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On the Equilibria of a Four-Parameter Rational Planar System of Difference Equations

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Abstract

In this paper, we consider aspects of equilibrium solutions of a planar system of difference equations defined on the open first quadrant and whose behavior is governed by four independent, nonnegative parameters. This system, indexed as (21,21) in the notation of Ladas (Open problems and conjectures, *J. Differential Equ. Appl.*, 15(3) (2009) pp. 303–323), is one of the open problems listed about which little is known.

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1 Introduction

Consider the system

$$x_{n+1} = \frac{a + bx_n}{y_n}, \tag{1.1}$$

$$y_{n+1} = \frac{c + dy_n}{x_n}, \tag{1.2}$$

where a, b, c, and d are nonnegative parameters, and the initial point (x_0, y_0) belongs to the open first quadrant $\mathbb{Q}_1^o = (0, \infty) \times (0, \infty)$. In [4], this is an open problem, and we will attempt to answer the questions:

a. What are the equilibrium points of the system?

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b. What is the local stability of the system's equilibrium points?

c. Does the system possess any prime period 2 solutions?

A similar analysis was done in [2] for a different system.

We can view the system as the mapping $F(x,y) : \mathbb{Q}_1^o \to \mathbb{Q}_1^o$ defined by $F(x,y) = (f(x,y), g(x,y))$, where

$$f(x,y) = \frac{a+bx}{y}$$
$$g(x,y) = \frac{c+dy}{x}.$$

Let $F^0(x,y) = (x,y)$, $F^1(x,y) = F(x,y)$, and define $F^k(x,y) = F(F^{k-1}(x,y))$. A solution of this system $\{(x_0,y_0),(x_1,y_1),\dots\}$ corresponds to the orbit of (x_0,y_0) under iterations of F:

$$\mathcal{O}_F(x_0,y_0) = \{F^k(x_0,y_0) | k = 0,1,2,\dots\}.$$

2 Basic Properties of $F(x,y)$ for $a,b,c,d > 0$

Lemma 2.1. *F is a continuous mapping of $\mathbb{Q}_1^o$ into $\mathbb{Q} = [0,\infty) \times [0,\infty)$. Moreover, if $\max\{a,b\} > 0$, and $\max\{c,d\} > 0$, then F is a continuous mapping of $\mathbb{Q}_1^o$ into $\mathbb{Q}_1^o$.*

Proof. The expression

$$\frac{\alpha+\beta r}{s}$$

with $\alpha,\beta \geq 0$ is a rational expression in r,s and therefore continuous for $s \neq 0$, which is satisfied by the domain of F. For the second statement, since $\max\{\alpha,\beta\} > 0$ and $r > 0$, the expression $\alpha + \beta r$ must be positive. Therefore, we have $\dfrac{\alpha+\beta r}{s} > 0$. □

Lemma 2.2. *Suppose that a, b, c, and d are all positive. Then*

a. $f(x,y) = x \iff y = \dfrac{a+bx}{x}.$

b. $g(x,y) = y \iff y = \dfrac{c}{x-d}.$

c. *$F(x,y)$ has a unique fixed point in $\mathbb{Q}_1^o$.*

d. *$F^2(x,y)$ has a unique fixed point in $\mathbb{Q}_1^o$.*

Proof. (a) and (b) are immediate. For part (c), to find the fixed point(s), we consider

$$\frac{a+bx}{x} = \frac{c}{x-d}.$$

This gives us the quadratic equation

$$bx^2 + (a - bd - c)x - ad = 0.$$

This equation has one positive and one negative root, the positive root is
$x = \dfrac{-a + bd + c + \sqrt{(a-bd-c)^2 + 4abd}}{2b}$. Hence the equation $F(x, y) = (x, y)$ has exactly one fixed point in $\mathbb{Q}_1^o$.

