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The Invariant Curve Caused by Neimark–Sacker Bifurcation of a Perturbed Beverton–Holt Difference Equation

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Abstract

We compute the direction of the Neimark–Sacker bifurcation for the difference equation $x_{n+1} = \dfrac{x_n}{Cx_{n-1}^2 + Dx_n + F}$, where C, D and F are positive numbers and the initial conditions x_{-1} and x_0 are nonnegative numbers. Moreover, we give the asymptotic approximation of the invariant curve. We give the necessary and sufficient conditions for global asymptotic stability of the zero equilibrium as well as sufficient conditions for global asymptotic stability of the positive equilibrium.

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1 Introduction and Preliminaries

In this paper, we consider the difference equation

$$x_{n+1} = \frac{x_n}{Cx_{n-1}^2 + Dx_n + F}, \quad n = 0, 1, \ldots, \tag{1.1}$$

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where the parameters C, D and F are positive numbers and the initial conditions x_{-1} and x_0 are positive numbers.

Equation (1.1) can be considered as a nonlinear perturbation of the Beverton–Holt difference equation

$$x_{n+1} = \frac{x_n}{Dx_n + F}, \quad n = 0, 1, \ldots, \tag{1.2}$$

which is a major mathematical model in population dynamics see [1, 13]. Furthermore, it is similar in appearance to the linear fractional equation of the form

$$x_{n+1} = \frac{x_n}{Cx_{n-1} + Dx_n + F}, \quad n = 0, 1, \ldots, \tag{1.3}$$

which was considered in [5]. Both equations (1.2) and (1.3) exhibit a global asymptotic stability of either zero or positive equilibrium solutions and exchange of stability bifurcation. As we will see in this paper the introduction of quadratic term will substantially change dynamics and will introduce the existence of a locally stable periodic solution and possibly chaos. We will show that local asymptotic stability of the zero equilibrium will also implies its global asymptotic stability. In the case of the positive equilibrium solution, we will show that such statement is true in some subspace of the parametric region of local asymptotic stability and we pose the conjecture that the same property holds in the complete region of local asymptotic stability. Our tool in proving global asymptotic stability of the positive equilibrium solution consists of embedding considered equation into higher order equation and using global attractivity results for maps with invariant boxes, see [3, 5, 7]. Related rational difference equations which exhibit similar behavior were considered in [4, 8].

Now, for the sake of completeness, we give the basic facts about the Neimark–Sacker bifurcation.

The Hopf bifurcation is a well-known phenomenon for a system of ordinary differential equations in two or more dimension, whereby, when some parameter is varied, a pair of complex conjugate eigenvalues of the Jacobian matrix at a fixed point crosses the imaginary axis, so that the fixed point changes its behavior from stable to unstable and a limit cycle is generated.

In the discrete setting, the Neimark–Sacker bifurcation is the discrete analogue of the Hopf bifurcation. The Neimark–Sacker bifurcation occurs for a discrete system in the plane depending on a parameter, λ say, with a fixed point whose Jacobian matrix has a pair of complex conjugate eigenvalues $\mu(\lambda)$, $\bar{\mu}(\lambda)$ which crosses the unit circle transversally at $\lambda = \lambda_0$. In this case the periodic solution, which is in general, of unknown period appears and is locally stable. In this paper, we use Murakami computational approach, see [12] to find an asymptotic formula for an invariant locally attracting curve in the phase plane, which represents a periodic solution.

The following result is referred as the Neimark–Sacker bifurcation theorem, see [2, 6, 9, 11, 15].

Theorem 1.1 (Neimark–Sacker bifurcation)**.** *Let*

$$\mathbf{F} : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2; \quad (\lambda, x) \to \mathbf{F}(\lambda, \mathbf{x})$$

be a C^4 map depending on real parameter λ satisfying the following conditions:

(i) $\mathbf{F}(\lambda, \mathbf{0}) = 0$ *for λ near some fixed λ_0;*

(ii) $D\mathbf{F}(\lambda, \mathbf{0})$ *has two nonreal eigenvalues $\mu(\lambda)$ and $\bar{\mu}(\lambda)$ for λ near λ_0 with $|\mu(\lambda_0)| = 1$;*

