \overline{y})$ is said to be unstable if it is not stable.

iii) An equilibrium point $(\overline{x}, \overline{y})$ is said to be asymptotically stable if there exists $r > 0$ such that $(x_n, y_n) \to (\overline{x}, \overline{y})$ as $n \to \infty$ for all (x_0, y_0) that satisfy $||(x_0, y_0) - (\overline{x}, \overline{y})|| < r$.

iv) An equilibrium point $(\overline{x}, \overline{y})$ is called global attractor if $(x_n, y_n) \to (\overline{x}, \overline{y})$ as $n \to \infty$.

v) An equilibrium point $(\overline{x}, \overline{y})$ is called asymptotic global attractor if it is a global attractor and stable.

Definition 2.2. Let $(\overline{x}, \overline{y})$ be an equilibrium point of a map $F = (f, g)$ where f and g are continuously differentiable functions at $(\overline{x}, \overline{y})$. The linearized system of (2.1) about the equilibrium point $(\overline{x}, \overline{y})$ is

$$X_{n+1} = J_F X_n$$

where $X_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$ and J_F is the Jacobian matrix of system (2.1) about the equilibrium point $(\overline{x}, \overline{y})$.

Lemma 2.3 (See [15])**.** *Assume that $X_{n+1} = F(X_n), n = 0, 1, ...,$ is a system of difference equations such that $\overline{X}$ is a fixed point of F. If all eigenvalues of the Jacobian matrix J_F about $\overline{X}$ lie inside the open unit disk $|\lambda| < 1$, then $\overline{X}$ is locally asymptotically stable. If one of them has a modulus greater than one, then $\overline{X}$ is unstable.*

The following results give the rate of convergence of solutions of a system of difference equations

$$X_{n+1} = [A + B(n)]X_n \tag{2.2}$$

where X_n is a m-dimensional vector, $A \in \mathbf{C}^{m\times m}$ is a constant matrix, and $B: \mathbb{Z}^+ \longrightarrow \mathbf{C}^{m\times m}$ is a matrix function satisfying

$$\|B(n)\| \to 0 \text{ when } n \to \infty, \tag{2.3}$$

where $||.||$ denotes any matrix norm which is associated with the vector norm

$$\|(x,\ y)\| = \sqrt{x^2 + y^2}.$$

Proposition 2.4 (Perron's theorem [14]). *Assume that condition* (2.3) *holds. If X_n is a solution of system* (2.2)*, then either $X_n = 0$ for all large n or*

$$\rho = \lim_{n\to\infty} \sqrt[n]{\|X_n\|} \tag{2.4}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A.

Proposition 2.5 (See [14]). *Assume that condition* (2.3) *holds. If X_n is a solution of system* (2.2)*, then either $X_n = 0$ for all large n or*

$$\rho = \lim_{n\to\infty} \frac{\|X_{n+1}\|}{\|X_n\|} \tag{2.5}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A.

3 Main Results

3.1 Boundedness and Persistence

In this section, we show the boundedness and persistence of the positive solutions of system (1.1).

Lemma 3.1. *Every positive solution $\{(x_n, y_n)\}$ of system* (1.1) *is bounded and persists.*

Proof. For any positive solution $\{(x_n, y_n)\}$ of system (1.1), one has

$$x_{n+1} \le \frac{\alpha_1 + \beta_1}{a_1} = U_1, \quad y_{n+1} \le \frac{\alpha_2 + \beta_2}{a_2} = U_2, \ n = 0, 1, 2, \dots \tag{3.1}$$

Furthermore, from system (1.1) and (3.1), we obtain that

$$x_{n+1} \ge \frac{\alpha_1 + \beta_1 e^{-U_2}}{a_1 + b_1 U_1} = L_1, \quad y_{n+1} \ge \frac{\alpha_2 + \beta_2 e^{-U_1}}{a_2 + b_2 U_2} = L_2, \ n = 1, 2, 3, \dots \tag{3.2}$$

From (3.1) and (3.2), it follows that

$$L_1 \le x_n \le U_1, \quad L_2 \le y_n \le U_2, \ n = 2, 3, 4, \dots$$

So the proof is complete. □

Lemma 3.2. *Let $\{(x_n, y_n)\}$ be a positive solution of system* (1.1)*. Then $[L_1, U_1] \times [L_2, U_2]$ is invariant set for system* (1.1)*.*

