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Oscillatory Solutions of Nonlinear Fractional Difference Equations

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Abstract

In this paper, we study the oscillatory behavior of the fractional difference equation of the form

$$\Delta(\Delta^\alpha x(t))^\gamma + q(t)f(x(t)) = 0, \quad t \in \mathbb{N}_{t_0+1-\alpha},$$

where Δ^α denotes the Riemann left fractional difference operator of order α, $0 < \alpha \leq 1$ and $\gamma > 0$ is a quotient of odd positive integers. We establish some oscillatory criteria for the above equation, using the Riccati transformation and Hardy type inequalities. Examples are provided to illustrate our theoretical results.

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1 Introduction

Fractional difference equations have received considerable attention during the recent years. Fractional calculus finds significant application in the fields of viscoelasticity,

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capacitor theory, electrical circuits, electro-analytical chemistry, tumor growth models, neurology, control theory, statistics and a review on this direction, see [16, 18, 20, 22, 23, 25, 26]. Significant progress has been made in the study of fractional differential equations, see [6, 7, 10, 12, 14, 27, 31]. In contrast, very little progress has been made in theory of fractional difference equations, see [1–5, 8, 9, 11, 15, 17, 21]. In particular, we observe that the oscillation of fractional difference equations has been studied by many authors in recent researches [13,19,24,28,29]. This is one of reasons to study difference equations with fractional order. Strong interest in the fractional difference equation (1.1) is motivated by the fact that it represents a discrete analogue of the following fractional differential equation

$$D_a^{\alpha}x(t) + q(t)f(x(t)) = 0 \text{ for } 0 \leq \alpha < 1, \quad t \in [a, +\infty], \;\; a > 0,$$

where D_a^{α} denotes the Riemann–Liouville differential operator of order α and the above problem was investigated by Wang et al [30]. The objective is to study the oscillatory behavior of the solutions of fractional difference equations of the form

$$\Delta(\Delta^{\alpha}x(t))^{\gamma} + q(t)f(x(t)) = 0 \text{ for } 0 < \alpha \leq 1, \quad t \in \mathbb{N}_{t_0+1-\alpha}. \tag{1.1}$$

Here Δ^{α} denotes the Riemann left fractional difference operator and $\gamma > 0$ is a quotient of odd positive integers.

In the paper, we assume the conditions

(H1) $q(t)$ is a positive sequence and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\dfrac{f(x)}{(x^n)} \geq k$ for a positive constant k, n is a natural number for all $x \neq 0$ and $\left[\dfrac{c}{q(t)}\right]^{\frac{1}{\gamma}} \leq -N$ for $t \geq t_0$ where $c < 0$ and $N > 0$.

(H2) $\dfrac{\Delta x(t)}{\Delta^{\alpha}x(t+1)} \geq M$, $\dfrac{\Delta x(t)}{\Delta^{\alpha}x(t)} \geq M^{\star}$, $t \geq t_0$ for some positive constants M, $M^{\star}$ for all $\Delta^{\alpha}x(t+1) \neq 0$, $\Delta^{\alpha}x(t) \neq 0$ and $\dfrac{(\Delta x(t))^2}{x(t)x(t+1)} \geq J$, $\Delta^2 x(t) \geq T$ for some positive constants J and T.

A solution $x(t)$ of (1.1) is said to be oscillatory if it has no last zero, i.e., if $x(t_1) = 0$, then there exists a $t_2 > t_1$ such that $x(t_2) = 0$. Equation (1.1) itself is said to be oscillatory if every solution of (1.1) is oscillatory. A solution $x(t)$ which is not oscillatory is called nonoscillatory.

2 Preliminaries

In this section, we present some preliminary results from discrete fractional calculus. We will make use of these results, throughout the paper.

Definition 2.1 (See [23]). Let $\nu > 0$. The νth fractional sum f is defined by

$$\Delta^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{\nu-1} f(s),$$

where f is defined for $s \equiv a \mod (1)$, $\Delta^{-\nu} f(t)$ is defined for $t \equiv (a+\nu) \mod (1)$ and $t^{(\nu)} = \dfrac{\Gamma(t+1)}{\Gamma(t-\nu+1)}$. The fractional sum $\Delta^{-\nu} f$ maps functions defined in $\mathbb{N}_a$ to functions defined in $\mathbb{N}_{a+\nu}$.

