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Observations on the Comparison Between the Behavior of Orbits in the $3x + 1$ Problem and the $5x + 1$ Problem

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Abstract

We heuristically address the $3x + 1$ problem and the corresponding $5x + 1$ problem. We make observations, based on a multitude of computations, which may, in turn, lend their support of the conjectures that no orbit under the $3x + 1$ map is divergent to $+\infty$ (and so every orbit is eventually periodic, in particular, as the cycle $(1, 4, 2)$) and *almost all orbits* under the $5x + 1$ map are divergent to $+\infty$. Our representation of the natural numbers (which we have as the domain of the $3x + 1$ map and of the $5x + 1$ map) is influenced by the Sharkovsky ordering of the natural numbers.

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1 Introduction

We consider the $3x + 1$ problem and its associated map,

$$\tilde{C}_1(x) = \begin{cases} \dfrac{x}{2}, & x = \text{even}, \\ 3x + 1, & x = \text{odd}, \end{cases} \quad x \in \mathbb{N}, \tag{1.1}$$

and the $5x + 1$ problem and its associated map,

$$\tilde{C}_2(x) = \begin{cases} \dfrac{x}{2}, & x = \text{even}, \\ 5x + 1, & x = \text{odd}, \end{cases} \quad x \in \mathbb{N}. \tag{1.2}$$

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We note that the difference equations corresponding to the maps (1.1) and (1.2) are

$$x_{n+1} = \begin{cases} \dfrac{x_n}{2}, & x_n = \text{even}, \\ & \qquad\qquad x_0 \in \mathbb{N} \\ 3x_n + 1, & x_n = \text{odd}, \end{cases} \tag{1.3}$$

and

$$x_{n+1} = \begin{cases} \dfrac{x_n}{2}, & x_n = \text{even}, \\ & \qquad\qquad x_0 \in \mathbb{N}, \\ 5x_n + 1, & x_n = \text{odd}, \end{cases} \tag{1.4}$$

respectively.

The $3x+1$ problem conjectures that no orbit under $\tilde{C}_1$ is divergent to $+\infty$ and that therefore every orbit is eventually periodic (as the cycle $(1, 4, 2)$). On the other hand, the $5x+1$ problem conjectures that almost all orbits under $\tilde{C}_2$ are divergent to $+\infty$ (where there exists a finite set of cycles to which an infinite set of natural numbers of density 0 is mapped by $\tilde{C}_2$). We make observations, based on a multitude of computations, on the comparison between the behavior of orbits under $\tilde{C}_1$ and the behavior of orbits under $\tilde{C}_2$, these observations having heuristic implications which offer support of the two conjectures simultaneously.

For a thorough treatment of the $3x+1$ and $5x+1$ problems, see the book by Lagarias [2], and, in particular, the article by Kontorovich and Lagarias in Part III of that book, who use a stochastic approach in comparing the $3x+1$ and $5x+1$ problems. While our approach ends up being probabilistic in nature, it is highly computational along the way and thereby elucidates in detail the actual composition of orbits under the $\tilde{C}_1$ map and under the $\tilde{C}_2$ map.

We place particular focus on the lower branch maps of $\tilde{C}_1$ and $\tilde{C}_2$, namely, on

$$C_1(x) \stackrel{\text{def}}{=} 3x + 1, \quad x = \text{odd},$$

and

$$C_2(x) \stackrel{\text{def}}{=} 5x + 1, \quad x = \text{odd},$$

respectively. We make these definitions because we wish to focus our attention on certain characteristics of the mappings of C_1 and C_2 of odd numbers, $2m+1$, $m \in \{0, 1, \ldots\}$, to even numbers, $2^k(2n+1)$, $k \in \{1, 2, \ldots\}$, $n \in \{0, 1, \ldots\}$.

Given that our main focus will be on how C_1 and C_2 map odd numbers (to even numbers of the form $2^k\times$ an odd number), we organize how we view the sequence of odd numbers in the following two ways:

When in the context of the $3x+1$ maps C_1 and $\tilde{C}_1$, we represent the sequence o_n of odd numbers (≥ 3) as

$$o_n = \begin{cases} 3(2\ell + 1), & n = 3\ell, \\ 3(2\ell + 1) + 2, & n = 3\ell + 1, \quad \ell = 0, 1, \ldots. \\ 3(2\ell + 1) + 4, & n = 3\ell + 2, \end{cases} \tag{1.5}$$

When in the context of the $5x+1$ maps C_2 and $\tilde{C}_2$, we represent the sequence o_n of odd numbers (≥ 5) as

$$o_n = \begin{cases} 5(2\ell+1), & n = 5\ell, \\ 5(2\ell+1)+2, & n = 5\ell+1, \\ 5(2\ell+1)+4, & n = 5\ell+2, \quad \ell = 0,1,\ldots. \\ 5(2\ell+1)+6, & n = 5\ell+3, \\ 5(2\ell+1)+8, & n = 5\ell+4, \end{cases} \tag{1.6}$$

For the C_1 and $\tilde{C}_1$ maps, with (1.5) in mind, we define the following three sets, which partition the odd numbers ≥ 3 into three categories:

$$A_1 =: \{x \in \mathbb{N} \colon x = 3m,\ m = 1,3,5,\ldots\},$$

$$A_2 =: \{x \in \mathbb{N} \colon x = 3m+2,\ m = 1,3,5,\ldots\},$$

$$A_3 =: \{x \in \mathbb{N} \colon x = 3m+4,\ m = 1,3,5,\ldots\}.$$

Observe that

$$A_1 \cup A_2 \cup A_3 \cup \{1\} = \text{all odd numbers.}$$

We describe the sets (of even numbers of the form $2^k(3n+j)$, $k \geq 1$, n odd and $n \geq 1$, $j = 0,2,4$)

$$C_1(A_1),\ \ C_1(A_2),\ \ C_1(A_3),$$

after an analysis that is computational. We do not consider $C_1(\{1\})$, for reasons given below, where we obtain a result that is already in agreement with what the $3x+1$ problem conjectures to be the case.

Specifically, the analysis entails consideration of the three categories of odd numbers ≥ 3,

$$3m+i,\ \ m \text{ odd and } m \geq 1,\ i = 0,2,4,$$

and determining to what even numbers of the form

$$2^k(1),\ \ k \geq 1,$$

or

$$2^k(3n+j),\ \ k \geq 1, n \text{ odd and } n \geq 1, j = 0,2,4,$$

C_1 maps $3m+i$ to. (See Sharkovsky's ordering of the natural numbers in Grove and Ladas [1], which inspired our depiction of odd and even numbers.)

However, in particular, we do not consider the following:

1. We do not consider $C_1(1)$ or $\tilde{C}_1(1)$, where, if we let $x_0 = 1$ in Eq.(1.3), then we obtain the periodic solution

$$\{x_n\}_{n=0}^{\infty} = 1,4,2,1,4,2,1,\ldots.$$

2. We do not consider $C_1\left(2^k(1)\right)$ or $\tilde{C}_1\left(2^k(1)\right)$, $k \geq 1$, where, if we let $x_0 = 2^k(1)$ in Eq.(1.3), then we obtain the eventually periodic solution

$$\{x_n\}_{n=0}^{\infty} = 2^k, 2^{k-1}, \ldots, 1, 4, 2, 1, 4, 2, 1, \ldots.$$

3. We do not consider $C_1\left(2^k(3n+j)\right)$, for $k \geq 1$, n odd and $n \geq 1$, $j = 0, 2, 4$, because we view the even numbers $2^k(3n+j)$ as negligible in that it only takes a finite number of iterations, k to be exact, for $\tilde{C}_1$ to map the even numbers $2^k(3n+j)$ to the odd numbers $3n+j$. We therefore limit the scope of the mappings of C_1 to the odd numbers ≥ 3.

For the C_2 and $\tilde{C}_2$ maps, with (1.6) in mind, we define the following five sets, which partition the odd numbers ≥ 5 into five categories:

$$B_1 =: \{x \in \mathbb{N} \colon x = 5m,\ m = 1, 3, 5, \ldots\},$$

$$B_2 =: \{x \in \mathbb{N} \colon x = 5m+2,\ m = 1, 3, 5, \ldots\},$$
$$B_3 =: \{x \in \mathbb{N} \colon x = 5m+4,\ m = 1, 3, 5, \ldots\},$$
$$B_4 =: \{x \in \mathbb{N} \colon x = 5m+6,\ m = 1, 3, 5, \ldots\},$$
$$B_5 =: \{x \in \mathbb{N} \colon x = 5m+8,\ m = 1, 3, 5, \ldots\}.$$

Observe that

$$B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5 \cup \{1\} \cup \{3\} = \text{all odd numbers.}$$

We describe the sets (of even numbers of the form $2^k(5n+j)$, $k \geq 1$, n odd and $n \geq 1$, $j = 0, 2, 4, 6, 8$)

$$C_2(B_1),\ \ C_2(B_2),\ \ C_2(B_3),\ \ C_2(B_4),\ \ C_2(B_5),$$

after an analysis that is computational. We do not consider $C_2(\{1\})$ or $C_2(\{3\})$, for reasons given below, where we readily obtain results in these two cases.

Specifically, the analysis entails consideration of the five categories of odd numbers ≥ 5,

$$5m+i,\ \ m \text{ odd and } m \geq 1,\ i = 0, 2, 4, 6, 8,$$

and determining to what even numbers of the form

$$2^k(1),\ \ k \geq 1,$$

$$2^k(3),\ \ k \geq 1,$$

or

$$2^k(5n+j),\ \ k \geq 1,\ n \text{ odd and } n \geq 1,\ j = 0, 2, 4, 6, 8,$$

C_2 maps $5m+i$ to. (Again see Sharkovsky's ordering of the natural numbers in Grove and Ladas [1], which inspired our depiction of odd and even numbers.)

However, in particular, we do not consider the following:

1. We do not consider $C_2(1)$ or $\tilde{C}_2(1)$, where, if we let $x_0 = 1$ in Eq.(1.4), then we obtain the periodic solution

$$\{x_n\}_{n=0}^{\infty} = 1, 6, 3, 16, 8, 4, 2, 1, 6, 3, 16, 8, 4, 2, 1, \ldots.$$

2. We do not consider $C_2(3)$ or $\tilde{C}_2(3)$, where, if we let $x_0 = 3$ in Eq.(1.4), then we obtain the periodic solution

$$\{x_n\}_{n=0}^{\infty} = 3, 16, 8, 4, 2, 1, 6, 3, 16, 8, 4, 2, 1, 6, 3, \ldots.$$

3. We do not consider $C_2\left(2^k(1)\right)$, $\tilde{C}_2\left(2^k(1)\right)$ or $C_2\left(2^k(3)\right)$, $\tilde{C}_2\left(2^k(3)\right)$, $k \geq 1$, where, if we let $x_0 = 2^k(1)$ or $x_0 = 2^k(3)$ in Eq.(1.4), then we obtain the eventually periodic solutions

$$\{x_n\}_{n=0}^{\infty} = 2^k, 2^{k-1}, \ldots, 1, 6, 3, 16, 8, 4, 2, 1, \ldots,$$

$$\{x_n\}_{n=0}^{\infty} = 2^k, 2^{k-1}, \ldots, 3, 16, 8, 4, 2, 1, 6, 3, \ldots.$$

4. We do not consider $C_2\left(2^k(5n+j)\right)$, for $k \geq 1$, n odd and $n \geq 1$, $j = 0, 2, 4, 6, 8$, because we view the even numbers $2^k(5n+j)$ as negligible in that it only takes a finite number of iterations, k to be exact, for $\tilde{C}_2$ to map the even numbers $2^k(5n+j)$ to the odd numbers $5n+j$. We therefore limit the scope of the mappings of C_2 to the odd numbers ≥ 5.

In light of our emphasis on the mappings of $\tilde{C}_1$ and $\tilde{C}_2$ of odd numbers to odd numbers, we give definitions whose language is highly suggestive of what we plan to conclude.