For part (d), we consider $f^2(x, y) = x$ and $g^2(x, y) = y$. For $f^2(x, y) = x$, we have

$$\frac{a + b\left(\frac{a+bx}{y}\right)}{\frac{c+dy}{x}} = x.$$

This reduces to the parabola

$$x = \frac{d}{b^2}y^2 + \frac{c-a}{b^2}y - \frac{a}{b}.$$

This parabola opens to the right and has one positive and one negative y-intercept. Similarly, $g^2(x, y) = y$ gives the parabola

$$y = \frac{b}{d^2}x^2 + \frac{a-c}{d^2}x - \frac{c}{d},$$

which is a parabola that opens up with one positive and one negative x-intercept. By the geometry of these two parabolas, there is exactly one point of intersection in $\mathbb{Q}_1^o$. $\square$

Since there is a unique fixed point of both F and F^2 in $\mathbb{Q}_1^o$, we will call it (u, v).

3 Properties of the Fixed Point (u, v)

In the proof of Lemma 2.2, we have the x-value of the fixed point, u, satisfies

$$bu^2 + (a - bd - c)u - ad = 0,$$

hence,

$$u = \frac{-a + bd + c + \sqrt{(a-bd-c)^2 + 4abd}}{2b}.$$

Note that (u, v) satisfies the equations

$$u = \frac{a+bu}{v} \quad v = \frac{c+dv}{u}.$$

Therefore, $v = \dfrac{a+bu}{u}$.

Proposition 3.1. *The point (u, v) satisfies $u > d$ and $v > b$.*

Proof. Since u is the x-value of the intersection of the hyperbolae

$$y = \frac{a + bx}{x} \text{ and } y = \frac{c}{x - d}$$

in the first quadrant, and $y = \dfrac{c}{x - d}$ has a vertical asymptote $x = d$, it must be the case that $u > d$. Similarly, $y = \dfrac{a + bx}{x}$ has a horizontal asymptote $y = b$, we have $v > b$. $\square$

Proposition 3.2. *If (u, v) is the fixed point of $F(x, y)$, then*

$$a + bu = c + dv.$$

Proof. We have that u satisfies

$$\begin{aligned}
bu^2 + (a - bd - c)u - ad &= 0 \\
bu^2 + au - bdu - cu - ad &= 0 \\
bu^2 + au &= bdu + cu + ad \\
u(a + bu) &= u\left(c + bd + \frac{ad}{u}\right) \\
a + bu &= c + \frac{bdu + ad}{u} \\
a + bu &= c + d\frac{a + bu}{u} \\
a + bu &= c + dv.
\end{aligned}$$

This concludes the proof. $\square$

Proposition 3.3. *Suppose $0 < \alpha < 1$ and $0 < \beta < 1$. Then*

$$-1 < \frac{\alpha + \beta - \sqrt{(\alpha - \beta)^2 + 4}}{2} < 0 < 1 < \frac{\alpha + \beta + \sqrt{(\alpha - \beta)^2 + 4}}{2}.$$

Proof. First, we show $-1 < \dfrac{\alpha + \beta - \sqrt{(\alpha - \beta)^2 + 4}}{2}$.

$$\begin{aligned}
-1 &< \frac{\alpha + \beta - \sqrt{(\alpha - \beta)^2 + 4}}{2} \\
-2 - \alpha - \beta &< -\sqrt{(\alpha - \beta)^2 + 4} \\
2 + \alpha + \beta &> \sqrt{(\alpha - \beta)^2 + 4} \\
4 + 4\alpha + 4\beta + 2\alpha\beta + \alpha^2 + \beta^2 &> \alpha^2 - 2\alpha\beta + \beta^2 + 4 \\
4\alpha + 4\beta + 4\alpha\beta &> 0.
\end{aligned}$$

Secondly, we show $\dfrac{\alpha+\beta-\sqrt{(\alpha-\beta)^2+4}}{2}<0$.

$$\begin{aligned}\alpha+\beta-\sqrt{(\alpha-\beta)^2+4}&<0\\ \alpha+\beta&<\sqrt{(\alpha-\beta)^2+4}\\ \alpha^2+2\alpha\beta+\beta^2&<\alpha^2-2\alpha\beta+\beta^2+4\\ 4\alpha\beta&<4.\end{aligned}$$