Definition 2.2 (See [23]). Let $\mu > 0$ and $m-1 < \mu < m$, where m denotes a positive integer, $m = \lceil \mu \rceil$. Set $\nu = m - \mu$. The μth order Riemann left fractional difference is defined as

$$\Delta^{\mu} f(t) = \Delta^{m-\nu} f(t) = \Delta^{m} \Delta^{-\nu} f(t),$$

where $\Delta^{-\nu} f(t)$ is νth fractional sum.

Lemma 2.3 (See [29]). *If*

$$G(t) = \sum_{s=t_0}^{t-1+\alpha} (t-s-1)^{(-\alpha)}\, x(s),$$

then

$$\Delta(G(t)) = \Gamma(1-\alpha) \Delta^{\alpha}(x(t)).$$

Lemma 2.4 (See [17]). *If X and Y are nonnegative, then*

$$mXY^{m-1} - X^m \leq (m-1)Y^m \text{ for } m > 1.$$

3 Main Results

Theorem 3.1. *Suppose that $(H1)$ and $(H2)$ hold and*

$$\sum_{s=t_0}^{\infty} q^{\frac{1}{\gamma}}(s) = \infty.$$

Furthermore, assume that there exists a positive sequence $r(t)$ such that

$$\limsup_{t\to\infty} \sum_{s=t_1}^{t-1} \left(kr(s)q(s) - \frac{(\Delta r_+(s))^2}{4r(s+1)M^{\gamma}} \right) = \infty, \tag{3.1}$$

where $\Delta r_+(s) = \max\{\Delta r(s), 0\}$. Then every solution of (1.1) *is oscillatory.*

Proof. Suppose to the contrary that $x(t)$ is a nonoscillatory solution of (1.1). Without loss of generality, we can assume that $x(t)$ is an eventually positive solution of (1.1). Then there exists $t_1 > t_0$ such that

$$x(t) > 0, G(t) > 0 \text{ and } f(x(t)) > 0 \text{ for } t \geq t_1,$$

where $G(t)$ is defined as in Lemma 2.3. From (1.1) we have

$$\Delta(\Delta^\alpha x(t))^\gamma = -q(t)f(x(t)) < 0 \text{ for } t \geq t_1.$$

Thus $(\Delta^\alpha x(t))^\gamma$ is an eventually non increasing sequence. Next we show that $(\Delta^\alpha x(t))^\gamma$ is eventually positive. Suppose there exists an integer $t_1 > t_0$ such that $(\Delta^\alpha x(t))^\gamma = c < 0$ for $t \geq t_1$, so that

$$(\Delta^\alpha x(t))^\gamma \leq (\Delta^\alpha x(t_1))^\gamma = c < 0,$$

or

$$(\Delta^\alpha x(t))^\gamma \leq c,$$

or

$$\Delta^\alpha x(t) \leq c^{\frac{1}{\gamma}}.$$

Applying Lemma 2.3, we get that

$$\frac{\Delta G(t)}{\Gamma(1-\alpha)} \leq c^{\frac{1}{\gamma}}, \left(\frac{q(t)^{\frac{1}{\gamma}}}{q(t)^{\frac{1}{\gamma}}}\right),$$

or

$$\Delta G(t) \leq \Gamma(1-\alpha)\left(\frac{c^{\frac{1}{\gamma}}}{q(t)^{\frac{1}{\gamma}}}\right) q(t)^{\frac{1}{\gamma}}.$$

Thus

$$\Delta G(t) \leq \Gamma(1-\alpha)\left(\frac{c}{q(t)}\right)^{\frac{1}{\gamma}} q(t)^{\frac{1}{\gamma}},$$

i.e.,

$$\Delta G(t) \leq \Gamma(1-\alpha)(-N)q(t)^{\frac{1}{\gamma}}.$$

Summing both sides of the last inequality from t_1 to $t-1$, we get

$$G(t) \leq G(t_1) + \sum_{s=t_1}^{t-1} \Gamma(1-\alpha)(-N)q(t)^{\frac{1}{\gamma}} \to -\infty \; as \; t \to \infty,$$

which contradicts the fact that $G(t) > 0$. Hence $(\Delta^\alpha x(t))^\gamma$ is eventually positive.