Definition 1.1. When $\tilde{C}_1$ or $\tilde{C}_2$ maps an odd iterate, $2m+1$, $m \in \{1, 2, \ldots\}$ or $m \in \{2, 3, \ldots\}$, respectively, to the very next odd iterate, $2n+1$, $n \in \{0, 1, \ldots\}$,

(i) we say that *$\tilde{C}_1$ or $\tilde{C}_2$ maps $2m+1$ back to the smaller $2n+1$* if $2n+1 < 2m+1$;

(ii) we say that *$\tilde{C}_1$ or $\tilde{C}_2$ maps $2m+1$ forward to the larger $2n+1$* if $2n+1 > 2m+1$.

Definition 1.2. With each occurrence of an odd-valued term x_n in the solution $\{x_n\}_{n=0}^{\infty}$ of Eq.(1.3) or (1.4), we take a tally and say that *an odd number has been visited.*

2 The C_1 and $\tilde{C}_1$ Maps

2.1 Characterization of the set $C_1(A_1)$

We first ask if C_1 can map odd numbers of the form $3m$, with m odd and $m \geq 1$, to even numbers of the form $2^k(1)$, with $k \geq 1$:

Note that we have

$$C_1(3m) = 2^k(1) \iff$$
$$3(3m) + 1 = 2^k \iff$$
$$m = \frac{2^k - 1}{9},$$

which holds when $9|(2^k - 1)$, where $2^k - 1$ is odd so that the right-hand side of this last equation is consistent with m being odd when $9|(2^k - 1)$. If the digits of $2^k - 1$ add up to 9, then $9|(2^k - 1)$. The set of values of k for which $9|(2^k - 1)$ is as follows:

$$k = 6, 12, 18, \ldots \implies k = 6\ell,\ \ell = 1, 2, \ldots \implies$$

$$k = 2(3\ell),\ \ell = 1, 2, \ldots, \text{ and } k \geq 6.$$

Therefore, we have the infinite set of images $P_1 \subset C_1(A_1)$ such that

$$P_1 = \left\{2^k(1)\colon k = 2(3\ell),\ \ \ell = 1, 2, \ldots\right\}.$$

After the mapping $C_1(3m) = 2^k(1)$, $\tilde{C}_1$ maps $2^k(1)$ to the cycle $(1, 4, 2)$.

We next ask if C_1 can map odd numbers of the form $3m$, with m odd and $m \geq 1$, to even numbers of the form $2^k(3n)$, with $k \geq 1$ and n odd and $n \geq 1$:

Note that we have

$$C_1(3m) = 2^k(3n) \iff$$
$$3(3m) + 1 = 3(2^k n) \iff$$
$$3(3m - 2^k n) = -1 \iff$$
$$3m - 2^k n = -\frac{1}{3},$$

which is a contradiction since $3m - 2^k n \in \mathbb{Z}$. Thus, C_1 does not map odd numbers of the form $3m$ to even numbers of the form $2^k(3n)$.

We then ask if C_1 can map odd numbers of the form $3m$, with m odd and $m \geq 1$, to even numbers of the form $2^k(3n + 2)$, with $k \geq 1$ and n odd and $n \geq 1$:

Note that we have

$$C_1(3m) = 2^k(3n + 2) \iff$$
$$3(3m) + 1 = 3(2^k n) + 2^{k+1} \iff$$
$$3(3m - 2^k n) = 2^{k+1} - 1 \iff$$
$$3m - 2^k n = \frac{2^{k+1} - 1}{3}, \tag{2.1}$$

which holds if

1. $3|(2^{k+1} - 1)$ for certain values of k;

2. for the certain values of k, there exist odd m, n pairs that satisfy Eq.(2.1).

We first determine all values of k for which $3|(2^{k+1} - 1)$:

$$k + 1 = 2\ell,\ \ell = 1, 2, \ldots \implies$$

$$k = 2\ell - 1,\ \ell = 1, 2, \ldots, \text{ and } k \geq 1. \tag{2.2}$$

In order to find the odd m, n pairs that satisfy Eq.(2.1) for values of k given by Eq.(2.2), we first determine the odd m, n pairs that satisfy the equation

$$3m - 2^k n = 1, \quad \text{for } k = 1, 2, \ldots. \tag{2.3}$$

In actuality, we are only able to find at least one odd m, n pair for each $k = 1, 2, \ldots$, with Eq.(2.3) rewritten as

$$2^k n = 3m - 1, \quad \text{for } k = 1, 2, \ldots.$$

We obtain

$$\begin{array}{lll} k = 1: & 2^1 \cdot 7 = 3 \cdot 5 - 1 & \implies n = 7, m = 5 \\ k = 2: & 2^2 \cdot 5 = 3 \cdot 7 - 1 & \implies n = 5, m = 7 \\ k = 3: & 2^3 \cdot 7 = 3 \cdot 19 - 1 & \implies n = 7, m = 19 \\ k = 4: & 2^4 \cdot 5 = 3 \cdot 27 - 1 & \implies n = 5, m = 27 \\ k = 5: & 2^5 \cdot 7 = 3 \cdot 75 - 1 & \implies n = 7, m = 75 \\ k = 6: & 2^6 \cdot 5 = 3 \cdot 107 - 1 & \implies n = 5, m = 107 \end{array} \tag{2.4}$$

Given (2.4), we are then able to define the infinite set

$$\begin{aligned} Q &= \{(m, n, k)\colon k = 1, 2, \ldots, \text{ and } m, n \text{ satisfy the equation} \\ &\quad\ 3m - 2^k n = 1 \text{ with } m, n \text{ odd and } m, n \geq 1\} \\ &\supset \{(m_k, n_k, k)\colon \text{(i) for } k = 1, 3, 5, \ldots, m_k = (2^k \cdot 7 + 1)/3, \\ &\quad\ n_k = 7; \text{ (ii) for } k = 2, 4, 6, \ldots, \\ &\quad\ m_k = (2^k \cdot 5 + 1)/3, n_k = 5\}. \end{aligned}$$

Now, for any value of k that satisfies Eq.(2.2), i.e., $k = 2\ell - 1,\ \ell \in \{1, 2, \ldots\}$, let

$$d =: \frac{2^{k+1} - 1}{3} \in \mathbb{N}. \tag{2.5}$$

Then d is odd since $2^{k+1} - 1$ is odd, and there exists at least one $(m_0, n_0, k) \in Q$ such that

$$3(dm_0) - 2^k(dn_0) = d(1). \tag{2.6}$$

Thus, for each value of $k = 2\ell - 1,\ \ell \in \{1, 2, \ldots\}$, there exists at least one odd pair, $m = dm_0$, $n = dn_0$, such that

$$3m - 2^k n = \frac{2^{k+1} - 1}{3}.$$

Therefore, we have the infinite set of images $R_1 \subset C_1(A_1)$ such that

$$\begin{aligned} R_1 &= \{2^k(3n+2)\colon k = 2\ell - 1,\ \ell = 1, 2, \ldots, \text{ and } n \text{ is } \ni \\ &\quad m, n \text{ satisfy the equation } 3m - 2^k n = (2^{k+1} - 1)/3 \\ &\quad \text{with } m, n \text{ odd and } m, n \geq 1\} \\ &\supset \{2^k(3n+2)\colon k = 2\ell - 1,\ \ell = 1, 2, \ldots, \text{ and } n \text{ is } \ni \\ &\quad m = (2^{k+1} - 1)m_0/3,\ n = (2^{k+1} - 1)n_0/3, \text{ with} \\ &\quad (m_0, n_0, k) \in Q\}\,. \end{aligned}$$

Furthermore, we make the following observations on the mapping

$$C_1(3m) = 2^k(3n+2)\ :$$

1. After the mapping $C_1(3m) = 2^k(3n+2)$ *with* $k \geq 3$, an additional sequence of k iterations of $\tilde{C}_1$ has odd numbers of the form $3m$ mapped back to smaller odd numbers of the form $3n+2$ since

$$3n + 2 = \frac{9m+1}{2^k} \leq \frac{9m+1}{2^3} < 3m \quad \Longleftarrow \quad \frac{9m+1}{8} < 3m \quad \Longleftrightarrow$$

$$9m + 1 < 9m + 15m,$$

   which is obviously true for $m \geq 1$.

2. After the mapping $C_1(3m) = 2^k(3n+2)$ *with* $k = 1$, an additional $k = 1$ iteration of $\tilde{C}_1$ has odd numbers of the form $3m$ mapped forward to larger odd numbers of the form $3n+2$ since

$$3n + 2 = \frac{9m+1}{2^1} > 3m \quad \Longleftarrow \quad \frac{9m+1}{2} > 3m \quad \Longleftrightarrow 9m + 1 > 6m,$$

   which is obviously true for $m \geq 1$.

We finally ask if C_1 can map odd numbers of the form $3m$, with m odd and $m \geq 1$, to even numbers of the form $2^k(3n+4)$, with $k \geq 1$ and n odd and $n \geq 1$:

Note that we have

$$\begin{aligned} C_1(3m) = 2^k(3n+4) \quad &\Longleftrightarrow \\ 3(3m) + 1 = 3(2^k n) + 2^{k+2} \quad &\Longleftrightarrow \\ 3(3m - 2^k n) = 2^{k+2} - 1 \quad &\Longleftrightarrow \\ 3m - 2^k n = \frac{2^{k+2} - 1}{3}. & \end{aligned} \tag{2.7}$$

We find all values of k for which $3 | (2^{k+2} - 1)$:

$$k + 2 = 2\ell,\ \ell = 1, 2, \ldots \quad \Longrightarrow$$

$$k = 2\ell - 2,\ \ \ell = 2, 3, \ldots, \text{ and } k \geq 2 \tag{2.8}$$

(where we must at least have $k \geq 1$ so we cannot have $k = 0$).

Here, we redefine the d given in Eq.(2.5) as

$$d =: \frac{2^{k+2} - 1}{3}. \tag{2.9}$$

Note that indeed $d \in \mathbb{N}$ since, for $k \geq 2$, $d \geq 5 > 0$.

By a similar argument as that on the mapping $C_1(3m) = 2^k(3n + 2)$ involving Eqs.(2.5) and (2.6), and based on Eqs.(2.7)–(2.9), we have the infinite set of images $S_1 \subset C_1(A_1)$ such that

$$\begin{aligned} S_1 \quad &= \quad \{2^k(3n + 4)\colon k = 2\ell - 2, \ell = 2, 3, \ldots, \text{ and } n \text{ is } \ni \\ &\qquad m, n \text{ satisfy the equation } 3m - 2^k n = (2^{k+2} - 1)/3 \\ &\qquad \text{with } m, n \text{ odd and } m, n \geq 1\} \\ &\supset \quad \{2^k(3n + 4)\colon k = 2\ell - 2, \ell = 2, 3, \ldots, \text{ and } n \text{ is } \ni \\ &\qquad m = (2^{k+2} - 1)m_0/3, n = (2^{k+2} - 1)n_0/3, \text{ with} \\ &\qquad (m_0, n_0, k) \in Q\}. \end{aligned}$$

Furthermore, we make the following observation on the mapping

$$C_1(3m) = 2^k(3n + 4) :$$

After the mapping $C_1(3m) = 2^k(3n + 4)$ *with* $k \geq 2$, an additional sequence of k iterations of $\widetilde{C}_1$ has odd numbers of the form $3m$ mapped back to smaller odd numbers of the form $3n + 4$ since

$$3n + 4 = \frac{9m + 1}{2^k} \leq \frac{9m + 1}{2^2} < 3m \iff \frac{9m + 1}{4} < 3m \iff$$
$$9m + 1 < 9m + 3m,$$

which is obviously true for $m \geq 1$.

We conclude that, since it is clear that the union of P_1, R_1, and S_1 contains all possible images to which C_1 maps odd numbers of the form $3m$, then

$$C_1(A_1) = P_1 \cup R_1 \cup S_1,$$

with the set definitions of P_1, R_1, and S_1 elucidating the structure of $C_1(A_1)$.