Lastly, consider $1<\dfrac{\alpha+\beta+\sqrt{(\alpha-\beta)^2+4}}{2}$. Since $\sqrt{(\alpha-\beta)^2+4}>2$, we have $\alpha+\beta+\sqrt{(\alpha-\beta)^2+4}>2$, hence $1<\dfrac{\alpha+\beta+\sqrt{(\alpha-\beta)^2+4}}{2}$. □

4 Main Results

Theorem 4.1. *F is linearizable in a neighborhood of (u,v).*

Proof. The Jacobian of the system F is given by

$$J_F(x,y)=\begin{bmatrix}\dfrac{b}{y} & \dfrac{-(a+bx)}{y^2}\\ \dfrac{-(c+dy)}{x^2} & \dfrac{d}{x}\end{bmatrix}.$$

Since each partial derivative is continuous in $\mathbb{Q}_1^o$, it follows that F is linearizable with linearization given by J_F above (c.f. [1, Theorem 20.7]). □

Theorem 4.2. *The fixed point of F is a saddle point with reflection.*

Proof. The Jacobian matrix evaluated at the fixed point is

$$J_F(u,v)=\begin{bmatrix}\dfrac{b}{v} & \dfrac{-(a+bu)}{v^2}\\ \dfrac{-(c+dv)}{u^2} & \dfrac{d}{u}\end{bmatrix}.$$

Thus, the characteristic equation is

$$\lambda^2-\left(\frac{b}{v}+\frac{d}{u}\right)\lambda+\frac{bd}{uv}-\frac{(a+bu)(c+dv)}{u^2v^2}=0.$$

Recall that $v=\dfrac{a+bu}{u}$, hence $u^2v^2=(a+bu)^2$. Also, by Lemma 3.2, we have $c+dv=a+bu$. Thus, this reduces to

$$\lambda^2-\left(\frac{b}{v}+\frac{d}{u}\right)\lambda+\frac{bd}{uv}-1=0.$$

The eigenvalues of $J_F(u,v)$ are

$$\lambda = \frac{\frac{b}{v} + \frac{d}{u} \pm \sqrt{\left(\frac{b}{v} + \frac{d}{u}\right)^2 - 4\left(\frac{bd}{uv} - 1\right)}}{2},$$

which reduces to

$$\lambda = \frac{\frac{b}{v} + \frac{d}{u} \pm \sqrt{\left(\frac{b}{v} - \frac{d}{u}\right)^2 + 4}}{2}.$$

In this case, by Lemmas 3.1 and 3.3, we have $-1 < \lambda_1 < 0 < 1 < \lambda_2$, which gives a saddle point with reflection (see [3, Chapter 4]). □

5 Analysis of Certain Zero Parameter Cases

Now, we will consider the case where two of the four parameters a, b, c and d are zero. First, note if $a = b = 0$, then $F(x,y) = \left(0, \dfrac{c+dy}{x}\right)$, thus $F^2(x,y)$ does not exist. So this case is not possible. Similarly, $c = d = 0$ is not possible.

We will consider the following cases:

i. $a = c = 0$, $b, d > 0$

ii. $b = d = 0$, $a, c > 0$

iii. $a = d = 0$, $b, c > 0$

iv. $b = c = 0$, $a, d > 0$.

First, we consider case i, where $a = c = 0$ and $b, d > 0$. In this case,

$$F(x,y) = \left(\frac{bx}{y}, \frac{dy}{x}\right).$$

Lemma 5.1. *If $a = c = 0$ and $b, d > 0$, then*

1. *F has a single fixed point, (d, b).*
2. *The fixed point (d, b) is unstable.*
3. *The end behavior of $F^k(x,y)$ is*

   (a) If $y < \dfrac{bx}{d}$, then $F^k(x,y) \to (\infty, 0)$ as $k \to \infty$.

   (b) If $y > \dfrac{bx}{d}$, then $F^k(x,y) \to (0, \infty)$ as $k \to \infty$.

(c) *If* $y = \dfrac{bx}{d}$, *then* $F^k(x, y) = (d, b)$ *for all* $k \geq 1$.