Define the function $w(t)$ by the Riccati substitution

$$w(t) = \frac{r(t)(\Delta^\alpha x(t))^\gamma}{x^\gamma(t)}.$$

Since $r(t) > 0,\ x(t) > 0$ and $(\Delta^\alpha x(t))^\gamma > 0$, we have $w(t) > 0$. Now

$$\begin{aligned}
\Delta w(t) &= \Delta\left[\frac{r(t)(\Delta^\alpha x(t))^\gamma}{x^\gamma(t)}\right] \\
&= r(t)\Delta\left[\frac{(\Delta^\alpha x(t))^\gamma}{x^\gamma(t)}\right] + \frac{(\Delta^\alpha x(t+1))^\gamma}{x^\gamma(t+1)}\Delta r(t) \\
&= r(t)\left[\frac{x^\gamma(t)\Delta(\Delta^\alpha x(t))^\gamma - (\Delta^\alpha x(t))^\gamma \Delta x^\gamma(t)}{x^\gamma(t)x^\gamma(t+1)}\right] + \Delta r(t)\frac{w(t+1)}{r(t+1)} \\
&\le \frac{\Delta r_+(t)}{r(t+1)}w(t+1) + r(t)\left[\frac{-q(t)f(x(t))}{x^\gamma(t+1)}\right] - r(t)\left[\frac{(\Delta^\alpha x(t))^\gamma \Delta x^\gamma(t)}{x^\gamma(t)x^\gamma(t+1)}\right] \\
&\le \frac{\Delta r_+(t)}{r(t+1)}w(t+1) - r(t)q(t)k \\
&- \frac{1}{r(t+1)}\left[\frac{(\Delta^\alpha x(t))^\gamma \Delta x^\gamma(t)}{(\Delta^\alpha x(t))^\gamma(\Delta^\alpha x(t+1))^\gamma}\right]w^2(t+1) \\
&\le \frac{\Delta r_+(t)}{r(t+1)}w(t+1) - r(t)q(t)k - \frac{1}{r(t+1)}\left[\frac{\Delta x(t)}{\Delta^\alpha x(t+1)}\right]^\gamma w^2(t+1) \\
&\le \frac{\Delta r_+(t)}{r(t+1)}w(t+1) - r(t)q(t)k - \frac{M^\gamma}{r(t+1)}w^2(t+1). \qquad (3.2)
\end{aligned}$$

Let $X = \sqrt{\dfrac{M^\gamma}{r(t+1)}}w(t+1)$ and $Y = \dfrac{\Delta r_+(t)}{2\sqrt{r(t+1)M^\gamma}}$. Using Lemma 2.4 and setting $m = 2$, we obtain

$$\begin{aligned}
2\sqrt{\frac{M^\gamma}{r(t+1)}}w(t+1)\frac{\Delta r_+(t)}{2\sqrt{r(t+1)M^\gamma}} - \frac{M^\gamma}{r(t+1)}w^2(t+1) &\le \frac{(\Delta r_+(t))^2}{4r(t+1)M^\gamma} \\
\frac{\Delta r_+(t)}{r(t+1)}w(t+1) - \frac{M^\gamma}{r(t+1)}w^2(t+1) &\le \frac{(\Delta r_+(t))^2}{4r(t+1)M^\gamma},
\end{aligned}$$

which implies that

$$\Delta w(t) \le -kr(t)q(t) + \frac{(\Delta r_+(t))^2}{4r(t+1)M^\gamma}.$$

Summing the above inequality from t_1 to $t-1$, we get

$$\sum_{s=t_1}^{t-1}\Delta w(s) \le \sum_{s=t_1}^{t-1}\left(-kr(s)q(s) + \frac{(\Delta r_+(s))^2}{4r(s+1)M^\gamma}\right),$$

or

$$w(t) - w(t_1) \le \sum_{s=t_1}^{t-1}\left(-kr(s)q(s) + \frac{(\Delta r_+(s))^2}{4r(s+1)M^\gamma}\right),$$

i.e.,

$$\sum_{s=t_1}^{t-1}\left(kr(s)q(s)-\frac{(\Delta r_+(s))^2}{4r(s+1)M^\gamma}\right)\le w(t_1)-w(t)\le w(t_1)<\infty \text{ for } t\ge t_1.$$