2.2 Characterization of the Set $C_1(A_2)$

We first ask if C_1 can map odd numbers of the form $3m + 2$, with m odd and $m \geq 1$, to even numbers of the form $2^k(1)$, with $k \geq 1$:

Note that we have

$$C_1(3m + 2) = 2^k(1) \iff$$

$$3(3m+2)+1=2^k \iff$$
$$m=\frac{2^k-7}{9},$$

which holds when $9|(2^k-7)$, where 2^k-7 is odd so that the right-hand side of this last equation is consistent with m being odd when $9|(2^k-7)$. If the digits of 2^k-7 add up to 9, then $9|(2^k-7)$. The set of values of k for which $9|(2^k-7)$ is as follows:

$$k=4,10,16,\ldots \implies k=6\ell+4,\ \ell=0,1,\ldots \implies$$

$$k=2(3\ell+2),\ \ell=0,1,\ldots, \text{ and } k\geq 4.$$

Therefore, we have the infinite set of images $P_2 \subset C_1(A_2)$ such that

$$P_2=\left\{2^k(1)\colon k=2(3\ell+2),\ \ \ell=0,1,\ldots\right\}.$$

After the mapping $C_1(3m+2)=2^k(1)$, $\tilde{C}_1$ maps $2^k(1)$ to the cycle $(1,4,2)$.

We next ask if C_1 can map odd numbers of the form $3m+2$, with m odd and $m\geq 1$, to even numbers of the form $2^k(3n)$, with $k\geq 1$ and n odd and $n\geq 1$:

Note that we have

$$C_1(3m+2)=2^k(3n) \iff$$
$$3(3m+2)+1=3(2^kn) \iff$$
$$3(3m-2^kn)=-7 \iff$$
$$3m-2^kn=-\frac{7}{3},$$

which is a contradiction since $3m-2^kn\in\mathbb{Z}$. Thus, C_1 does not map odd numbers of the form $3m+2$ to even numbers of the form $2^k(3n)$.

We then ask if C_1 can map odd numbers of the form $3m+2$, with m odd and $m\geq 1$, to even numbers of the form $2^k(3n+2)$, with $k\geq 1$ and n odd and $n\geq 1$:

Note that we have

$$C_1(3m+2)=2^k(3n+2) \iff$$
$$3(3m+2)+1=3(2^kn)+2^{k+1} \iff$$
$$3(3m-2^kn)=2^{k+1}-7 \iff$$
$$3m-2^kn=\frac{2^{k+1}-7}{3}. \tag{2.10}$$

We first find all values of k for which $3|(2^{k+1}-7)$:

$$k+1=2\ell,\ \ell=1,2,\ldots \implies$$

$$k=2\ell-1,\ \ell=1,2,\ldots, \text{ and } k\geq 1. \tag{2.11}$$

The discussion on the mapping $C_1(3m) = 2^k(3n+2)$ involving Eqs.(2.5) and (2.6) applies here *for* $k \geq 3$ and for d redefined as

$$d =: \frac{2^{k+1}-7}{3}, \quad k \geq 3 \tag{2.12}$$

(for $k \geq 3$, $d \in \mathbb{N}$).

However, for $k = 1$, $(2^{k+1}-7)/3 = -1 < 0$. Hence, we must separately consider the case when odd m, n pairs satisfy the equation

$$3m - 2^1 n = -1. \tag{2.13}$$

So, we let $m = 2s+1$, $s \geq 0$, and $n = 2t+1$, $t \geq 0$, in Eq.(2.13) and obtain the following:

$$3m - 2n = -1 \iff$$

$$3(2s+1) - 2(2t+1) = -1 \iff$$

$$3s - 2t = -1 \iff$$

$$3s = 2t - 1. \tag{2.14}$$

Notice that in Eq.(2.14), s must be odd. The ordered pairs (s, t) that satisfy Eq.(2.14) are infinite in number and are

$$(s,t) = (1,2),\ (3,5),\ (5,8),\ (7,11), \ldots \implies$$

$$(s,t) = (2u+1, 3u+2), \quad u = 0, 1, \ldots. \tag{2.15}$$

Therefore, there exist infinitely many odd m, n pairs that satisfy Eq.(2.13).

Based on Eqs.(2.10)–(2.15), we have the infinite set of images $R_2 \subset C_1(A_2)$ such that

$$\begin{aligned} R_2 \;=\; & \{2^k(3n+2)\colon k = 2\ell-1, \ell = 1, 2, \ldots, \text{ and } n \text{ is } \ni \\ & m, n \text{ satisfy the equation } 3m - 2^k n = (2^{k+1}-7)/3 \\ & \text{with } m, n \text{ odd and } m, n \geq 1\} \\ \supset\; & \{2^k(3n+2)\colon k = 2\ell-1, \ell = 2, 3, \ldots, \text{ and } n \text{ is } \ni \\ & m = (2^{k+1}-7)m_0/3,\ n = (2^{k+1}-7)n_0/3, \text{ with} \\ & (m_0, n_0, k) \in Q\} \\ & \cup \{2^1(3n+2)\colon n \text{ is } \ni m, n \text{ satisfy the equation} \\ & \quad 3m - 2n = -1, \text{ where } m = 2s+1,\ n = 2t+1, \\ & \quad (s,t) = (2u+1, 3u+2),\ u = 0, 1, \ldots\}. \end{aligned}$$

Furthermore, we make the following observations on the mapping

$$C_1(3m+2) = 2^k(3n+2):$$

1. After the mapping $C_1(3m+2) = 2^k(3n+2)$ *with* $k \geq 3$, an additional sequence of k iterations of $\tilde{C}_1$ has odd numbers of the form $3m+2$ mapped back to smaller odd numbers of the form $3n+2$ since

$$3n+2 = \frac{9m+7}{2^k} \leq \frac{9m+7}{2^3} < 3m+2 \quad \Longleftarrow \quad \frac{9m+7}{8} < 3m+2 \quad \Longleftrightarrow$$

$$9m+7 < 24m+16,$$

which is obviously true for $m \geq 1$.

2. After the mapping $C_1(3m+2) = 2^k(3n+2)$ *with* $k = 1$, an additional $k = 1$ iteration of $\tilde{C}_1$ has odd numbers of the form $3m+2$ mapped forward to larger odd numbers of the form $3n+2$ since

$$3n+2 = \frac{9m+7}{2^1} > 3m+2 \quad \Longleftarrow \quad \frac{9m+7}{2} > 3m+2 \quad \Longleftrightarrow 9m+7 > 6m+4,$$

which is obviously true for $m \geq 1$.

We finally ask if C_1 can map odd numbers of the form $3m+2$, with m odd and $m \geq 1$, to even numbers of the form $2^k(3n+4)$, with $k \geq 1$ and n odd and $n \geq 1$:

Note that we have

$$C_1(3m+2) = 2^k(3n+4) \quad \Longleftrightarrow$$

$$3(3m+2)+1 = 3(2^k n) + 2^{k+2} \quad \Longleftrightarrow$$

$$3(3m - 2^k n) = 2^{k+2} - 7 \quad \Longleftrightarrow$$

$$3m - 2^k n = \frac{2^{k+2}-7}{3}. \tag{2.16}$$

We first find all values of k for which $3|(2^{k+2}-7)$:

$$k+2 = 2\ell, \; \ell = 1, 2, \ldots \quad \Longrightarrow$$

$$k = 2\ell - 2, \; \ell = 2, 3, \ldots, \text{ and } k \geq 2 \tag{2.17}$$

(where we must at least have $k \geq 1$ so we cannot have $k = 0$).

Here, we redefine the d given in Eq.(2.5) as

$$d =: \frac{2^{k+2}-7}{3}. \tag{2.18}$$

Note that indeed $d \in \mathbb{N}$ since, for $k \geq 2$, $d \geq 3 > 0$.

By a similar argument as that on the mapping $C_1(3m) = 2^k(3n + 2)$ involving Eqs.(2.5) and (2.6), and based on Eqs.(2.16)–(2.18), we have the infinite set of images $S_2 \subset C_1(A_2)$ such that

$$\begin{aligned} S_2 &= \{2^k(3n+4)\colon k = 2\ell - 2,\ \ell = 2, 3, \ldots, \text{ and } n \text{ is } \ni \\ &\quad\ m, n \text{ satisfy the equation } 3m - 2^k n = (2^{k+2} - 7)/3 \\ &\quad\ \text{with } m, n \text{ odd and } m, n \geq 1\} \\ &\supset \{2^k(3n+4)\colon k = 2\ell - 2,\ \ell = 2, 3, \ldots, \text{ and } n \text{ is } \ni \\ &\quad\ m = (2^{k+2} - 7)m_0/3,\ n = (2^{k+2} - 7)n_0/3, \text{ with} \\ &\quad\ (m_0, n_0, k) \in Q\}\,. \end{aligned}$$

Furthermore, we make the following observation on the mapping

$$C_1(3m + 2) = 2^k(3n + 4) :$$

After the mapping $C_1(3m + 2) = 2^k(3n + 4)$ *with* $k \geq 2$, an additional sequence of k iterations of $\tilde{C}_1$ has odd numbers of the form $3m + 2$ mapped back to smaller odd numbers of the form $3n + 4$ since

$$3n + 4 = \frac{9m+7}{2^k} \leq \frac{9m+7}{2^2} < 3m + 2 \iff \frac{9m+7}{4} < 3m + 2 \iff$$
$$9m + 7 < 12m + 8,$$

which is obviously true for $m \geq 1$.

We conclude that, since it is clear that the union of P_2, R_2, and S_2 contains all possible images to which C_1 maps odd numbers of the form $3m + 2$, then

$$C_1(A_2) = P_2 \cup R_2 \cup S_2,$$

with the set definitions of P_2, R_2, and S_2 elucidating the structure of $C_1(A_2)$.

2.3 Characterization of the Set $C_1(A_3)$

We first ask if C_1 can map odd numbers of the form $3m + 4$, with m odd and $m \geq 1$, to even numbers of the form $2^k(1)$, with $k \geq 1$:

Note that we have

$$\begin{aligned} C_1(3m + 4) = 2^k(1) &\iff \\ 3(3m + 4) + 1 = 2^k &\iff \\ m = \frac{2^k - 13}{9}, \end{aligned}$$

which holds when $9|(2^k - 13)$, where $2^k - 13$ is odd so that the right-hand side of this last equation is consistent with m being odd when $9|(2^k - 13)$. If the digits of $2^k - 13$ add up to 9, then $9|(2^k - 13)$. The set of values of k for which $9|(2^k - 13)$ is as follows:

$$k = 8, 14, 20, \ldots \implies k = 6\ell + 8,\ \ell = 0, 1, \ldots \implies$$

$$k = 2(3\ell + 4),\ \ell = 0, 1, \ldots, \text{ and } k \geq 8$$

(note that when $k = 2$, $(2^k - 13)/9 = -1 < 0$, which we cannot have since we must have $m \geq 1 > 0$).

Therefore, we have the infinite set of images $P_3 \subset C_1(A_3)$ such that

$$P_3 = \left\{2^k(1) \colon k = 2(3\ell + 4),\ \ \ell = 0, 1, \ldots\right\}.$$

After the mapping $C_1(3m + 4) = 2^k(1)$, $\tilde{C}_1$ maps $2^k(1)$ to the cycle $(1, 4, 2)$.

We next ask if C_1 can map odd numbers of the form $3m+4$, with m odd and $m \geq 1$, to even numbers of the form $2^k(3n)$, with $k \geq 1$ and n odd and $n \geq 1$:

Note that we have

$$C_1(3m + 4) = 2^k(3n) \iff$$
$$3(3m + 4) + 1 = 3(2^k n) \iff$$
$$3(3m - 2^k n) = -13 \iff$$
$$3m - 2^k = -\frac{13}{3},$$

which is a contradiction since $3m - 2^k n \in \mathbb{Z}$. Thus, C_1 does not map odd numbers of the form $3m + 4$ to even numbers of the form $2^k(3n)$.