Proof. The proof follows by induction. □

3.2 Stability Analysis

In this section, we shall investigate the asymptotic behavior of system (1.1). Similar method can be found in [16].
Let $(\overline{x}, \overline{y})$ be the equilibrium point of system (1.1) then

$$\overline{x} = \frac{\alpha_1 + \beta_1 e^{-\overline{y}}}{a_1 + b_1\overline{x}}, \quad \overline{y} = \frac{\alpha_2 + \beta_2 e^{-\overline{x}}}{a_2 + b_2\overline{y}}.$$

The linearized form of system (1.1) about the equilibrium point $(\overline{x}, \overline{y})$ is given by

$$X_{n+1} = J_F(\overline{x}, \overline{y})X_n,$$

where $X_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$ and $J_F(\overline{x}, \overline{y}) = \begin{pmatrix} -\dfrac{b_1\overline{x}}{a_1 + b_1\overline{x}} & -\dfrac{\beta_1 e^{-\overline{y}}}{a_1 + b_1\overline{x}} \\ -\dfrac{\beta_2 e^{-\overline{x}}}{a_2 + b_2\overline{y}} & -\dfrac{b_2\overline{y}}{a_2 + b_2\overline{y}} \end{pmatrix}$.

In order to study the asymptotic behavior of positive equilibrium of system (1.1), we state and proof the following theorem.

Theorem 3.3. *Assume that $f : (0, \infty) \times (0, \infty) \to (0, \infty)$ and $g : (0, \infty) \times (0, \infty) \to (0, \infty)$ be continuous functions and a, b, c, d are positive real numbers with $a < b, c < d$. Moreover, suppose that $f : [a, b] \times [c, d] \to [a, b]$ and $g : [a, b] \times [c, d] \to [c, d]$ such that following conditions are satisfied:*

i) $f(x, y), g(x, y)$ are decreasing in both x and y.

ii) Let m_1, M_1, m_2, M_2 are real numbers such that

$$m_1 = f(M_1, M_2), M_1 = f(m_1, m_2), m_2 = g(M_1, M_2), M_2 = g(m_1, m_2) \quad (3.3)$$

then $m_1 = M_1$ and $m_2 = M_2$.

Then the system of difference equations (2.1) *has a unique positive equilibrium point $(\overline{x}, \overline{y})$ and every solution (x_n, y_n) of the system* (2.1) *with $(x_0, y_0) \in [a, b] \times [c, d]$ converges to the unique equilibrium $(\bar{x}, \bar{y})$.*

Proof. Consider the function

$$T : [a, b] \times [c, d] \longrightarrow [a, b] \times [c, d]$$

given by

$$T(x, y) = [f(x, y), g(x, y)].$$

Following Brouwer's fixed point theorem, the function T has a fixed point $(\bar{x}, \bar{y})$ which is clearly an equilibrium point of system (2.1), and so it true that the system (2.1) has at

least one equilibrium point.
Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of system (2.1). It suffices to show that

$$\lim_{n\to\infty}(x_n, y_n) = (\bar{x}, \bar{y}).$$

Set

$$m_1^0 = a, M_1^0 = b, m_2^0 = c, M_2^0 = d,$$

and for $k = 1, 2, 3, \ldots$ define

$$\begin{cases} m_1^k = f(M_1^{k-1}, M_2^{k-1}), & M_1^k = f(m_1^{k-1}, m_2^{k-1}), \\ m_2^k = g(M_1^{k-1}, M_2^{k-1}), & M_2^k = g(m_1^{k-1}, m_2^{k-1}). \end{cases} \tag{3.4}$$

Using the assumption (i), we have

$$m_1^0 = a \le f(M_1^0, M_2^0) \le f(m_1^0, m_2^0) \le b = M_1^0,$$

and

$$m_2^0 = c \le g(M_1^0, M_2^0) \le g(m_1^0, m_2^0) \le d = M_2^0,$$

and so we see that

$$m_1^0 \le m_1^1 \le M_1^1 \le M_1^0, \quad \text{and} \quad m_2^0 \le m_2^1 \le M_2^1 \le M_2^0.$$