Letting $t\to\infty$, we have

$$\limsup_{t\to\infty}\sum_{s=t_1}^{t-1}\left(kr(s)q(s)-\frac{(\Delta r_+(s))^2}{4r(s+1)M^\gamma}\right)\le w(t_1)<\infty,$$

which contradicts (3.1). The proof is complete. □

Theorem 3.2. *Suppose that* $(H1)$ *and* $(H2)$ *hold and*

$$\sum_{s=t_0}^{\infty} q^{\frac{1}{\gamma}}(s)=\infty.$$

Furthermore, assume that there exists a positive sequence $r(t)$, *and a double positive sequence* $H(t,s)$ *such that*

$$\begin{aligned} H(t,t) &= 0 \textit{ for } t\ge t_0,\ H(t,s)>0 \textit{ for } t>s\ge t_0,\\ \Delta_s H(t,s) &= H(t,s+1)-H(t,s)\le 0 \textit{ for } t>s\ge t_0. \end{aligned}$$

If

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\sum_{s=t_0}^{t-1}\left(r(s)q(s)H(t,s)-k^{-1}\frac{h_+^2(t,s)r(s+1)}{4H(t,s)M^\gamma}\right)=\infty, \tag{3.3}$$

where $h_+(t,s)=\Delta_2 H(t,s)+H(t,s)\dfrac{\Delta r_+(s)}{r(s+1)}$ *and* $\Delta r_+(s)=\max\{\Delta r(s),0\}$,
then every solution of equation (1.1) *is oscillatory.*

Proof. Suppose to the contrary that $x(t)$ is a non-oscillatory solution of (1.1). Without loss of generality, we can assume that $x(t)$ is an eventually positive solution of (1.1). Proceeding as in Theorem 3.1, we arrive at equation (3.2). Multiplying (3.2) by $H(t,s)$ and summing from t_1 to $t-1$, we get

$$\begin{aligned} \sum_{s=t_1}^{t-1}(H(t,s)\Delta w(s)) &\le \sum_{s=t_1}^{t-1}\left(\frac{\Delta r_+(s)}{r(s+1)}w(s+1)H(t,s)\right.\\ &- \left. r(s)q(s)kH(t,s)-H(t,s)\frac{M^\gamma}{r(s+1)}w^2(s+1)\right), \end{aligned}$$

or

$$\sum_{s=t_1}^{t-1} r(s)q(s)kH(t,s) \leq -\sum_{s=t_1}^{t-1}(H(t,s)\Delta w(s)) + \sum_{s=t_1}^{t-1}\left(\frac{\Delta r_+(s)}{r(s+1)}w(s+1)H(t,s)\right) - \sum_{s=t_1}^{t-1}\left(H(t,s)\frac{M^\gamma}{r(s+1)}w^2(s+1)\right).$$

Using the summation by parts formula, we have that

$$\begin{aligned} -\sum_{s=t_1}^{t-1}(H(t,s)\Delta w(s)) &= -[H(t,s)w(s)]_{s=t_1}^{t} + \sum_{s=t_1}^{t-1} w(s+1)\Delta_2 H(t,s) \\ &= H(t,t_1)w(t_1) + \sum_{s=t_1}^{t-1} w(s+1)\Delta_2 H(t,s), \end{aligned}$$

which implies that

$$\begin{aligned} & k\sum_{s=t_1}^{t-1} r(s)q(s)H(t,s) \\ \leq \ & H(t,t_1)w(t_1) + \sum_{s=t_1}^{t-1} w(s+1)\Delta_2 H(t,s) \\ + \ & \sum_{s=t_1}^{t-1}\left(\frac{\Delta r_+(s)}{r(s+1)}w(s+1)H(t,s)\right) - \sum_{s=t_1}^{t-1}\left(H(t,s)\frac{M^\gamma}{r(s+1)}w^2(s+1)\right) \\ \leq \ & H(t,t_1)w(t_1) + \sum_{s=t_1}^{t-1}\left(\Delta_2 H(t,s) + \frac{\Delta r_+(s)}{r(s+1)}H(t,s)\right) w(s+1) \\ - \ & \sum_{s=t_1}^{t-1}\left(H(t,s)\frac{M^\gamma}{r(s+1)}w^2(s+1)\right) \\ \leq \ & H(t,t_1)w(t_1) + \sum_{s=t_1}^{t-1} h_+(t,s)w(s+1) - \sum_{s=t_1}^{t-1}\left(H(t,s)\frac{M^\gamma}{r(s+1)}w^2(s+1)\right), \end{aligned}$$