We then ask if C_1 can map odd numbers of the form $3m+4$, with m odd and $m \geq 1$, to even numbers of the form $2^k(3n + 2)$, with $k \geq 1$ and n odd and $n \geq 1$:

Note that we have

$$C_1(3m + 4) = 2^k(3n + 2) \iff$$
$$3(3m + 4) + 1 = 3(2^k n) + 2^{k+1} \iff$$
$$3(3m - 2^k n) = 2^{k+1} - 13 \iff$$
$$3m - 2^k n = \frac{2^{k+1} - 13}{3}. \tag{2.19}$$

We first find all values of k for which $3|(2^{k+1} - 13)$:

$$k + 1 = 2\ell,\ \ell = 1, 2, \ldots \implies$$

$$k = 2\ell - 1,\ \ell = 1, 2, \ldots, \text{ and } k \geq 1. \tag{2.20}$$

The discussion on the mapping $C_1(3m) = 2^k(3n + 2)$ involving Eqs.(2.5) and (2.6) applies here *for* $k \geq 3$ and for d redefined as

$$d =: \frac{2^{k+1} - 13}{3},\ \ k \geq 3 \tag{2.21}$$

(for $k \geq 3$, $d \in \mathbb{N}$).

However, for $k=1$, $(2^{k+1}-13)/3=-3<0$. Hence, we must separately consider the case when odd m,n pairs satisfy the equation

$$3m-2^1n=-3. \tag{2.22}$$

So, we let $m=2s+1$, $s\geq 0$, and $n=2t+1$, $t\geq 0$, in Eq.(2.22) and obtain the following:

$$\begin{aligned} 3m-2n=-3 &\iff \\ 3(2s+1)-2(2t+1)=-3 &\iff \\ 3s-2t=-2 &\iff \\ 3s=2(t-1). & \end{aligned} \tag{2.23}$$

Notice that in Eq.(2.23), s must be even. The ordered pairs (s,t) that satisfy Eq.(2.23) are infinite in number and are

$$(s,t)=(0,1),\ (2,4),\ (4,7),\ (6,10),\ldots \implies$$

$$(s,t)=(2u,3u+1),\quad u=0,1,\ldots. \tag{2.24}$$

Therefore, there exist infinitely many odd m,n pairs that satisfy Eq.(2.22).

Based on Eqs.(2.19)–(2.24), we have the infinite set of images $R_3\subset C_1(A_3)$ such that

$$\begin{aligned} R_3 \;=\; & \{2^k(3n+2)\colon k=2\ell-1, \ell=1,2,\ldots, \text{and } n \text{ is} \ni \\ & m,n \text{ satisfy the equation } 3m-2^kn=(2^{k+1}-13)/3 \\ & \text{with } m,n \text{ odd and } m,n\geq 1\} \\ \supset\; & \{2^k(3n+2)\colon k=2\ell-1, \ell=2,3,\ldots, \text{and } n \text{ is} \ni \\ & m=(2^{k+1}-13)m_0/3,\ n=(2^{k+1}-13)n_0/3, \text{ with} \\ & (m_0,n_0,k)\in Q\} \\ & \cup\{2^1(3n+2)\colon n \text{ is} \ni m,n \text{ satisfy the equation} \\ & \quad 3m-2n=-3, \text{ where } m=2s+1,\ n=2t+1, \\ & \quad (s,t)=(2u,3u+1),\ u=0,1,\ldots\}. \end{aligned}$$

Furthermore, we make the following observations on the mapping

$$C_1(3m+4)=2^k(3n+2):$$

1. After the mapping $C_1(3m+4)=2^k(3n+2)$ *with* $k\geq 3$, an additional sequence of k iterations of $\tilde{C}_1$ has odd numbers of the form $3m+4$ mapped back to smaller odd numbers of the form $3n+2$ since

$$3n+2=\frac{9m+13}{2^k}\leq\frac{9m+13}{2^3}<3m+4 \quad\Longleftarrow\quad \frac{9m+13}{8}<3m+4 \iff$$

$$9m+13<24m+32,$$

which is obviously true for $m\geq 1$.

2. After the mapping $C_1(3m+4) = 2^k(3n+2)$ *with* $k = 1$, an additional $k = 1$ iteration of $\tilde{C}_1$ has odd numbers of the form $3m+4$ mapped forward to larger odd numbers of the form $3n+2$ since

$$3n+2 = \frac{9m+13}{2^1} > 3m+4 \iff \frac{9m+13}{2} > 3m+4 \iff$$
$$9m+13 > 6m+8,$$

which is obviously true for $m \geq 1$.

We finally ask if C_1 can map odd numbers of the form $3m+4$, with m odd and $m \geq 1$, to even numbers of the form $2^k(3n+4)$, with $k \geq 1$ and n odd and $n \geq 1$:

Note that we have

$$\begin{aligned} C_1(3m+4) = 2^k(3n+4) &\iff \\ 3(3m+4)+1 = 3(2^k n) + 2^{k+2} &\iff \\ 3(3m - 2^k n) = 2^{k+2} - 13 &\iff \\ 3m - 2^k n = \frac{2^{k+2}-13}{3}. & \end{aligned} \tag{2.25}$$

We first find all values of k for which $3|(2^{k+2} - 13)$:

$$k+2 = 2\ell, \ \ell = 1, 2, \ldots \implies$$
$$k = 2\ell - 2, \ \ell = 2, 3, \ldots, \text{ and } k \geq 2 \tag{2.26}$$

(where we must at least have $k \geq 1$ so we cannot have $k = 0$).

Here, we redefine the d given in Eq.(2.5) as

$$d =: \frac{2^{k+2}-13}{3}. \tag{2.27}$$

Note that indeed $d \in \mathbb{N}$ since, for $k \geq 2$, $d \geq 1 > 0$.

By a similar argument as that on the mapping $C_1(3m) = 2^k(3n+2)$ involving Eqs.(2.5) and (2.6), and based on Eqs.(2.25)–(2.27), we have the infinite set of images $S_3 \subset C_1(A_3)$ such that

$$\begin{aligned} S_3 &= \{2^k(3n+4) \colon k = 2\ell - 2, \ell = 2, 3, \ldots, \text{ and } n \text{ is} \ni \\ &\quad m, n \text{ satisfy the equation } 3m - 2^k n = (2^{k+2} - 13)/3 \\ &\quad \text{with } m, n \text{ odd and } m, n \geq 1\} \\ &\supset \{2^k(3n+4) \colon k = 2\ell - 2, \ell = 2, 3, \ldots, \text{ and } n \text{ is} \ni \\ &\quad m = (2^{k+2} - 13)m_0/3, n = (2^{k+2} - 13)n_0/3, \text{ with} \\ &\quad (m_0, n_0, k) \in Q\}. \end{aligned}$$

Furthermore, we make the following observation on the mapping

$$C_1(3m+4) = 2^k(3n+4):$$

After the mapping $C_1(3m + 4) = 2^k(3n + 4)$ *with* $k \geq 2$, an additional sequence of k iterations of $\tilde{C}_1$ has odd numbers of the form $3m + 4$ mapped back to smaller odd numbers of the form $3n + 4$ since

$$3n + 4 = \frac{9m + 13}{2^k} \leq \frac{9m + 13}{2^2} < 3m + 4 \iff \frac{9m + 13}{4} < 3m + 4 \iff$$
$$9m + 13 < 12m + 16,$$

which is obviously true for $m \geq 1$.

We conclude that, since it is clear that the union of P_3, R_3, and S_3 contains all possible images to which C_1 maps odd numbers of the form $3m + 4$, then

$$C_1(A_3) = P_3 \cup R_3 \cup S_3,$$

with the set definitions of P_3, R_3, and S_3 elucidating the structure of $C_1(A_3)$.

3 The C_2 and $\tilde{C}_2$ Maps

For the sake of brevity, we analyze in detail the set $C_2(B_1)$, but only present the results of the analyses of the sets $C_2(B_i)$, for $i = 2, 3, 4, 5$, together with the results on how a certain number of iterations of $\tilde{C}_2$ maps odd numbers of the form $5m + i$, $i = 2, 4, 6, 8$, to 1, 3, or odd numbers of the form $5n + j$, n odd, $n \geq 1$, $j = 2, 4, 6, 8$. The analyses, which were computational in nature, of the sets $C_2(B_i)$, for $i = 2, 3, 4, 5$, were made in the same way as the analyses of the sets $C_1(A_i)$, for $i = 1, 2, 3$ in Section 2 and as the analysis of the set $C_2(B_1)$ in this section.

3.1 Characterization of the Set $C_2(B_1)$

Recall that

$$B_1 = \{x \in \mathbb{N} \colon x = 5m, \ \ m = 1, 3, 5, \ldots\}.$$

First, we ask if C_2 can map odd numbers of the form $5m$, with m odd and $m \geq 1$, to even numbers of the form $2^k(1)$, with $k \geq 1$:

Note that we have

$$C_1(5m) = 2^k(1) \iff$$
$$5(5m) + 1 = 2^k \iff$$
$$m = \frac{2^k - 1}{25},$$

which holds when $25|(2^k - 1)$, where $2^k - 1$ is odd so that the right-hand side of this last equation is consistent with m being odd when $25|(2^k - 1)$. The set of values of k for which $25|(2^k - 1)$ is as follows:

$$k = 4\ell, \ \ell = 5, 10, 15, 20, \ldots \implies$$

$$k = 4(5\ell),\ \ell = 1, 2, \ldots, \text{ and } k \geq 20.$$

Therefore, we have the infinite set of images $M_1 \subset C_2(B_1)$ such that

$$M_1 = \left\{2^k(1)\colon k = 4(5\ell),\ \ \ell = 1, 2, \ldots\right\}.$$

After the mapping $C_2(5m) = 2^k(1)$, $\tilde{C}_2$ maps $2^k(1)$ to the cycle $(1, 6, 3, 16, 8, 4, 2)$.

Second, we ask if C_2 can map odd numbers of the form $5m$, with m odd and $m \geq 1$, to even numbers of the form $2^k(3)$, with $k \geq 1$:

Note that we have

$$C_2(5m) = 2^k(3) \iff$$

$$5(5m) + 1 = 2^k(3) \iff$$

$$m = \frac{2^k \cdot 3 - 1}{25},$$

which holds when $25|(2^k \cdot 3 - 1)$, where $2^k \cdot 3 - 1$ is odd so that the right-hand side of this last equation is consistent with m being odd when $25|(2^k \cdot 3 - 1)$. The set of values of k for which $25|(2^k \cdot 3 - 1)$ is as follows:

$$k = 20\ell + 13,\ \ell = 0, 1, \ldots, \text{ and } k \geq 13.$$

Therefore, we have the infinite set of images $N_1 \subset C_2(B_1)$ such that

$$N_1 = \left\{2^k(3)\colon k = 20\ell + 13,\ \ \ell = 0, 1, \ldots\right\}.$$

After the mapping $C_2(5m) = 2^k(3)$, $\tilde{C}_2$ maps $2^k(3)$ to the cycle $(3, 16, 8, 4, 2, 1, 6)$.

Third, we ask if C_2 can map odd numbers of the form $5m$, with m odd and $m \geq 1$, to even numbers of the form $2^k(5n)$, with $k \geq 1$ and n odd and $n \geq 1$:

Note that we have

$$C_2(5m) = 2^k(5n) \iff$$

$$5(5m) + 1 = 5(2^k n) \iff$$

$$5(5m - 2^k n) = -1 \iff$$

$$5m - 2^k n = -\frac{1}{5},$$

which is a contradiction since $5m - 2^k n \in \mathbb{Z}$. Thus, C_2 does not map odd numbers of the form $5m$ to even numbers of the form $2^k(5n)$.

Fourth, we ask if C_2 can map odd numbers of the form $5m$, with m odd and $m \geq 1$, to even numbers of the form $2^k(5n + 2)$, with $k \geq 1$ and n odd and $n \geq 1$:

Note that we have

$$C_2(5m) = 2^k(5n + 2) \iff$$

$$5(5m) + 1 = 5(2^k n) + 2^{k+1} \iff$$

$$5(5m - 2^k n) = 2^{k+1} - 1 \iff$$
$$5m - 2^k n = \frac{2^{k+1} - 1}{5}, \tag{3.1}$$

which holds if

1. $5|(2^{k+1} - 1)$ for certain values of k;
2. for the certain values of k, there exist odd m, n pairs that satisfy Eq.(3.1).