Similarly, we have

$$m_1^1 = f(M_1^0, M_2^0) \le f(M_1^1, M_2^1) \le f(m_1^1, m_2^1) \le f(m_1^0, m_2^0) = M_1^1,$$

and

$$m_2^1 = g(M_1^0, M_2^0) \le g(M_1^1, M_2^1) \le g(m_1^1, m_2^1) \le g(m_1^0, m_2^0) = M_2^1,$$

and so we see that

$$m_1^0 \le m_1^1 \le m_1^2 \le M_1^2 \le M_1^1 \le M_1^0,$$

and

$$m_2^0 \le m_2^1 \le m_2^2 \le M_2^2 \le M_2^1 \le M_2^0.$$

Note that

$$m_1^0 = a \le x_n \le b = M_1^0, \text{ and } m_2^0 = c \le y_n \le d = M_2^0 \text{ for all } n \ge 0.$$

For all $n \ge 0$, we have

$$m_1^1 = f(M_1^0, M_2^0) \le f(x_n, y_n) \le f(m_1^0, m_2^0) = M_1^1,$$

and

$$m_2^1 = g(M_1^0, M_2^0) \le g(x_n, y_n) \le g(m_1^0, m_2^0) = M_2^1$$

and so

$$m_1^1 \leq x_n \leq M_1^1, \text{ and } m_2^1 \leq y_n \leq M_2^1 \text{ for all } n \geq 1.$$

For all $n \geq 1$, we have

$$m_1^2 = f(M_1^1, M_2^1) \leq f(x_n, y_n) \leq f(m_1^1, m_2^1) = M_1^2,$$

and

$$m_2^2 = g(M_1^1, M_2^1) \leq g(x_n, y_n) \leq g(m_1^1, m_2^1) = M_2^2.$$

So that

$$m_1^2 \leq x_n \leq M_1^2, \text{ and } m_2^2 \leq y_n \leq M_2^2 \text{ for all } n \geq 2.$$

It follows by induction that for $k \geq 0$, the following statements that true:

(1) $a = m_1^0 \leq m_1^1 \leq m_1^2 \leq ... \leq m_1^k \leq M_1^k \leq \ldots \leq M_1^2 \leq M_1^1 \leq M_1^0 = b.$

(2) $c = m_2^0 \leq m_2^1 \leq m_2^2 \leq ... \leq m_2^k \leq M_2^k \leq \ldots \leq M_2^2 \leq M_2^1 \leq M_2^0 = d.$

(3) $m_1^k \leq x_n \leq M_1^k$ for all $n \geq k.$

(4) $m_2^k \leq y_n \leq M_2^k$ for all $n \geq k.$

Set

$$m_1 = \lim_{k\to\infty} m_1^k, M_1 = \lim_{k\to\infty} M_1^k, m_2 = \lim_{k\to\infty} m_2^k, \text{ and } M_2 = \lim_{k\to\infty} M_2^k.$$

Then

$$a \leq m_1 \leq \underline{\lim}_{n\to\infty} x_n \leq \overline{\lim}_{n\to\infty} x_n \leq M_1 \leq b,$$
$$c \leq m_2 \leq \underline{\lim}_{n\to\infty} y_n \leq \overline{\lim}_{n\to\infty} y_n \leq M_2 \leq d.$$

By the continuity of f and g, from (3.4), we obtain

$$\begin{cases} m_1 = f(M_1, M_2), & M_1 = f(m_1, m_2), \\ m_2 = g(M_1, M_2), & M_2 = g(m_1, m_2). \end{cases}$$

Using the assumption (ii), implies $m_1 = M_1 = \overline{x},\ m_2 = M_2 = \overline{y}$. This completes the proof. □

In the next theorem, we show the asymptotic behavior of the positive solutions of system (1.1).

Theorem 3.4. *Suppose that the following relation holds true:*

$$\beta_1\beta_2 < a_1a_2. \tag{3.5}$$

Then system (1.1) *has a unique positive equilibrium* $(\overline{x}, \overline{y})$ *and every positive solution of system* (1.1) *tends to the unique positive equilibrium as* $n \to \infty$.