where $h_+(t,s) = \Delta_2 H(t,s) + \dfrac{\Delta r_+(s)}{r(s+1)}H(t,s)$

$$\leq H(t,t_1)w(t_1) + \sum_{s=t_1}^{t-1}\left(h_+(t,s)w(s+1) - H(t,s)\frac{M^\gamma}{r(s+1)}w^2(s+1)\right). \tag{3.4}$$

Set

$$X = \sqrt{H(t,s)\frac{M^\gamma}{r(s+1)}}w(s+1) \text{ and } Y = \frac{h_+(t,s)}{2\sqrt{H(t,s)\dfrac{M^\gamma}{r(s+1)}}}.$$

Using Lemma 2.4 with $m = 2$, we have that

$$
\begin{aligned}
2\sqrt{H(t,s)\frac{M^\gamma}{r(s+1)}}w(s+1)\frac{h_+(t,s)}{2\sqrt{H(t,s)\frac{M^\gamma}{r(s+1)}}}&\\
-H(t,s)\frac{M^\gamma}{r(s+1)}w^2(s+1) &\leq \frac{h_+^2(t,s)r(s+1)}{4H(t,s)M^\gamma},\\
h_+(t,s)w(s+1) - H(t,s)\frac{M^\gamma}{r(s+1)}w^2(s+1) &\leq \frac{h_+^2(t,s)r(s+1)}{4H(t,s)M^\gamma}.
\end{aligned}
$$

From equation (3.4), we have

$$
\Delta_2 H(t,s) \leq 0 \text{ for } t > s \geq t_0,\ 0 < H(t,t_1) \leq H(t,t_0) \text{ for } t > s \geq t_0,
$$

$$
\begin{aligned}
\sum_{s=t_1}^{t-1} r(s)q(s)H(t,s) \leq k^{-1}H(t,t_1)w(t_1) &+ k^{-1}\sum_{s=t_1}^{t-1}\frac{h_+^2(t,s)r(s+1)}{4H(t,s)M^\gamma},\\
\sum_{s=t_1}^{t-1}\left(r(s)q(s)H(t,s) - k^{-1}\frac{h_+^2(t,s)r(s+1)}{4H(t,s)M^\gamma}\right) &\leq k^{-1}H(t,t_1)w(t_1)\\
&\leq k^{-1}H(t,t_0)w(t_1).
\end{aligned}
$$

Since $0 < H(t,s) \leq H(t,t_0)$ for $t > s \geq t_0$ then we have $0 < \dfrac{H(t,s)}{H(t,t_0)} \leq 1$ for $t > s \geq t_0$. Hence it follows that

$$
\begin{aligned}
&\frac{1}{H(t,t_0)}\sum_{s=t_0}^{t-1}\left(r(s)q(s)H(t,s) - k^{-1}\frac{h_+^2(t,s)r(s+1)}{4H(t,s)M^\gamma}\right)\\
\leq\ &\frac{1}{H(t,t_0)}\sum_{s=t_0}^{t_1-1}\left(r(s)q(s)H(t,s) - k^{-1}\frac{h_+^2(t,s)r(s+1)}{4H(t,s)M^\gamma}\right)\\
+\ &\frac{1}{H(t,t_0)}\sum_{s=t_1}^{t-1}\left(r(s)q(s)H(t,s) - k^{-1}\frac{h_+^2(t,s)r(s+1)}{4H(t,s)M^\gamma}\right)\\
\leq\ &\frac{1}{H(t,t_0)}\sum_{s=t_0}^{t_1-1}(r(s)q(s)H(t,s)) + k^{-1}w(t_1)\\
\leq\ &\sum_{s=t_0}^{t_1-1} r(s)q(s) + k^{-1}w(t_1).
\end{aligned}
$$