We first determine all values of k for which $5|(2^{k+1} - 1)$:

$$k + 1 = 4\ell,\ \ell = 1, 2, \ldots \implies$$
$$k = 4\ell - 1,\ \ell = 1, 2, \ldots, \text{ and } k \geq 3. \tag{3.2}$$

In order to find the odd m, n pairs that satisfy Eq.(3.1) for values of k given by Eq.(3.2), we first determine the odd m, n pairs that satisfy the equation

$$5m - 2^k n = 1, \text{ for } k = 1, 2, \ldots. \tag{3.3}$$

In actuality, we are only able to find at least one odd m, n pair for each $k = 1, 2, \ldots$, with Eq.(3.3) rewritten as

$$2^k n = 5m - 1, \text{ for } k = 1, 2, \ldots.$$

We obtain

$$\begin{array}{lll} k = 1: & 2^1 \cdot 7 = 2^1 \cdot 7 \cdot 3^0 = 14 = 5 \cdot 3 - 1 & \implies m = 3, n = 7 \cdot 3^0 \\ k = 2: & 2^2 \cdot 21 = 2^2 \cdot 7 \cdot 3^1 = 84 = 5 \cdot 17 - 1 & \implies m = 17, n = 7 \cdot 3^1 \\ k = 3: & 2^3 \cdot 189 = 2^3 \cdot 7 \cdot 3^2 = 504 = 5 \cdot 101 - 1 & \implies m = 101, n = 7 \cdot 3^2 \\ k = 4: & 2^4 \cdot 567 = 2^4 \cdot 7 \cdot 3^3 = 18144 = 5 \cdot 3629 - 1 & \implies m = 3629, n = 7 \cdot 3^3 \end{array} \tag{3.4}$$

Given (3.4), we are then able to define the infinite set

$$\begin{aligned} U &= \{(m, n, k)\colon k = 1, 2, \ldots, \text{ and } m, n \text{ satisfy the equation} \\ &\quad 5m - 2^k n = 1 \text{ with } m, n \text{ odd and } m, n \geq 1\} \\ &\supset \left\{(m_k, n_k, k)\colon \text{for } k = 1, 2, \ldots, m_k = (2^k \cdot 7 \cdot 3^{k-1} + 1)/3, \right. \\ &\quad \left. n_k = 7 \cdot 3^k\right\}. \end{aligned}$$

Now, for any value of k that satisfies Eq.(3.2), i.e., $k = 4\ell - 1,\ \ell \in \{1, 2, \ldots\}$, let

$$d =: \frac{2^{k+1} - 1}{5} \in \mathbb{N}. \tag{3.5}$$

Then d is odd since $2^{k+1} - 1$ is odd, and there exists at least one ordered triple, $(m_0, n_0, k) \in U$, such that

$$5(dm_0) - 2^k(dn_0) = d(1). \tag{3.6}$$

Thus, for each value of $k = 4\ell - 1,\ \ell \in \{1, 2, \ldots\}$, there exists at least one odd pair, $m = dm_0$, $n = dn_0$, such that

$$5m - 2^k n = \frac{2^{k+1} - 1}{5}.$$

Therefore, we have the infinite set of images $V_1 \subset C_2(B_1)$ such that

$$\begin{aligned} V_1 \;=\;& \big\{2^k(5n+2)\colon k = 4\ell - 1, \ell = 1, 2, \ldots, \text{ and } n \text{ is} \ni \\ & m, n \text{ satisfy the equation } 5m - 2^k n = (2^{k+1} - 1)/5 \\ & \text{with } m, n \text{ odd and } m, n \geq 1\big\} \\ \supset\;& \big\{2^k(5n+2)\colon k = 4\ell - 1, \ell = 1, 2, \ldots, \text{ and } n \text{ is} \ni \\ & m = (2^{k+1} - 1)m_0/5,\ n = (2^{k+1} - 1)n_0/5, \text{ with} \\ & (m_0, n_0, k) \in U\big\}. \end{aligned}$$

Furthermore, we make the following observation on the mapping

$$C_2(5m) = 2^k(5n + 2):$$

After the mapping $C_2(5m) = 2^k(5n + 2)$ *with* $k \geq 3$, an additional sequence of k iterations of $\tilde{C}_2$ has odd numbers of the form $5m$ mapped back to smaller odd numbers of the form $5n + 2$ since

$$5n + 2 = \frac{25m + 1}{2^k} \leq \frac{25m + 1}{2^3} < 5m \iff \frac{25m + 1}{8} < 5m \iff$$
$$25m + 1 < 25m + 15m,$$

which is obviously true for $m \geq 1$.

Fifth, we ask if C_2 can map odd numbers of the form $5m$, with m odd and $m \geq 1$, to even numbers of the form $2^k(5n + 4)$, with $k \geq 1$ and n odd and $n \geq 1$:

Note that we have

$$\begin{aligned} C_2(5m) = 2^k(5n + 4) &\iff \\ 5(5m) + 1 = 5(2^k n) + 2^{k+2} &\iff \\ 5(5m - 2^k n) = 2^{k+2} - 1 &\iff \\ 5m - 2^k n = \frac{2^{k+2} - 1}{5}. & \end{aligned} \tag{3.7}$$

We find all values of k for which $5|(2^{k+2} - 1)$:

$$k + 2 = 4\ell,\ \ell = 1, 2, \ldots \implies$$

$$k = 4\ell - 2,\ \ell = 1, 2, \ldots, \text{ and } k \geq 2 \tag{3.8}$$

Here, we redefine the d given in Eq.(3.5) as

$$d =: \frac{2^{k+2} - 1}{5}. \tag{3.9}$$

Note that indeed $d \in \mathbb{N}$ since, for $k \geq 2$, $d \geq 3 > 0$.

By a similar argument as that on the mapping $C_2(5m) = 2^k(5n+2)$ involving Eqs.(3.5) and (3.6), and based on Eqs.(3.7)–(3.9), we have the infinite set of images $W_1 \subset C_2(B_1)$ such that

$$\begin{aligned} W_1 &= \{2^k(5n+4)\colon k = 4\ell - 2,\ \ell = 1, 2, \ldots, \text{ and } n \text{ is } \ni \\ &\quad m, n \text{ satisfy the equation } 5m - 2^k n = (2^{k+2}-1)/5 \\ &\quad \text{with } m, n \text{ odd and } m, n \geq 1\} \\ &\supset \{2^k(5n+4)\colon k = 4\ell - 2,\ \ell = 1, 2, \ldots, \text{ and } n \text{ is } \ni \\ &\quad m = (2^{k+2}-1)m_0/5,\ n = (2^{k+2}-1)n_0/5, \text{ with} \\ &\quad (m_0, n_0, k) \in U\}\,. \end{aligned}$$

Furthermore, we make the following observations on the mapping

$$C_2(5m) = 2^k(5n+4):$$

1. After the mapping $C_2(5m) = 2^k(5n+4)$ *with* $k \geq 6$, an additional sequence of k iterations of $\tilde{C}_2$ has odd numbers of the form $5m$ mapped back to smaller odd numbers of the form $5n+4$ since

$$5n+4 = \frac{25m+1}{2^k} \leq \frac{25m+1}{2^6} < 5m \iff \frac{25m+1}{64} < 5m \iff$$

$$25m+1 < 25m+295m,$$

which is obviously true for $m \geq 1$.

2. After the mapping $C_2(5m) = 2^k(5n+4)$ *with* $k = 2$, an additional sequence of $k = 2$ iterations of $\tilde{C}_2$ has odd numbers of the form $5m$ mapped forward to larger odd numbers of the form $5n+4$ since

$$5n+4 = \frac{25m+1}{2^2} > 5m \iff \frac{25m+1}{4} > 5m \iff 25m+1 > 20m,$$

which is obviously true for $m \geq 1$.

Sixth, we ask if C_2 can map odd numbers of the form $5m$, with m odd and $m \geq 1$, to even numbers of the form $2^k(5n+6)$, with $k \geq 1$ and n odd and $n \geq 1$:

Note that we have

$$C_2(5m) = 2^k(5n+6) \iff$$

$$5(5m)+1 = 5(2^k n) + 2^{k+1}(3) \iff$$

$$5(5m - 2^k n) = 2^{k+1}(3) - 1 \iff$$

$$5m - 2^k n = \frac{2^{k+1}\cdot 3 - 1}{5}. \tag{3.10}$$

We find all values of k for which $5|(2^{k+1}\cdot 3-1)$:

$$k+1=4\ell+1,\ \ell=0,1,\ldots \implies$$

$$k=4\ell,\ \ell=0,1,\ldots \implies$$

$$k=4\ell,\ \ell=1,2,\ldots,\text{ and } k\geq 4 \tag{3.11}$$

(We cannot have $k=0$. Otherwise, Eq.(3.10) becomes

$$5m-n=1,$$

which is a contradiction since $5m-n$ is even for m,n odd.)

Here, we redefine the d given in Eq.(3.5) as

$$d=:\frac{2^{k+1}\cdot 3-1}{5}. \tag{3.12}$$

Note that indeed $d\in\mathbb{N}$ since, for $k\geq 4$, $d\geq 19>0$.

By a similar argument as that on the mapping $C_2(5m)=2^k(5n+2)$ involving Eqs.(3.5) and (3.6), and based on Eqs.(3.10)–(3.12), we have the infinite set of images $X_1\subset C_2(B_1)$ such that

$$\begin{aligned} X_1 \;=\; & \{2^k(5n+6)\colon k=4\ell,\ \ell=1,2,\ldots,\text{ and } n \text{ is } \ni \\ & m,n \text{ satisfy the equation } 5m-2^k n=(2^{k+1}\cdot 3-1)/5 \\ & \text{with } m,n \text{ odd and } m,n\geq 1\} \\ \supset\; & \{2^k(5n+6)\colon k=4\ell,\ \ell=1,2,\ldots,\text{ and } n \text{ is } \ni \\ & m=(2^{k+1}\cdot 3-1)m_0/5,\ n=(2^{k+1}\cdot 3-1)n_0/5,\text{ with} \\ & (m_0,n_0,k)\in U\}\,. \end{aligned}$$

Furthermore, we make the following observation on the mapping

$$C_2(5m)=2^k(5n+6)\ :$$

After the mapping $C_2(5m)=2^k(5n+6)$ *with* $k\geq 4$, an additional sequence of k iterations of $\tilde{C}_2$ has odd numbers of the form $5m$ mapped back to smaller odd numbers of the form $5n+6$ since

$$5n+6=\frac{25m+1}{2^k}\leq\frac{25m+1}{2^4}<5m \iff \frac{25m+1}{16}<5m \iff$$

$$25m+1<25m+65m,$$

which is obviously true for $m\geq 1$.

Lastly, we ask if C_2 can map odd numbers of the form $5m$, with m odd and $m\geq 1$, to even numbers of the form $2^k(5n+8)$, with $k\geq 1$ and n odd and $n\geq 1$:

Note that we have

$$C_2(5m)=2^k(5n+8) \iff$$

$$\begin{aligned} 5(5m)+1 = 5(2^k n) + 2^{k+3} &\iff \\ 5(5m - 2^k n) = 2^{k+3} - 1 &\iff \\ 5m - 2^k n = \frac{2^{k+3}-1}{5}. & \end{aligned} \tag{3.13}$$

We find all values of k for which $5|(2^{k+3}-1)$:

$$k+3 = 4\ell,\ \ell = 1,2,\ldots \implies$$

$$k = 4\ell - 3,\ \ell = 1,2,\ldots, \text{ and } k \geq 1 \tag{3.14}$$

Here, we redefine the d given in Eq.(3.5) as

$$d =: \frac{2^{k+3}-1}{5}. \tag{3.15}$$

Note that indeed $d \in \mathbb{N}$ since, for $k \geq 1$, $d \geq 3 > 0$.