Proof. Consider the following functions:

$$f(x,y)=\frac{\alpha_1+\beta_1 e^{-y}}{a_1+b_1x},\quad g(x,y)=\frac{\alpha_2+\beta_2 e^{-x}}{a_2+b_2y}$$

where $x\in I_1=[L_1,U_1], y\in I_2=[L_2,U_2]$ which implies that $f(x,y)\in I_1, g(x,y)\in I_2$, and so that $f: I_1\times I_2\to I_1, g: I_1\times I_2\to I_2$. Then, it is easy to see that $f(x,y), g(x,y)$ are decreasing in both x and y. Let (m,M,r,R) be a solution of the system

$$\begin{aligned} m=f(M,R),&\quad M=f(m,r)\\ r=g(M,R),&\quad R=g(m,r). \end{aligned}$$

Then, one has

$$\begin{aligned} m=\frac{\alpha_1+\beta_1e^{-R}}{a_1+b_1M},&\quad M=\frac{\alpha_1+\beta_1e^{-r}}{a_1+b_1m},\\ r=\frac{\alpha_2+\beta_2e^{-M}}{a_2+b_2R},&\quad R=\frac{\alpha_2+\beta_2e^{-m}}{a_2+b_2r}. \end{aligned}\tag{3.6}$$

From (3.6), we get

$$\beta_1e^{-r}=M(a_1+b_1m)-\alpha_1,\quad \beta_1e^{-R}=m(a_1+b_1M)-\alpha_1,$$

$$\beta_2e^{-m}=R(a_2+b_2r)-\alpha_2,\quad \beta_2e^{-M}=r(a_2+b_2R)-\alpha_2,$$

which imply that

$$\begin{aligned} M-m&=\frac{\beta_1}{a_1}(e^{-r}-e^{-R})=\frac{\beta_1}{a_1}e^{-r-R}(e^R-e^r),\\ R-r&=\frac{\beta_2}{a_2}(e^{-m}-e^{-M})=\frac{\beta_2}{a_2}e^{-m-M}(e^M-e^m). \end{aligned}\tag{3.7}$$

Moreover, we get

$$\begin{aligned} e^R-e^r&=e^{\xi}(R-r),\ \min\{R,r\}\le\xi\le\max\{R,r\},\\ e^M-e^m&=e^{\theta}(M-m),\ \min\{M,m\}\le\theta\le\max\{M,m\}. \end{aligned}\tag{3.8}$$

Then relations (3.7) and (3.8) imply that

$$M-m=\frac{\beta_1}{a_1}e^{-r-R+\xi}(R-r),\quad R-r=\frac{\beta_2}{a_2}e^{-m-M+\theta}(M-m)$$

and so

$$|M-m|\le\frac{\beta_1}{a_1}|R-r|,|R-r|\le\frac{\beta_2}{a_2}|M-m|.\tag{3.9}$$

In addition, observe that relations (3.5) and (3.9) imply that

$$\left(1-\frac{\beta_1\beta_2}{a_1a_2}\right)|M-m|\le 0,\quad \left(1-\frac{\beta_1\beta_2}{a_1a_2}\right)|R-r|\le 0$$

from which we see that $M=m$ and $R=r$. From Theorem 3.3, it follows that system (1.1) has a unique positive equilibrium $(\overline{x},\overline{y})$ and every positive solution of system (1.1) tends to the unique positive equilibrium as $n\to\infty$. This completes the proof of the theorem. □

In the next theorem of this section, we will study the global asymptotic stability of the positive equilibrium of system (1.1).

Theorem 3.5. *Consider system* (1.1) *where* (3.5) *holds true. Also suppose that*

$$\frac{b_1U_1}{a_1+b_1L_1}+\frac{b_2U_2}{a_2+b_2L_2}+\frac{b_1b_2U_1U_2+\beta_1\beta_2e^{-L_1-L_2}}{(a_1+b_1L_1)(a_2+b_2L_2)}<1. \tag{3.10}$$

Then the unique positive equilibrium point $(\overline{x},\overline{y})$ *of system* (1.1) *is globally asymptotically stable.*