Letting $t \to \infty$, we have

$$
\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\sum_{s=t_0}^{t-1}\left(r(s)q(s)H(t,s) - k^{-1}\frac{h_+^2(t,s)r(s+1)}{4H(t,s)M^\gamma}\right)
$$

$$\leq \sum_{s=t_0}^{t_1-1} r(s)q(s) + k^{-1}w(t_1) < \infty,$$

which contradicts (3.3). The proof is complete. □

Theorem 3.3. *Suppose that* $(H1)$ *and* $(H2)$ *hold. Furthermore assume that there exists a positive sequence* $r(t)$ *such that*

$$\limsup_{t\to\infty} \sum_{s=t_1}^{t-1} \left[q(s)k + \left[\frac{J}{M^\star}\right]^\gamma - \frac{T}{M^\star} - \Delta r_+(s) \right] = \infty, \tag{3.5}$$

where $\Delta r_+(s) = \max\{\Delta r(s), 0\}$. *Then every solution of* (1.1) *is oscillatory.*

Proof. Suppose to the contrary that $x(t)$ is a nonoscillatory solution of (1.1). Without loss of generality, we can assume that $x(t)$ is an eventually positive solution of (1.1). We proceed as in Theorem 3.1 to get that $(\Delta^\alpha x(t))^\gamma$ is positive. Now define the following function, using Riccati substitution

$$w(t) = \frac{(\Delta^\alpha x(t))^\gamma}{x^\gamma(t)} + \Delta^\alpha x(t) + r(t).$$

Thus

$$\Delta w(t) = \Delta \left[\frac{(\Delta^\alpha x(t))^\gamma}{x^\gamma(t)} + \Delta^\alpha x(t) + r(t) \right],$$

or

$$\Delta w(t) = \Delta \left[\frac{(\Delta^\alpha x(t))^\gamma}{x^\gamma(t)} \right] + \Delta[\Delta^\alpha x(t)] + \Delta r(t).$$

Hence,

$$\begin{aligned}
\Delta w(t) &\leq \frac{x^\gamma(t)\Delta[(\Delta^\alpha x(t))^\gamma] - (\Delta^\alpha x(t))^\gamma \Delta x^\gamma(t)}{x^\gamma(t)x^\gamma(t+1)} + \Delta\left[\frac{\Delta x(t)}{M^\star}\right] + \Delta r(t) \\
&= \frac{-q(t)f(x(t))}{x^\gamma(t+1)} - \frac{(\Delta^\alpha x(t))^\gamma}{x^\gamma(t)} \frac{\Delta x^\gamma(t)}{x^\gamma(t+1)} + \Delta\left[\frac{\Delta x(t)}{M^\star}\right] + \Delta r(t) \\
&\leq -q(t)k - \left[\frac{\Delta^\alpha x(t)}{x(t)}\right]^\gamma \left[\frac{\Delta x(t)}{x(t+1)}\right]^\gamma + \frac{1}{M^\star}\Delta^2 x(t) + \Delta r(t) \\
&\leq -q(t)k - \left[\frac{\Delta x(t)}{M^\star x(t)}\right]^\gamma \left[\frac{\Delta x(t)}{x(t+1)}\right]^\gamma + \frac{T}{M^\star} + \Delta r(t) \\
&= -q(t)k - \frac{1}{M^{\star\gamma}} \left[\frac{(\Delta x(t))^2}{x(t)x(t+1)}\right]^\gamma + \frac{T}{M^\star} + \Delta r(t) \\
&\leq -q(t)k - \frac{J^\gamma}{M^{\star\gamma}} + \frac{T}{M^\star} + \Delta r_+(t) \\
&= -q(t)k - \left[\frac{J}{M^\star}\right]^\gamma + \frac{T}{M^\star} + \Delta r_+(t).
\end{aligned}$$

Summing the above inequality from t_1 to $t-1$, we get

$$w(t) - w(t_1) \leq \sum_{s=t_1}^{t-1} \left[-q(s)k - \left[\frac{J}{M^\star}\right]^\gamma + \frac{T}{M^\star} + \Delta r_+(s) \right],$$

or

$$\sum_{s=t_1}^{t-1} \left[q(s)k + \left[\frac{J}{M^\star}\right]^\gamma - \frac{T}{M^\star} - \Delta r_+(s) \right] \leq w(t_1) - w(t) \leq w(t_1) < \infty.$$