By a similar argument as that on the mapping $C_2(5m) = 2^k(5n+2)$ involving Eqs.(3.5) and (3.6), and based on Eqs.(3.13)–(3.15), we have the infinite set of images $Y_1 \subset C_2(B_1)$ such that

$$\begin{aligned} Y_1 &= \{2^k(5n+8) \colon k = 4\ell-3, \ell = 1,2,\ldots, \text{ and } n \text{ is} \ni \\ &\quad m,n \text{ satisfy the equation } 5m - 2^k n = (2^{k+3}-1)/5 \\ &\quad \text{with } m,n \text{ odd and } m,n \geq 1\} \\ &\supset \{2^k(5n+8) \colon k = 4\ell-3, \ell = 1,2,\ldots, \text{ and } n \text{ is} \ni \\ &\quad m = (2^{k+3}-1)m_0/5,\ n = (2^{k+3}-1)n_0/5, \text{ with} \\ &\quad (m_0, n_0, k) \in U\}. \end{aligned}$$

Furthermore, we make the following observations on the mapping

$$C_2(5m) = 2^k(5n+8):$$

1. After the mapping $C_2(5m) = 2^k(5n+8)$ *with* $k \geq 5$, an additional sequence of k iterations of $\tilde{C}_2$ has odd numbers of the form $5m$ mapped back to smaller odd numbers of the form $5n+8$ since
$$5n+8 = \frac{25m+1}{2^k} \leq \frac{25m+1}{2^5} < 5m \iff \frac{25m+1}{32} < 5m \iff$$
$$25m+1 < 25m + 135m,$$
which is obviously true for $m \geq 1$.

2. After the mapping $C_2(5m) = 2^k(5n+8)$ *with* $k = 1$, an additional $k = 1$ iteration of $\tilde{C}_2$ has odd numbers of the form $5m$ mapped forward to larger odd numbers of the form $5n+8$ since
$$5n+8 = \frac{25m+1}{2^1} > 5m \iff \frac{25m+1}{2} > 5m \iff 25m+1 > 10m,$$
which is obviously true for $m \geq 1$.

We conclude that, since it is clear that the union of M_1, N_1, V_1, W_1, X_1, and Y_1 contains all possible images to which C_2 maps odd numbers of the form $5m$, then

$$C_2(B_1) = M_1 \cup N_1 \cup V_1 \cup W_1 \cup X_1 \cup Y_1,$$

with the set definitions of M_1, N_1, V_1, W_1, X_1, and Y_1 elucidating the structure of $C_2(B_1)$.

3.2 Characterization of the Set $C_2(B_2)$

Recall that

$$B_2 = \{x \in \mathbb{N}\colon x = 5m + 2, \ \ m = 1, 3, 5, \ldots\}.$$

After an analysis in the same spirit as those on the characterizations of $C_1(A_i)$, $i = 1, 2, 3$, in Section 2 and of $C_2(B_2)$ in Subsection 3.1, we obtain

$$C_2(B_2) = M_2 \cup N_2 \cup V_2 \cup W_2 \cup X_2 \cup Y_2,$$

where each of the sets in the union on the right-hand side is as follows:

$$M_2 = \left\{2^k(1)\colon k = 4(5\ell + 4), \ \ \ell = 0, 1, \ldots\right\};$$

$$N_2 = \left\{2^k(3)\colon k = 20\ell + 9, \ \ \ell = 0, 1, \ldots\right\};$$

$$\begin{aligned} V_2 &= \big\{2^k(5n+2)\colon k = 4\ell - 1, \ell = 1, 2, \ldots, \text{ and } n \text{ is } \ni \\ &\quad\ m, n \text{ satisfy the equation } 5m - 2^k n = (2^{k+1} - 11)/5 \\ &\quad\ \text{with } m, n \text{ odd and } m, n \geq 1\} \\ &\supset \big\{2^k(5n+2)\colon k = 4\ell - 1, \ell = 1, 2, \ldots, \text{ and } n \text{ is } \ni \\ &\quad\ m = (2^{k+1} - 11)m_0/5, n = (2^{k+1} - 11)n_0/5, \text{ with} \\ &\quad\ (m_0, n_0, k) \in U\}; \end{aligned}$$

$$\begin{aligned} W_2 &= \big\{2^k(5n+4)\colon k = 4\ell - 2, \ell = 1, 2, \ldots, \text{ and } n \text{ is } \ni \\ &\quad\ m, n \text{ satisfy the equation } 5m - 2^k n = (2^{k+2} - 11)/5 \\ &\quad\ \text{with } m, n \text{ odd and } m, n \geq 1\} \\ &\supset \big\{2^k(5n+4)\colon k = 4\ell - 2, \ell = 1, 2, \ldots, \text{ and } n \text{ is } \ni \\ &\quad\ m = (2^{k+2} - 11)m_0/5, n = (2^{k+2} - 11)n_0/5, \text{ with} \\ &\quad\ (m_0, n_0, k) \in U\}; \end{aligned}$$

$$\begin{aligned} X_2 &= \big\{2^k(5n+6)\colon k = 4\ell, \ell = 1, 2, \ldots, \text{ and } n \text{ is } \ni \\ &\quad\ m, n \text{ satisfy the equation } 5m - 2^k n = (2^{k+1} \cdot 3 - 11)/5 \\ &\quad\ \text{with } m, n \text{ odd and } m, n \geq 1\} \\ &\supset \big\{2^k(5n+6)\colon k = 4\ell, \ell = 1, 2, \ldots, \text{ and } n \text{ is } \ni \\ &\quad\ m = (2^{k+1} \cdot 3 - 11)m_0/5, n = (2^{k+1} \cdot 3 - 11)n_0/5, \text{ with} \\ &\quad\ (m_0, n_0, k) \in U\}; \end{aligned}$$

$$\begin{aligned} Y_2 &= \{2^k(5n+8)\colon k=4\ell-3,\ \ell=1,2,\ldots,\text{ and } n \text{ is } \ni \\ &\quad m,n \text{ satisfy the equation } 5m-2^k n=(2^{k+3}-11)/5 \\ &\quad \text{with } m,n \text{ odd and } m,n\geq 1\} \\ &\supset \{2^k(5n+8)\colon k=4\ell-3,\ \ell=1,2,\ldots,\text{ and } n \text{ is } \ni \\ &\quad m=(2^{k+3}-11)m_0/5,\ n=(2^{k+3}-11)n_0/5, \text{ with} \\ &\quad (m_0,n_0,k)\in U\}\,. \end{aligned}$$

Furthermore, we make the following observations on the mapping of C_2 of $5m+2$:

1. After the mapping $C_2(5m+2)=2^k(1)$, $\tilde{C}_2$ maps $2^k(1)$ to the cycle

$$(1,6,3,16,8,4,2).$$

2. After the mapping $C_2(5m+2)=2^k(3)$, $\tilde{C}_2$ maps $2^k(3)$ to the cycle

$$(3,16,8,4,2,1,6).$$

3. After the mapping $C_2(5m+2)=2^k(5n+2)$ *with* $k\geq 3$, an additional sequence of k iterations of $\tilde{C}_2$ has odd numbers of the form $5m+2$ mapped back to smaller odd numbers of the form $5n+2$.

4. After the mapping $C_2(5m+2)=2^k(5n+4)$ *with* $k\geq 6$ (*with* $k=2$), an additional sequence of k iterations of $\tilde{C}_2$ has odd numbers of the form $5m+2$ mapped back to smaller (forward to larger) odd numbers of the form $5n+4$.

5. After the mapping $C_2(5m+2)=2^k(5n+6)$ *with* $k\geq 4$, an additional sequence of k iterations of $\tilde{C}_2$ has odd numbers of the form $5m+2$ mapped back to smaller odd numbers of the form $5n+6$.

6. After the mapping $C_2(5m+2)=2^k(5n+8)$ *with* $k\geq 5$ (*with* $k=1$), an additional sequence of k iterations (an additional $k=1$ iteration) of $\tilde{C}_2$ has odd numbers of the form $5m+2$ mapped back to smaller (forward to larger) odd numbers of the form $5n+8$.

3.3 Characterization of the Set $C_2(B_3)$

Recall that

$$B_3=\{x\in\mathbb{N}\colon x=5m+4,\ \ m=1,3,5,\ldots\}\,.$$

After an analysis in the same spirit as those on the characterizations of $C_1(A_i)$, $i=1,2,3$, in Section 2 and of $C_2(B_2)$ in Subsection 3.1, we obtain

$$C_2(B_3)=M_3\cup N_3\cup V_3\cup W_3\cup X_3\cup Y_3,$$

where each of the sets in the union on the right-hand side is as follows:

$$M_3=\{2^k(1)\colon k=4(5\ell+3),\ \ \ell=0,1,\ldots\}\,;$$

$$N_3 = \left\{2^k(3) \colon k = 20\ell + 5, \;\; \ell = 0, 1, \ldots\right\};$$

$$\begin{aligned}
V_3 \;&=\; \big\{2^k(5n+2) \colon k = 4\ell - 1,\ \ell = 1, 2, \ldots, \text{ and } n \text{ is} \ni \\
&\quad m, n \text{ satisfy the equation } 5m - 2^k n = (2^{k+1} - 21)/5 \\
&\quad \text{with } m, n \text{ odd and } m, n \geq 1\big\} \\
&\supset\; \big\{2^k(5n+2) \colon k = 4\ell - 1,\ \ell = 2, 3, \ldots, \text{ and } n \text{ is} \ni \\
&\quad m = (2^{k+1} - 21)m_0/5,\ n = (2^{k+1} - 21)n_0/5, \text{ with} \\
&\quad (m_0, n_0, k) \in U\big\} \\
&\quad \cup \big\{2^3(5n+2) \colon n \text{ is} \ni m, n \text{ satisfy the equation} \\
&\qquad 5m - 8n = -1, \text{ where } m = 2s+1,\ n = 2t+1, \\
&\qquad (s,t) = (8u+5, 5u+3),\ u = 0, 1, \ldots\big\};
\end{aligned}$$

$$\begin{aligned}
W_3 \;&=\; \big\{2^k(5n+4) \colon k = 4\ell - 2,\ \ell = 1, 2, \ldots, \text{ and } n \text{ is} \ni \\
&\quad m, n \text{ satisfy the equation } 5m - 2^k n = (2^{k+2} - 21)/5 \\
&\quad \text{with } m, n \text{ odd and } m, n \geq 1\big\} \\
&\supset\; \big\{2^k(5n+4) \colon k = 4\ell - 2,\ \ell = 2, 3, \ldots, \text{ and } n \text{ is} \ni \\
&\quad m = (2^{k+2} - 21)m_0/5,\ n = (2^{k+2} - 21)n_0/5, \text{ with} \\
&\quad (m_0, n_0, k) \in U\big\} \\
&\quad \cup \big\{2^2(5n+4) \colon n \text{ is} \ni m, n \text{ satisfy the equation} \\
&\qquad 5m - 4n = -1, \text{ where } m = 2s+1,\ n = 2t+1, \\
&\qquad (s,t) = (4u+3, 5u+4),\ u = 0, 1, \ldots\big\};
\end{aligned}$$

$$\begin{aligned}
X_3 \;&=\; \big\{2^k(5n+6) \colon k = 4\ell,\ \ell = 1, 2, \ldots, \text{ and } n \text{ is} \ni \\
&\quad m, n \text{ satisfy the equation } 5m - 2^k n = (2^{k+1} \cdot 3 - 21)/5 \\
&\quad \text{with } m, n \text{ odd and } m, n \geq 1\big\} \\
&\supset\; \big\{2^k(5n+6) \colon k = 4\ell,\ \ell = 1, 2, \ldots, \text{ and } n \text{ is} \ni \\
&\quad m = (2^{k+1} \cdot 3 - 21)m_0/5,\ n = (2^{k+1} \cdot 3 - 21)n_0/5, \text{ with} \\
&\quad (m_0, n_0, k) \in U\big\};
\end{aligned}$$

$$\begin{aligned}
Y_3 \;&=\; \big\{2^k(5n+8) \colon k = 4\ell - 3,\ \ell = 1, 2, \ldots, \text{ and } n \text{ is} \ni \\
&\quad m, n \text{ satisfy the equation } 5m - 2^k n = (2^{k+3} - 21)/5 \\
&\quad \text{with } m, n \text{ odd and } m, n \geq 1\big\} \\
&\supset\; \big\{2^k(5n+8) \colon k = 4\ell - 3,\ \ell = 2, 3, \ldots, \text{ and } n \text{ is} \ni \\
&\quad m = (2^{k+3} - 21)m_0/5,\ n = (2^{k+3} - 21)n_0/5, \text{ with} \\
&\quad (m_0, n_0, k) \in U\big\} \\
&\quad \cup \big\{2^1(5n+8) \colon n \text{ is} \ni m, n \text{ satisfy the equation} \\
&\qquad 5m - 2n = -1, \text{ where } m = 2s+1,\ n = 2t+1, \\
&\qquad (s,t) = (2u+2, 5u+6),\ u = 0, 1, \ldots\big\}.
\end{aligned}$$

Furthermore, we make the following observations on the mapping of C_2 of $5m+4$:

1. After the mapping $C_2(5m+4) = 2^k(1)$, $\tilde{C}_2$ maps $2^k(1)$ to the cycle

$$(1, 6, 3, 16, 8, 4, 2).$$

2. After the mapping $C_2(5m + 4) = 2^k(3)$, $\tilde{C}_2$ maps $2^k(3)$ to the cycle

$$(3, 16, 8, 4, 2, 1, 6).$$

3. After the mapping $C_2(5m + 4) = 2^k(5n + 2)$ *with* $k \geq 3$, an additional sequence of k iterations of $\tilde{C}_2$ has odd numbers of the form $5m+4$ mapped back to smaller odd numbers of the form $5n + 2$.