Proof. First, we will prove that $(\overline{x},\overline{y})$ is locally asymptotically stable. The characteristic equation of the Jacobian matrix $J_F(\overline{x},\overline{y})$ about $(\overline{x},\overline{y})$ is given by

$$\lambda^2+p_1\lambda+p_2=0, \tag{3.11}$$

where

$$p_1=\frac{b_1\overline{x}}{a_1+b_1\overline{x}}+\frac{b_2\overline{y}}{a_2+b_2\overline{y}},$$

$$p_2=\frac{b_1b_2\overline{xy}}{(a_1+b_1\overline{x})(a_2+b_2\overline{y})}-\frac{\beta_1e^{-\overline{y}}}{a_1+b_1\overline{x}}.\frac{\beta_2e^{-\overline{x}}}{a_2+b_2\overline{y}}.$$

From condition (3.10), we get

$$\begin{aligned}|p_1|+|p_2|&=\frac{b_1\overline{x}}{a_1+b_1\overline{x}}+\frac{b_2\overline{y}}{a_2+b_2\overline{y}}+\frac{b_1b_2\overline{xy}}{(a_1+b_1\overline{x})(a_2+b_2\overline{y})}+\frac{\beta_1e^{-\overline{y}}}{a_1+b_1\overline{x}}.\frac{\beta_2e^{-\overline{x}}}{a_2+b_2\overline{y}}\\&\le\frac{b_1U_1}{a_1+b_1L_1}+\frac{b_2U_2}{a_2+b_2L_2}+\frac{b_1b_2U_1U_2}{(a_1+b_1L_1)(a_2+b_2L_2)}\\&+\frac{\beta_1e^{-L_2}}{a_1+b_1L_1}.\frac{\beta_2e^{-L_1}}{a_2+b_2L_2}<1.\end{aligned}$$

Therefore, follows [7, Remark 1.3.1], all the roots of equation (3.11) are of modulus less than 1, and it follows from Lemma 2.3 that the unique positive equilibrium point $(\overline{x},\overline{y})$ of system (1.1) is locally asymptotically stable. Using Theorem 3.4, we obtain that $(\overline{x},\overline{y})$ is globally asymptotically stable. This completes the proof of the theorem. □

3.3 Rate of Convergence

In this section, we give the rate of convergence of a solution that converges to the equilibrium of the systems (1.1). Similar method can be found in [16].

Let $\{(x_n, y_n)\}$ be an arbitrary solution of system (1.1) such that $\lim_{n\to\infty} x_n = \overline{x}$, and $\lim_{n\to\infty} y_n = \overline{y}$, where $\overline{x} \in [L_1, U_1]$, and $\overline{y} \in [L_2, U_2]$. To find the error terms, one has from the system (1.1)

$$\begin{aligned}
x_{n+1} - \overline{x} &= \frac{\alpha_1 + \beta_1 e^{-y_n}}{a_1 + b_1 x_n} - \frac{\alpha_1 + \beta_1 e^{-\overline{y}}}{a_1 + b_1 \overline{x}} \\
&= \frac{(\alpha_1 + \beta_1 e^{-y_n})(a_1 + b_1\overline{x}) - (\alpha_1 + \beta_1 e^{-\overline{y}})(a_1 + b_1 x_n)}{(a_1 + b_1 x_n)(a_1 + b_1\overline{x})} \\
&= \frac{-\alpha_1 b_1(x_n - \overline{x}) + \beta_1 a_1(e^{-y_n} - e^{-\overline{y}}) + \beta_1 b_1(e^{-y_n}\overline{x} - e^{-\overline{y}} x_n)}{(a_1 + b_1 x_n)(a_1 + b_1\overline{x})} \\
&= \frac{-\alpha_1 b_1(x_n - \overline{x}) - \beta_1 a_1 e^{-y_n}(e^{y_n - \overline{y}} - 1)}{(a_1 + b_1 x_n)(a_1 + b_1\overline{x})} \\
&+ \frac{\beta_1 b_1(e^{-y_n}\overline{x} - e^{-y_n} x_n + e^{-y_n} x_n - e^{-\overline{y}} x_n)}{(a_1 + b_1 x_n)(a_1 + b_1\overline{x})} \\
&= -\frac{(\alpha_1 + \beta_1 e^{-y_n}) b_1}{(a_1 + b_1 x_n)(a_1 + b_1\overline{x})}(x_n - \overline{x}) - \frac{\beta_1 e^{-y_n}(e^{y_n - \overline{y}} - 1)}{(a_1 + b_1\overline{x})(y_n - \overline{y})}(y_n - \overline{y}),
\end{aligned}$$