Taking $t \to \infty$ and $\sup$, we get

$$\limsup_{t\to\infty} \sum_{s=t_1}^{t-1} \left[q(s)k + \left[\frac{J}{M^\star}\right]^\gamma - \frac{T}{M^\star} - \Delta r_+(s) \right] \leq w(t_1) < \infty,$$

which contradicts (3.5). The proof is complete. □

4 Examples

Example 4.1. Consider the nonlinear fractional difference equation

$$\Delta(\Delta^{0.5} x(t))^\gamma + t f(x(t)) = 0 \ \text{ for } \ t \in \mathbb{N}_{t_0+0.5}, \tag{4.1}$$

where $\alpha = 0.5,\ q(t) = t$ and $\gamma > 0$ is a quotient of odd positive integers. We apply Theorem 3.3 with $r(t) = \frac{1}{t^{\gamma+1}},\ q(t) = t,\ T = 1,\ M^\star = 1$ and $J = 1$. It is easy to see that $(H1)$ and $(H2)$ hold. Then we have

$$\begin{aligned} \limsup_{t\to\infty} \sum_{s=t_1}^{t-1} \left[q(s)k + \left[\frac{J}{M^\star}\right]^\gamma - \frac{T}{M^\star} - \Delta r_+(s) \right] &= \limsup_{t\to\infty} \sum_{s=t_1}^{t-1} \left[s - \frac{1}{s^{\gamma+1}} \right] \\ &= \limsup_{t\to\infty} \sum_{s=t_1}^{t-1} \frac{1}{s} \left[s^2 - \frac{1}{s^\gamma} \right] \\ &= \limsup_{t\to\infty} \sum_{s=t_1}^{t-1} \frac{1}{s} \left[\frac{s^{\gamma+2} - 1}{s^\gamma} \right] \\ &= \infty, \end{aligned}$$

that is condition (3.5) of Theorem 3.3 is satisfied. Therefore, all solutions of (4.1) are oscillatory.

Example 4.2. Consider the nonlinear fractional difference equation

$$\Delta(\Delta^{0.5} x(t))^\gamma + t f(x(t)) = 0 \ \text{ for } \ t \in \mathbb{N}_{t_0+0.5}, \tag{4.2}$$

where $q(t) = t,\ \alpha = 0.5$, and $\gamma > 0$ is a quotient of odd positive integers. Clearly $\sum_{s=t_0}^{\infty} t = \infty$ and conditions $(H1)$ and $(H2)$ hold. We apply Theorem 3.1 with $r(t) = \frac{1}{t^2},\ k = 1$ and $M = 1$, we obtain

$$\begin{aligned}
\limsup_{t\to\infty} \sum_{s=t_1}^{t-1} \left(kr(s)q(s) - \frac{(\Delta r_+(s))^2}{4r(s+1)M^\gamma} \right) &= \limsup_{t\to\infty} \sum_{s=t_1}^{t-1} \left(\frac{s}{s^2} - \frac{(s+1)^2}{4s^4} \right) \\
&= \limsup_{t\to\infty} \sum_{s=t_1}^{t-1} \frac{1}{s} \left(1 - \frac{(s+1)^2}{4s^3} \right) \\
&= \infty,
\end{aligned}$$

that is, condition (3.1) of Theorem 3.1 is satisfied. Therefore all solutions of (4.2) are oscillatory.

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References

[1] G. A. Anastassiou, *Discrete fractional calculus and inequalities,* http://arxiv.org /abs/0911. 3370v1.

[2] F. M. Atici and P. W. Eloe, A transform method in discrete fractional calculus, *Int. J. Difference Equ.,* 2 (2) (2007) 165–176.

[3] F. M. Atici and P. W. Eloe, *Discrete fractional calculus with the nabla operator, Electron. J. Qual. Theory Differ. Equ.*, 3 (2009) 1–12.

[4] F. M. Atici and P. W. Eloe, Initial value problems in discrete fractional calculus, *Proc. Am. Math. Soc.* 137 (3) (2009) 981–989.

[5] F. Chen, Fixed points and asymptotic stability of nonlinear fractional difference equations, *Electron. J. Qual. Theory Differ. Equ.*, 39 (2001) 1–18.

[6] D. Chen, Oscillatory behavior of a class of Fractional differential equations with damping, *U. Politeh. Buch. Ser. A*, 75 (1) (2013) 107–118.

[7] D. X. Chen, Oscillation criteria of fractional differential equations, *Adv. Differ. Equ.*, 2012 (13) (2012) 18–33.