Proof. For part (1), solving

$$x = \frac{bx}{y}, \quad y = \frac{dy}{x}$$

gives us $y = b$ and $x = d$, hence, (d, b) is the only fixed point.

For (2), consider the Jacobian evaluated at (d, b):

$$J_F(d, b) = \begin{bmatrix} 1 & \dfrac{-b}{d} \\ \dfrac{-d}{b} & 1 \end{bmatrix}.$$

The characteristic equation is $\lambda^2 - 2\lambda = 0$, thus the eigenvalues are $\lambda = 0$ and $\lambda = 2$. As the spectral radius $r(J) > 1$, we get that the fixed point is unstable.

For part (3), note that

$$F^k(x, y) = \left(d\left(\frac{bx}{dy}\right)^{2^{k-1}}, b\left(\frac{dy}{bx}\right)^{2^{k-1}} \right).$$

Inserting the inequalities in part (3) gives the results. $\square$

Next, we consider case ii, $b = d = 0$ and $a, c > 0$. Here, we have

$$F(x, y) = \left(\frac{a}{y}, \frac{c}{x}\right).$$

There are two cases, $a = c$ and $a \neq c$.

Lemma 5.2. *If* $b = d = 0$ *and* $a = c > 0$, *then*

1. *Every point on the hyperbola* $y = \dfrac{a}{x}$ *is a fixed point for* F.

2. *Every point in* $\mathbb{Q}_1^o$ *is fixed for* F^{2k}.

3. $\mathcal{O}_F(x, y) = \left\{ \left(\dfrac{a}{y}, \dfrac{a}{x}\right), (x, y), \left(\dfrac{a}{y}, \dfrac{a}{x}\right), (x, y), \ldots \right\}$.

Proof. For (1), we need to solve

$$x = \frac{a}{y}, \quad y = \frac{a}{x}.$$

In both cases we get $y = \dfrac{a}{x}$. For part (2), notice

$$F^2(x, y) = \left(\frac{a}{a/x}, \frac{a}{a/y}\right) = (x, y).$$

Finally, combining the facts that $F(x,y) = \left(\frac{a}{y}, \frac{a}{x}\right)$ and $F^2(x,y) = (x,y)$ gives the result in part (3). $\square$

Lemma 5.3. *If $b = d = 0$ and $a \neq c > 0$, then*

1. *F has no fixed points.*
2. *F^2 has no fixed points.*
3. *The end behavior of $F^k(x,y)$ is*
   (a) *If $a < c$, then $F^k(x,y) \to (0,\infty)$ as $k \to \infty$.*
   (b) *If $a > c$, then $F^k(x,y) \to (\infty,0)$ as $k \to \infty$.*

Proof. For (1), we need to solve

$$x = \frac{a}{y}, \quad y = \frac{c}{x}.$$

This gives $y = \frac{a}{x}$ and $y = \frac{c}{x}$. These hyperbolae do not intersect if $a \neq c$.

Similarly,

$$F^2(x,y) = \left(\frac{ax}{c}, \frac{cy}{a}\right),$$

so if we set

$$x = \frac{ax}{c}, \quad y = \frac{cy}{a},$$

we get $1 = \frac{a}{c}$ and $1 = \frac{c}{a}$, neither of which is true. Thus F^2 has no fixed points.

For part (3), notice

$$F^{2k}(x,y) = \left(\left(\frac{a}{c}\right)^k x, \left(\frac{c}{a}\right)^k y\right) \text{ and } F^{2k+1}(x,y) = \left(\frac{a}{y}\left(\frac{a}{c}\right)^k, \frac{c}{x}\left(\frac{c}{a}\right)^k y\right).$$

Thus, if $a < c$, then $F^{2k}(x,y) \to (0,\infty)$ as $k \to \infty$ and $F^{2k+1}(x,y) \to (0,\infty)$ as $k \to \infty$. Hence, if $a < c$, then $F^k(x,y) \to (0,\infty)$ as $k \to \infty$. Similarly, if $a > c$, then $F^k(x,y) \to (\infty,0)$ as $k \to \infty$. $\square$