and

$$\begin{aligned}
y_{n+1} - \overline{y} &= \frac{\alpha_2 + \beta_2 e^{-x_n}}{a_2 + b_2 y_n} - \frac{\alpha_2 + \beta_2 e^{-\overline{x}}}{a_2 + b_2 \overline{y}} \\
&= \frac{(\alpha_2 + \beta_2 e^{-x_n})(a_2 + b_2\overline{y}) - (\alpha_2 + \beta_2 e^{-\overline{x}})(a_2 + b_2 y_n)}{(a_2 + b_2 y_n)(a_2 + b_2\overline{y})} \\
&= \frac{-\alpha_2 b_2(y_n - \overline{y}) + \beta_2 a_2(e^{-x_n} - e^{-\overline{x}}) + \beta_2 b_2(e^{-x_n}\overline{y} - e^{-\overline{x}} y_n)}{(a_2 + b_2 y_n)(a_2 + b_2\overline{y})} \\
&= \frac{-\alpha_2 b_2(y_n - \overline{y}) - \beta_2 a_2 e^{-x_n}(e^{x_n - \overline{x}} - 1)}{(a_2 + b_2 y_n)(a_2 + b_2\overline{y})} \\
&+ \frac{\beta_2 b_2(e^{-x_n}\overline{y} - e^{-x_n} y_n + e^{-x_n} y_n - e^{-\overline{x}} y_n)}{(a_2 + b_2 y_n)(a_2 + b_2\overline{y})} \\
&= -\frac{\beta_2 e^{-x_n}(e^{x_n - \overline{x}} - 1)}{(a_2 + b_2\overline{y})(x_n - \overline{x})}(x_n - \overline{x}) - \frac{(\alpha_2 + \beta_2 e^{-x_n}) b_2}{(a_2 + b_2 y_n)(a_2 + b_2\overline{y})}(y_n - \overline{y}).
\end{aligned}$$

Let $e_n^1 = x_n - \overline{x}$, and $e_n^2 = y_n - \overline{y}$, then one has

$$\begin{aligned}
e_{n+1}^1 &= a_n e_n^1 + b_n e_n^2, \\
e_{n+1}^2 &= c_n e_n^1 + d_n e_n^2,
\end{aligned}$$

where

$$a_n = -\frac{(\alpha_1 + \beta_1 e^{-y_n})b_1}{(a_1 + b_1 x_n)(a_1 + b_1 \overline{x})},$$
$$b_n = -\frac{\beta_1 e^{-y_n}(e^{y_n - \overline{y}} - 1)}{(a_1 + b_1 \overline{x})(y_n - \overline{y})},$$
$$c_n = -\frac{\beta_2 e^{-x_n}(e^{x_n - \overline{x}} - 1)}{(a_2 + b_2 \overline{y})(x_n - \overline{x})},$$
$$d_n = -\frac{(\alpha_2 + \beta_2 e^{-x_n})b_2}{(a_2 + b_2 y_n)(a_2 + b_2 \overline{y})}.$$

Moreover,

$$\lim_{n\to\infty} a_n = -\frac{b_1 \overline{x}}{a_1 + b_1 \overline{x}}, \quad \lim_{n\to\infty} b_n = -\frac{\beta_1 e^{-\overline{y}}}{a_1 + b_1 \overline{x}},$$
$$\lim_{n\to\infty} c_n = -\frac{\beta_2 e^{-\overline{x}}}{a_2 + b_2 \overline{y}}, \quad \lim_{n\to\infty} d_n = -\frac{b_2 \overline{y}}{a_2 + b_2 \overline{y}}.$$

So, the limiting system of the error terms can be written as

$$\begin{pmatrix} e^1_{n+1} \\ e^2_{n+1} \end{pmatrix} = \begin{pmatrix} -\frac{b_1 \overline{x}}{a_1 + b_1 \overline{x}} & -\frac{\beta_1 e^{-\overline{y}}}{a_1 + b_1 \overline{x}} \\ -\frac{\beta_2 e^{-\overline{x}}}{a_2 + b_2 \overline{y}} & -\frac{b_2 \overline{y}}{a_2 + b_2 \overline{y}} \end{pmatrix} \begin{pmatrix} e^1_n \\ e^2_n \end{pmatrix}$$

which similar to the linearized system of (1.1) about the equilibrium point $(\overline{x}, \overline{y})$. Using Proposition 2.4 and 2.5, one has following result.