4. After the mapping $C_2(5m + 4) = 2^k(5n + 4)$ *with* $k \geq 6$ (*with* $k = 2$), an additional sequence of k iterations of $\tilde{C}_2$ has odd numbers of the form $5m + 4$ mapped back to smaller (forward to larger) odd numbers of the form $5n + 4$.

5. After the mapping $C_2(5m + 4) = 2^k(5n + 6)$ *with* $k \geq 4$, an additional sequence of k iterations of $\tilde{C}_2$ has odd numbers of the form $5m+4$ mapped back to smaller odd numbers of the form $5n + 6$.

6. After the mapping $C_2(5m+4) = 2^k(5n+8)$ *with* $k \geq 5$ (*with* $k = 1$), an additional sequence of k iterations (an additional $k = 1$ iteration) of $\tilde{C}_2$ has odd numbers of the form $5m + 4$ mapped back to smaller (forward to larger) odd numbers of the form $5n + 8$.

3.4 Characterization of the Set $C_2(B_4)$

Recall that

$$B_4 = \{x \in \mathbb{N} \colon x = 5m + 6, \ \ m = 1, 3, 5, \ldots\}.$$

After an analysis in the same spirit as those on the characterizations of $C_1(A_i)$, $i = 1, 2, 3$, in Section 2 and of $C_2(B_2)$ in Subsection 3.1, we obtain

$$C_2(B_4) = M_4 \cup N_4 \cup V_4 \cup W_4 \cup X_4 \cup Y_4,$$

where each of the sets in the union on the right-hand side is as follows:

$$M_4 = \left\{2^k(1) \colon k = 4(5\ell + 2), \ \ \ell = 0, 1, \ldots\right\};$$

$$N_4 = \left\{2^k(3) \colon k = 20\ell + 1, \ \ \ell = 1, 2, \ldots\right\};$$

$$\begin{aligned} V_4 \ = \ & \big\{2^k(5n + 2) \colon k = 4\ell - 1, \ell = 1, 2, \ldots, \text{ and } n \text{ is } \ni \\ & m, n \text{ satisfy the equation } 5m - 2^k n = (2^{k+1} - 31)/5 \\ & \text{with } m, n \text{ odd and } m, n \geq 1\} \\ \supset \ & \big\{2^k(5n + 2) \colon k = 4\ell - 1, \ell = 2, 3, \ldots, \text{ and } n \text{ is } \ni \\ & m = (2^{k+1} - 31)m_0/5, n = (2^{k+1} - 31)n_0/5, \text{ with} \\ & (m_0, n_0, k) \in U\} \\ & \cup \big\{2^3(5n + 2) \colon n \text{ is } \ni m, n \text{ satisfy the equation} \\ & \quad 5m - 8n = -3, \text{ where } m = 2s + 1, n = 2t + 1, \\ & \quad (s, t) = (8u, 5u), u = 1, 2, \ldots\}; \end{aligned}$$

$$\begin{aligned}
W_4 \;=\;& \big\{2^k(5n+4)\colon\ k = 4\ell - 2,\ \ell = 1, 2, \ldots, \text{ and } n \text{ is} \ni \\
& m, n \text{ satisfy the equation } 5m - 2^k n = (2^{k+2} - 31)/5 \\
& \text{with } m, n \text{ odd and } m, n \geq 1\big\} \\
\supset\;& \big\{2^k(5n+4)\colon\ k = 4\ell - 2,\ \ell = 2, 3, \ldots, \text{ and } n \text{ is} \ni \\
& m = (2^{k+2} - 31)m_0/5,\ n = (2^{k+2} - 31)n_0/5, \text{ with} \\
& (m_0, n_0, k) \in U\big\} \\
& \cup \big\{2^2(5n+4)\colon\ n \text{ is} \ni m, n \text{ satisfy the equation} \\
& \quad 5m - 4n = -3, \text{ where } m = 2s + 1,\ n = 2t + 1, \\
& \quad (s, t) = (4u + 2, 5u + 3),\ u = 0, 1, \ldots\big\}\,;
\end{aligned}$$

$$\begin{aligned}
X_4 \;=\;& \big\{2^k(5n+6)\colon\ k = 4\ell,\ \ell = 1, 2, \ldots, \text{ and } n \text{ is} \ni \\
& m, n \text{ satisfy the equation } 5m - 2^k n = (2^{k+1} \cdot 3 - 31)/5 \\
& \text{with } m, n \text{ odd and } m, n \geq 1\big\} \\
\supset\;& \big\{2^k(5n+6)\colon\ k = 4\ell,\ \ell = 1, 2, \ldots, \text{ and } n \text{ is} \ni \\
& m = (2^{k+1} \cdot 3 - 31)m_0/5,\ n = (2^{k+1} \cdot 3 - 31)n_0/5, \text{ with} \\
& (m_0, n_0, k) \in U\big\}\,;
\end{aligned}$$

$$\begin{aligned}
Y_4 \;=\;& \big\{2^k(5n+8)\colon\ k = 4\ell - 3,\ \ell = 1, 2, \ldots, \text{ and } n \text{ is} \ni \\
& m, n \text{ satisfy the equation } 5m - 2^k n = (2^{k+3} - 31)/5 \\
& \text{with } m, n \text{ odd and } m, n \geq 1\big\} \\
\supset\;& \big\{2^k(5n+8)\colon\ k = 4\ell - 3,\ \ell = 2, 3, \ldots, \text{ and } n \text{ is} \ni \\
& m = (2^{k+3} - 31)m_0/5,\ n = (2^{k+3} - 31)n_0/5, \text{ with} \\
& (m_0, n_0, k) \in U\big\} \\
& \cup \big\{2^1(5n+8)\colon\ n \text{ is} \ni m, n \text{ satisfy the equation} \\
& \quad 5m - 2n = -3, \text{ where } m = 2s + 1,\ n = 2t + 1, \\
& \quad (s, t) = (2u + 1, 5u + 4),\ u = 0, 1, \ldots\big\}\,.
\end{aligned}$$

Furthermore, we make the following observations on the mapping of C_2 of $5m + 6$:

1. After the mapping $C_2(5m + 6) = 2^k(1)$, $\tilde{C}_2$ maps $2^k(1)$ to the cycle

$$(1, 6, 3, 16, 8, 4, 2).$$

2. After the mapping $C_2(5m + 6) = 2^k(3)$, $\tilde{C}_2$ maps $2^k(3)$ to the cycle

$$(3, 16, 8, 4, 2, 1, 6).$$

3. After the mapping $C_2(5m + 6) = 2^k(5n + 2)$ *with* $k \geq 3$, an additional sequence of k iterations of $\tilde{C}_2$ has odd numbers of the form $5m + 6$ mapped back to smaller odd numbers of the form $5n + 2$.

4. After the mapping $C_2(5m + 6) = 2^k(5n + 4)$ *with* $k \geq 6$ (*with* $k = 2$), an additional sequence of k iterations of $\tilde{C}_2$ has odd numbers of the form $5m + 6$ mapped back to smaller (forward to larger) odd numbers of the form $5n + 4$.

5. After the mapping $C_2(5m+6) = 2^k(5n+6)$ *with* $k \geq 4$, an additional sequence of k iterations of $\tilde{C}_2$ has odd numbers of the form $5m+6$ mapped back to smaller odd numbers of the form $5n+6$.

6. After the mapping $C_2(5m+6) = 2^k(5n+8)$ *with* $k \geq 5$ (*with* $k=1$), an additional sequence of k iterations (an additional $k=1$ iteration) of $\tilde{C}_2$ has odd numbers of the form $5m+6$ mapped back to smaller (forward to larger) odd numbers of the form $5n+8$.

3.5 Characterization of the Set $C_2(B_5)$

Recall that

$$B_5 = \{x \in \mathbb{N} \colon x = 5m+8,\ \ m = 1,3,5,\ldots\}.$$

After an analysis in the same spirit as those on the characterizations of $C_1(A_i)$, $i = 1,2,3$, in Section 2 and of $C_2(B_2)$ in Subsection 3.1, we obtain

$$C_2(B_5) = M_5 \cup N_5 \cup V_5 \cup W_5 \cup X_5 \cup Y_5,$$

where each of the sets in the union on the right-hand side is as follows:

$$M_5 = \left\{2^k(1) \colon k = 4(5\ell+1),\ \ \ell = 1,2,\ldots\right\};$$

$$N_5 = \left\{2^k(3) \colon k = 20\ell - 3,\ \ \ell = 1,2,\ldots\right\};$$

$$\begin{aligned} V_5 \ = \ & \big\{2^k(5n+2) \colon k = 4\ell-1, \ell = 1,2,\ldots, \text{ and } n \text{ is } \ni \\ & m,n \text{ satisfy the equation } 5m - 2^k n = (2^{k+1}-41)/5 \\ & \text{with } m,n \text{ odd and } m,n \geq 1\} \\ \supset \ & \big\{2^k(5n+2) \colon k = 4\ell-1, \ell = 2,3,\ldots, \text{ and } n \text{ is } \ni \\ & m = (2^{k+1}-41)m_0/5,\ n = (2^{k+1}-41)n_0/5, \text{ with} \\ & (m_0,n_0,k) \in U\} \\ & \cup \big\{2^3(5n+2) \colon n \text{ is } \ni m,n \text{ satisfy the equation} \\ & \quad 5m - 8n = -5, \text{ where } m = 2s+1,\ n = 2t+1, \\ & \quad (s,t) = (8u+3, 5u+2),\ u = 1,2,\ldots\}; \end{aligned}$$

$$\begin{aligned} W_5 \ = \ & \big\{2^k(5n+4) \colon k = 4\ell-2, \ell = 1,2,\ldots, \text{ and } n \text{ is } \ni \\ & m,n \text{ satisfy the equation } 5m - 2^k n = (2^{k+2}-41)/5 \\ & \text{with } m,n \text{ odd and } m,n \geq 1\} \\ \supset \ & \big\{2^k(5n+4) \colon k = 4\ell-2, \ell = 2,3,\ldots, \text{ and } n \text{ is } \ni \\ & m = (2^{k+2}-41)m_0/5,\ n = (2^{k+2}-41)n_0/5, \text{ with} \\ & (m_0,n_0,k) \in U\} \\ & \cup \big\{2^2(5n+4) \colon n \text{ is } \ni m,n \text{ satisfy the equation} \\ & \quad 5m - 4n = -5, \text{ where } m = 2s+1,\ n = 2t+1, \\ & \quad (s,t) = (4u+1, 5u+2),\ u = 0,1,\ldots\}; \end{aligned}$$

$$\begin{aligned} X_5 &= \{2^k(5n+6)\colon k = 4\ell,\ \ell = 1,2,\ldots, \text{ and } n \text{ is } \ni \\ &\quad m,n \text{ satisfy the equation } 5m - 2^k n = (2^{k+1}\cdot 3 - 41)/5 \\ &\quad \text{with } m,n \text{ odd and } m,n \geq 1\} \\ &\supset \{2^k(5n+6)\colon k = 4\ell,\ \ell = 1,2,\ldots, \text{ and } n \text{ is } \ni \\ &\quad m = (2^{k+1}\cdot 3 - 41)m_0/5,\ n = (2^{k+1}\cdot 3 - 41)n_0/5, \text{ with} \\ &\quad (m_0, n_0, k) \in U\}\,; \end{aligned}$$

$$\begin{aligned} Y_5 &= \{2^k(5n+8)\colon k = 4\ell - 3,\ \ell = 1,2,\ldots, \text{ and } n \text{ is } \ni \\ &\quad m,n \text{ satisfy the equation } 5m - 2^k n = (2^{k+3} - 41)/5 \\ &\quad \text{with } m,n \text{ odd and } m,n \geq 1\} \\ &\supset \{2^k(5n+8)\colon k = 4\ell - 3,\ \ell = 2,3,\ldots, \text{ and } n \text{ is } \ni \\ &\quad m = (2^{k+3} - 41)m_0/5,\ n = (2^{k+3} - 41)n_0/5, \text{ with} \\ &\quad (m_0, n_0, k) \in U\} \\ &\quad \cup \{2^1(5n+8)\colon n \text{ is } \ni m,n \text{ satisfy the equation} \\ &\qquad 5m - 2n = -5, \text{ where } m = 2s+1,\ n = 2t+1, \\ &\qquad (s,t) = (2u, 5u+2),\ u = 0,1,\ldots\}\,. \end{aligned}$$