Lemma 5.4. *If $a = d = 0$ and $b, c > 0$, then*

1. *F has a single fixed point, $\left(\frac{c}{b}, b\right)$.*
2. *The fixed point is a saddle point with reflection.*
3. *$F^k(x,y) = \left(\frac{b^{r_{k+2}-1}x^{r_{k+1}}}{c^{r_{k+1}-1}y^{r_k}}, \frac{c^{r_k}y^{r_{k-1}}}{b^{r_{k+1}-1}x^{r_k}}\right)$, where r_k is the kth entry in the Fibonacci sequence $\{1,1,2,3,5,8,\dots\}$.*

Proof. For part (1), solving

$$x = \frac{bx}{y}, \quad y = \frac{c}{x}$$

gives us $y = b$ and $x = \frac{c}{b}$, thus, the fixed point is $\left(\frac{c}{b}, b\right)$.

For part (2), the Jacobian is

$$J_F\left(\frac{c}{b}, b\right) = \begin{bmatrix} 1 & \frac{-c}{b^2} \\ \frac{-b^2}{c} & 0 \end{bmatrix}.$$

Thus, the eigenvalues satisfy

$$\lambda^2 - \lambda - 1 = 0$$

which gives $\lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2}$, thus $-1 < \lambda_1 < 0 < 1 < \lambda_2$.

We prove part (3) by induction. The base case is $k = 2$. By direct computation,

$$F^2(x, y) = \left(\frac{b^2 x^2}{cy}, \frac{cy}{bx}\right) = \left(\frac{b^{r_4 - 1} x^{r_3}}{c^{r_3 - 1} y^{r_2}}, \frac{c^{r_2} y^{r_1}}{b^{r_3 - 1} x^{r_2}}\right).$$

Now, assume that

$$F^k(x, y) = \left(\frac{b^{r_{k+2} - 1} x^{r_{k+1}}}{c^{r_{k+1} - 1} y^{r_k}}, \frac{c^{r_k} y^{r_{k-1}}}{b^{r_{k+1} - 1} x^{r_k}}\right),$$

and consider $F^{k+1}(x, y)$.

$$F^{k+1}(x, y) = F(F^k(x, y)) = \left(\frac{b\frac{b^{r_{k+2}-1} x^{r_{k+1}}}{c^{r_{k+1}-1} y^{r_k}}}{\frac{c^{r_k} y^{r_{k-1}}}{b^{r_{k+1}-1} x^{r_k}}}, \frac{c}{\frac{b^{r_{k+2}-1} x^{r_{k+1}}}{c^{r_{k+1}-1} y^{r_k}}}\right)$$

$$= \left(\frac{b^{r_{k+1} + r_{k+2} - 1} x^{r_k + r_{k+1}}}{c^{r_k + r_{k+1} - 1} y^{r_{k-1} + r_k}}, \frac{c^{r_{k+1}} y^{r_k}}{b^{r_{k+2} - 1} x^{r_{k+1}}}\right) = \left(\frac{b^{r_{k+3} - 1} x^{r_{k+2}}}{c^{r_{k+2} - 1} y^{r_{k+1}}}, \frac{c^{r_{k+1}} y^{r_k}}{b^{r_{k+2} - 1} x^{r_{k+1}}}\right).$$

This concludes the proof. □

Lemma 5.5. *If $b = c = 0$ and $a, d > 0$, then*

1. *F has a single fixed point, $\left(d, \frac{a}{d}\right)$.*
2. *The fixed point is a saddle point with reflection.*
3. *$F^k(x, y) = \left(\frac{a^{r_k} x^{r_{k-1}}}{d^{r_{k+1} - 1} y^{r_k}}, \frac{d^{r_{k+2} - 1} y^{r_{k+1}}}{a^{r_{k+1} - 1} x^{r_k}}\right)$, where r_k is the kth entry in the Fibonacci sequence $\{1, 1, 2, 3, 5, 8, \dots\}$.*

The proof of this lemma is very similar to the proof of Lemma 5.4.

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