[8] F. Chen, Z. Liu, Asymptotic Stability Results for Nonlinear Fractional Difference Equations, *J. Appl. Math.*, 2012 (2012) 1–14. doi:10.1155/2012/879657

[9] F. Chen, X. Luo, Y. Zhou, Existence Results for Nonlinear Fractional Difference Equation, *Adv. Differ. Equ.*, 2011 (2011) 1–12. doi:10.1155/2012/879657

[10] D. Chen, P. Qu, Y. Lan, Forced oscillation of certain fractional differential equations, *Adv. Differ. Equ.*, 2013 (125) (2012) 1–10.

[11] J. B. Diaz and T. J. Olser, Differences of Fractional Order, *Math. Comput.*, 28 (125) (1974) 185–202.

[12] K. Diethelm, *The Analysis of Fractional Differential Equations,* Springer, Berlin, 2010.

[13] A. George Maria Selvam, M. Reni Sagayaraj and M. Paul Loganathan, Oscillatory behavior of a class of fractional difference equations with damping, *Int. J. Appl. Math. Res.* , 3 (3) (2014) 220–224.

[14] S. R. Grace, R. P. Agarwal, Patricia J. Y. Wong, A. Zafer, On the Oscillation of Fractional Differential Equations, *Fract. Calc. Appl. Anal.*, 15 (2) (2012) 222–231.

[15] H. Gray and N. Zhang, On a new definition of the fractional difference, *Math. Comput.*, 50 (182) (1988) 513–529.

[16] Z. Han, Y. Zhao, Y. Sun, C. Zhang, Oscillation for a class of fractional differential equation, *Discrete Dyn. Nat. Soc.* 2013 (2013) 1–6.

[17] G. H. Hardy, J. E. Littlewood, G.Polya, *Inequalities,* Cambridge University Press, Cambridge 1959.

[18] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations,* North-Holland Math. Studies 204, Elsevier, Amsterdam, 2006.

[19] W. N. Li, Oscillation results for certain forced fractional difference equations with damping term, *Adv. Differ. Equ.*, 2016 (70) (2016) 1–9.

[20] R. Matusu, Application of fractional order calculus to control theory, *Int. J. Math. Models Methods Appl. Sci.*, 5 (7) (2011) 1162–1169.

[21] H. T. Michael, The Theory of Discrete Fractional Calculus: Development and Application (Spring 4-2011), Dissertations, Theses, and Student Research Papers in Mathematics, Paper 27.

[22] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations,* John Wiley and Sons, New York, USA, 1993.

[23] K. S. Miller and B. Ross, *Fractional Difference Calculus*, Proceedings of the International Symposium on Univalent Functions, Fractional Calculus and their Applications, Nihon University, Japan, 1988.

[24] M. Paul Loganathan, M. Reni Sagayaraj, A. George Maria Selvam, On the oscillation of non-linear fractional difference equations, *Math. Aeterna*, 4 (1) (2014) 91–99.

[25] I. Petras, *Control of Fractional-Order Chuas System,* arXiv:nlin/0008029v1.

[26] I. Podlubny, *Fractional Differential Equations,* Academic Press, USA, 1999.

[27] C. Qi and J. Cheng, Interval oscillation criteria for a class of fractional differential equations with damping term, *Math. Probl. Eng.*,2013 (2013) 1–8.

[28] M. Reni Sagayaraj, A. George Maria Selvam and M. Paul Loganathan, Oscillation criteria for a class of discrete nonlinear fractional equations, *Bull. Soc. Math. Serv. Stand.*, 3 (1) (2014) 27–35.

[29] A. Secer and H. Adiguzel, Oscillation of solutions for a class of nonlinear fractional difference equations, *J. Nonlinear Sci. Appl.*, 9 (2016) 5862–5869.

[30] Y. Wang, Z. Han, P. Zhao and S. Sun, On the oscillation and asymptotic behavior for a kind of fractional differential equations, *Adv. Differ. Equ.*, 2014 (50) (2014) 1–11.

[31] S. Xiang, Z. Han, P. Zhao and Y. Sun, Oscillatory behavior for a class of differential equations with fractional order derivatives, *Abstr. Appl. Anal.*, 2014 (2014) 1–9.