Furthermore, we make the following observations on the mapping of C_2 of $5m+8$:

1. After the mapping $C_2(5m+8) = 2^k(1)$, $\tilde{C}_2$ maps $2^k(1)$ to the cycle
$$(1, 6, 3, 16, 8, 4, 2).$$
2. After the mapping $C_2(5m+8) = 2^k(3)$, $\tilde{C}_2$ maps $2^k(3)$ to the cycle
$$(3, 16, 8, 4, 2, 1, 6).$$
3. After the mapping $C_2(5m+8) = 2^k(5n+2)$ *with* $k \geq 3$, an additional sequence of k iterations of $\tilde{C}_2$ has odd numbers of the form $5m+8$ mapped back to smaller odd numbers of the form $5n+2$.
4. After the mapping $C_2(5m+8) = 2^k(5n+4)$ *with* $k \geq 6$ (*with* $k = 2$), an additional sequence of k iterations of $\tilde{C}_2$ has odd numbers of the form $5m+8$ mapped back to smaller (forward to larger) odd numbers of the form $5n+4$.
5. After the mapping $C_2(5m+8) = 2^k(5n+6)$ *with* $k \geq 4$, an additional sequence of k iterations of $\tilde{C}_2$ has odd numbers of the form $5m+8$ mapped back to smaller odd numbers of the form $5n+6$.
6. After the mapping $C_2(5m+8) = 2^k(5n+8)$ *with* $k \geq 5$ (*with* $k = 1$), an additional sequence of k iterations (an additional $k = 1$ iteration) of $\tilde{C}_2$ has odd numbers of the form $5m+8$ mapped back to smaller (forward to larger) odd numbers of the form $5n+8$.

4 Conclusion

Keeping Definition 1.2 in mind and considering orbits of the maps $\tilde{C}_1$ and $\tilde{C}_2$ as the solutions $\{x_n\}_{n=0}^{\infty}$ of the corresponding difference equations, Eqs.(1.3) and (1.4), respectively, we make observations based on all the previous data on

1. the set definitions $C_1(A_i)$, $i = 1, 2, 3$, in Section 2, and the set definitions $C_2(B_i)$, $i = 1, 2, 3, 4, 5$, in Section 3;

2. the mappings $\tilde{C}_1^k(3m + i) = 1$ and $\tilde{C}_1^k(3m + i) = 3n + j$, for $i = 0, 2, 4,\ j = 2, 4$, in Section 2, and the mappings $\tilde{C}_2^k(5m + i) = 1$, $\tilde{C}_2^k(5m + i) = 3$, and $\tilde{C}_2^k(5m + i) = 5n + j$, for $i = 0, 2, 4, 6, 8,\ j = 2, 4, 6, 8$, in Section 3.

Consider the map C_1 and observe the following:

1. With the mapping $C_1(3m+i) = 2^k(1)$, m odd and $m \geq 1$, $i \in \{0, 2, 4\}$, solutions m to this equation only exist when $k = 2(3\ell + s)$, $s \in \{0, 2, 4\}$, $\ell = 0, 1, \ldots$.

2. With the mapping $C_1(3m+i) = 2^k(3n+j)$, m, n odd and $m, n \geq 1$, $i \in \{0, 2, 4\}$, $j \in \{2, 4\}$, such that $3n + j < 3m + i$, solutions m, n to this equation only exist when either $k = 2\ell - 1$, $\ell = 2, 3, \ldots$, or $k = 2\ell - 2$, $\ell = 2, 3, \ldots$.

Consider the map C_2 and observe the following:

1. With the mapping $C_2(5m + i) = 2^k(1)$, m odd and $m \geq 1$, $i \in \{0, 2, 4, 6, 8\}$, solutions m to this equation only exist when either $k = 4(5\ell + r_1)$, $r_1 \in \{0, 1\}$, $\ell = 1, 2, \ldots$, or $k = 4(5\ell + r_2)$, $r_2 \in \{2, 3, 4\}$, $\ell = 0, 1, \ldots$.

2. With the mapping $C_2(5m + i) = 2^k(3)$, m odd and $m \geq 1$, $i \in \{0, 2, 4, 6, 8\}$, solutions m to this equation only exist when either $k = 20\ell + s_1$, $s_1 \in \{-3, 1\}$, $\ell = 1, 2, \ldots$, or $k = 20\ell + s_2$, $s_2 \in \{5, 9, 13\}$, $\ell = 0, 1, \ldots$.

3. With the mapping $C_2(5m + i) = 2^k(5n + j)$, m, n odd and $m, n \geq 1$, $i \in \{0, 2, 4, 6, 8\}$, $j \in \{2, 4, 6, 8\}$, such that $5n + j < 5m + i$, solutions m, n to this equation only exist when $k \neq 1, 2$ and $k = 4\ell - t$, $t \in \{0, 1, 2, 3\}$, $\ell = 1, 2, \ldots$.

Then note that each of the sets of numbers

$$\{k = 2(3\ell + s),\, s \in \{0, 2, 4\},\, \ell = 0, 1, \ldots\},$$

$$\{k = 2\ell - 1,\, \ell = 2, 3, \ldots\},$$

$$\{k = 2\ell - 2,\, \ell = 2, 3, \ldots\}$$

is much less sparsely distributed among the natural numbers than each of the following corresponding sets of numbers

$$\{k = 4(5\ell + r_1),\, r_1 \in \{0,1\},\, \ell = 1,2,\ldots\},$$

$$\{k = 4(5\ell + r_2),\, r_2 \in \{2,3,4\},\, \ell = 0,1,\ldots\},$$

$$\{k = 20\ell + s_1,\, s_1 \in \{-3,1\},\, \ell = 1,2,\ldots\},$$

$$\{k = 20\ell + s_2,\, s_2 \in \{5,9,13\},\, \ell = 0,1,\ldots\},$$

$$\{k \neq 1,2 \text{ and } k = 4\ell - t,\, t \in \{0,1,2,3\},\, \ell = 1,2,\ldots\}.$$

These observations heuristically imply that, given an arbitrary odd number $2m+1$, $m \geq 0$, and given that $C_1(2m+1) = 2^{k_1}(2n_1+1)$ and $C_2(2m+1) = 2^{k_2}(2n_2+1)$, $k_1, k_2 \geq 1, n_1, n_2 \geq 0$, we have that the likelihood is greater in having $2n_1+1 < 2m+1$ than in having $2n_2+1 < 2m+1$.

Indeed, the table below supports this heuristic assertion, and possesses the following features:

1. The first column consists of a listing of the odd numbers from 1 to 51.

2. The second column has a check mark placed in the row of an odd number, $2m+1$, if $C_1(2m+1) = 2^{k_1}(2n_1+1)$ such that $2n_1+1 < 2m+1$; and there is no check mark if $2m+1 \leq 2n_1+1$.

3. The third column has a check mark placed in the row of an odd number, $2m+1$, if $C_2(2m+1) = 2^{k_2}(2n_2+1)$ such that $2n_2+1 < 2m+1$; and there is no check mark if $2m+1 \leq 2n_2+1$.

Apparently, while the second column has check marks in rows

$$2\ell - 1, \quad \ell = 2,3,\ldots$$

(i.e., every other row), the third column has check marks in rows

$$4\ell - 2, \quad \ell = 1,2,\ldots$$

(i.e., every fourth row)!

$2m + 1$	$C_1(2m + 1) = 2^{k_1}(2n_1 + 1)$ $\ni 2n_1 + 1 < 2m + 1$	$C_2(2m + 1) = 2^{k_2}(2n_2 + 1)$ $\ni 2n_2 + 1 < 2m + 1$
1		
3		$\surd$
5	$\surd$	
7		
9	$\surd$	
11		$\surd$
13	$\surd$	
15		
17	$\surd$	
19		$\surd$
21	$\surd$	
23		
25	$\surd$	
27		$\surd$
29	$\surd$	
31		
33	$\surd$	
35		$\surd$
37	$\surd$	
39		
41	$\surd$	
43		$\surd$
45	$\surd$	
47		
49	$\surd$	
51		$\surd$

Now, the heuristic assertion above, in turn, suggests the following:

Let $2^k(2m + 1) \in \mathbb{N}$ be arbitrarily chosen, where $k, m \geq 0$. Let $\{x_n\}_{n=0}^{\infty}$ be a solution of Eq.(1.3) (corresponding to the map $\tilde{C}_1$) and let $\{y_n\}_{n=0}^{\infty}$ be a solution of Eq.(1.4) (corresponding to the map $\tilde{C}_2$) such that

$$x_0 = y_0 = 2^k(2m + 1).$$

Observe that $x_k = y_k = 2m+1$. Then the subsequent number of times that odd numbers *less than* $2m + 1$ are visited is likely to be greater in $\{x_n\}_{n=0}^{\infty}$ than in $\{y_n\}_{n=0}^{\infty}$.

(We remark that the situation where odd numbers less than $2m + 1$ are visited is not vacuous. For, letting $x_0 = y_0 = 11$, we see that $\{x_n\}_{n=0}^{\infty}$ is eventually periodic as

$$\{x_n\}_{n=0}^{\infty} = 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 4, 2, 1, 4, 2, 1, \ldots,$$

where the odd numbers $1, 5$, which are less than 11, are visited; and we see that $\{y_n\}_{n=0}^{\infty}$ is apparently divergent to $+\infty$ as

$$\begin{aligned}\{y_n\}_{n=0}^{\infty} \quad = \quad & 11, 56, 28, 14, 7, 36, 18, 9, 46, 23, 116, 58, 29, 146, 73, \\ & 366, 183, 916, 458, 229, 1146, 573, 2866, 1433, 7166, \\ & 3583, 17916, 8958, 4479, \ldots,\end{aligned}$$

where the odd numbers $7, 9$, which are less than 11, are visited.)

Note that the greater the number of times that odd numbers less than $2m + 1$ are visited in a solution $\{x_n\}_{n=0}^{\infty}$, the greater the chances are that an odd number is visited more than once in $\{x_n\}_{n=0}^{\infty}$, which, in turn, results in $\{x_n\}_{n=0}^{\infty}$ being *eventually periodic*.

Therefore, we heuristically conclude that, letting $\{x_n\}_{n=0}^{\infty}$ be a solution of Eq.(1.3) and $\{y_n\}_{n=0}^{\infty}$ be a solution of Eq.(1.4) with $x_0 = y_0 \in \mathbb{N}$, the likelihood of having an odd number visited more than once (so that the solution is eventually periodic) is greater with $\{x_n\}_{n=0}^{\infty}$ than with $\{y_n\}_{n=0}^{\infty}$.

We thereby offer simultaneous support of the conjecture of the $3x + 1$ problem (i.e., all orbits are eventually a cycle) and of the conjecture of the $5x + 1$ problem (i.e., almost all orbits diverge to $+\infty$).

References

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[2] J.C. Lagarias. *The Ultimate Challenge: The* $3x + 1$ *Problem*. The American Mathematical Society, Providence, Rhode Island, 2010.