Theorem 3.6. *Assume that* $\{(x_n, y_n)\}$ *be a positive solution of system* (1.1) *such that* $\lim_{n\to\infty} x_n = \overline{x}$*, and* $\lim_{n\to\infty} y_n = \overline{y}$*, where* $\overline{x} \in [L_1, U_1]$ *and* $\overline{y} \in [L_2, U_2]$*. Then the error vector* $e_n = \begin{pmatrix} e^1_n \\ e^2_n \end{pmatrix}$ *of every solution of* (1.1) *satisfies both of the following asymptotic relations:*

$$\lim_{n\to\infty} (||e_n||)^{\frac{1}{n}} = |\lambda_{1,2} J_F(\overline{x}, \overline{y})|, \quad \lim_{n\to\infty} \frac{||e_{n+1}||}{||e_n||} = |\lambda_{1,2} J_F(\overline{x}, \overline{y})|,$$

where $\lambda_{1,2} J_F(\overline{x}, \overline{y})$ *are the characteristic roots of Jacobian matrix* $J_F(\overline{x}, \overline{y})$.

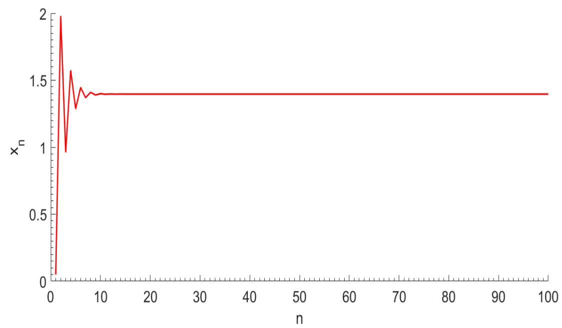
4 Numerical Simulations

In order to verify our theoretical results and to support our theoretical discussion, we consider several interesting numerical examples. These examples represent different types of qualitative behavior of solutions of the systems (1.1). All plots in this section are drawn with MATLAB.

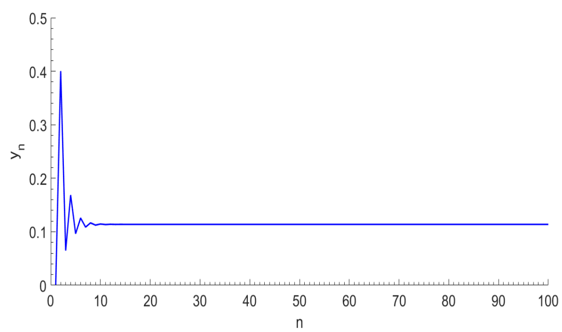
Example 4.1. Let $\alpha_1 = 0.1, \beta_1 = 90, a_1 = 45, b_1 = 9, \alpha_2 = 0.1, \beta_2 = 2, a_2 = 5$, and $b_2 = 2$. Then system (1.1) can be written as

$$x_{n+1} = \frac{0.1 + 90e^{-y_n}}{45 + 9x_n}, \; y_{n+1} = \frac{0.1 + 2e^{-x_n}}{5 + 2y_n} \tag{4.1}$$

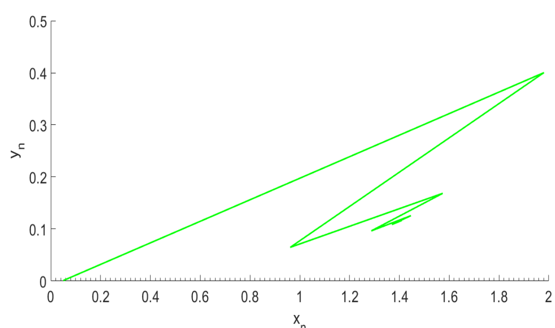
with initial conditions $x_0 = 0.05$, and $y_0 = 0.001$.



(a) Plot of x_n for the system (4.1)



(b) Plot of y_n for the system (4.1)

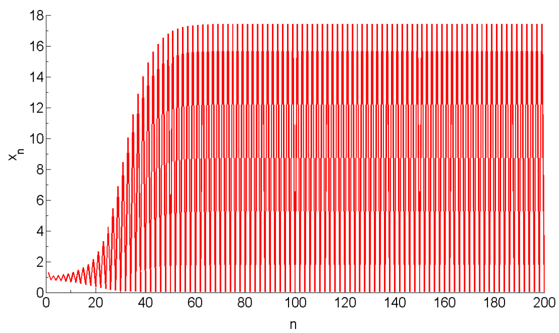


(c) An attractor of the system (4.1)

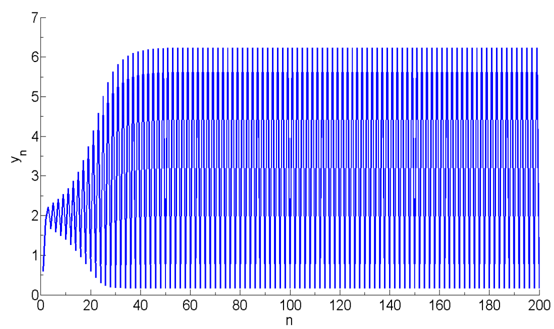
Figure 4.1: Plots for the system (4.1)

In this case, the unique positive equilibrium point of the system (4.1) is given by $(\overline{x}, \overline{y}) = (1.396890, 0.113769)$. Moreover, in Figure 4.1, the plot of x_n is shown in

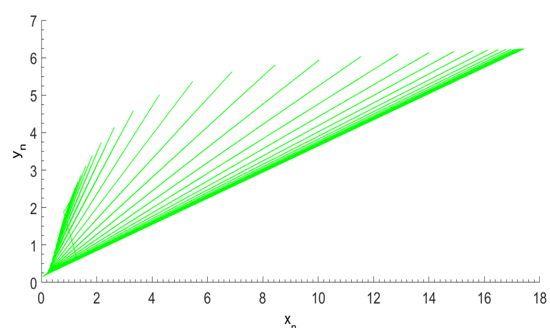
Figure (a), the plot of y_n is shown in Figure (b), and an attractor of the system (4.1) is shown in Figure (c).



(a) Plot of x_n for the system (4.2)



(b) Plot of y_n for the system (4.2)



(c) Phase portrait of system (4.2)

Figure 4.2: Plots for the system (4.2)

Example 4.2. Let $\alpha_1 = 66, \beta_1 = 35, a_1 = 1.7, b_1 = 80, \alpha_2 = 23.8, \beta_2 = 350, a_2 = 55$, and $b_2 = 14$. Then system (1.1) can be written as

$$x_{n+1} = \frac{66 + 35e^{-y_n}}{1.7 + 80x_n}, \; y_{n+1} = \frac{23.8 + 350e^{-x_n}}{55 + 14y_n} \tag{4.2}$$

with initial conditions $x_0 = 1.3$, and $y_0 = 0.6$.

In this case, the unique positive equilibrium point of the system (4.2) is unstable. Moreover, in Figure 4.2, the plot of x_n is shown in Figure (a), the plot of y_n is shown in Figure (b), and a phase portrait of system (4.2) is shown in Figure (c).

5 Conclusion

This work is related to qualitative behavior of a system of exponential difference equations. We have proved the boundedness and persistence of positive solutions of system (1.1). Moreover, we have shown that unique positive equilibrium point of system (1.1) is locally as well as globally asymptotically stable under certain parametric conditions. The main objective of dynamical systems theory is to predict the global behavior of a system based on the knowledge of its present state. An approach to this problem consists of determining the possible global behaviors of the system and determining which parametric conditions lead to these long-term behaviors. Furthermore, the rate of convergence of positive solutions of (1.1) which converges to its unique positive equilibrium point is demonstrated. Finally, some illustrative numerical examples are provided to support our theoretical discussion. Examples 4.1 show that the unique positive equilibrium point of system (1.1) is stable with different parametric values whereas Example 4.2 show that the unique positive equilibrium point of system (1.1) is unstable with suitable parametric choices.

Acknowledgments

The authors would like to sincerely thank the anonymous referee for carefully reading the paper and for valuable comments.

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