(iii) $\dfrac{d}{d\lambda}|\mu(\lambda)| = d(\lambda_0) \neq 0$ *at $\lambda = \lambda_0$ (transversality condition);*

(iv) $\mu^k(\lambda_0) \neq 1$ *for $k = 1, 2, 3, 4$ (nonresonance condition).*

Then there is a smooth λ-dependent change of coordinate bringing $\mathbf{F}$ into the form

$$\mathbf{F}(\lambda, \mathbf{x}) = \mathcal{F}(\lambda, \mathbf{x}) + O(\| \mathbf{x} \|^5)$$

and there are smooth functions $a(\lambda)$, $b(\lambda)$, and $\omega(\lambda)$ so that in polar coordinates the function $\mathcal{F}(\lambda, x)$ is given by

$$\mathcal{F} : \begin{pmatrix} r \\ \theta \end{pmatrix} \longrightarrow \begin{pmatrix} |\mu(\lambda)|r + a(\lambda)r^3 \\ \theta + \omega(\lambda) + b(\lambda)r^2 \end{pmatrix}. \tag{1.4}$$

If $a(\lambda_0) < 0$, then there is a neighborhood U of the origin and a $\delta > 0$ such that for $|\lambda - \lambda_0| < \delta$ and $x_0 \in U$, then ω-limit set of x_0 is the origin if $\lambda < \lambda_0$ and belongs to a closed invariant C^1 curve $\Gamma(\lambda)$ encircling the origin if $\lambda > \lambda_0$. Furthermore, $\Gamma(\lambda_0) = 0$. If $a(\lambda_0) > 0$, then there is a neighborhood U of the origin and a $\delta > 0$ such that for $|\lambda - \lambda_0| < \delta$ and $x_0 \in U$, then α-limit set of x_0 is the origin if $\lambda > \lambda_0$ and belongs to a closed invariant C^1 curve $\Gamma(\lambda)$ encircling the origin if $\lambda < \lambda_0$. Furthermore, $\Gamma(\lambda_0) = 0$.

Consider a general map $\mathbf{F}(\lambda_0, \mathbf{x})$ that has a fixed point at the origin with complex eigenvalues $\mu(\lambda_0) = \alpha(\lambda_0) + i\beta(\lambda_0)$ and $\bar{\mu}(\lambda_0) = \alpha(\lambda_0) - i\beta(\lambda_0)$ satisfying $\alpha(\lambda_0)^2 + \beta(\lambda_0)^2 = 1$ and $\beta(\lambda_0) \neq 0$. Assume that

$$\mathbf{F}(\lambda_0, \mathbf{x}) = \mathbf{A}(\lambda_0)\mathbf{x} + \mathbf{G}(\lambda_0, \mathbf{x}), \tag{1.5}$$

where $\mathbf{A}$ is the Jacobian matrix of $\mathbf{F}$ evaluated at the fixed point $(0, 0)$, and

$$\mathbf{G}(\lambda_0, \mathbf{x}) := \begin{pmatrix} g_1(\lambda_0, x_1, x_2) \\ g_2(\lambda_0, x_1, x_2) \end{pmatrix}.$$

Here we denote $\mu(\lambda_0) = \mu$, $\mathbf{A}(\lambda_0) = \mathbf{A}$ and $\mathbf{G}(\lambda_0, x) = \mathbf{G}(\mathbf{x})$. We let $\mathbf{p}$ and $\mathbf{q}$ be the eigenvectors of A associated with μ satisfying

$$\mathbf{A}\mathbf{q} = \mu\mathbf{q}, \quad \mathbf{p}A = \mu\mathbf{p}, \quad \mathbf{p}\mathbf{q} = 1$$

and $\mathbf{\Phi} = (\mathbf{q}, \bar{\mathbf{q}})$. Assume that

$$\mathbf{G}\left(\mathbf{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix}\right) = \frac{1}{2}(\mathbf{g}_{20}z^2 + 2\mathbf{g}_{11}z\bar{z} + \mathbf{g}_{02}\bar{z}^2) + O(|z|^3)$$

and

$$\begin{array}{l} \mathbf{K}_{20} = (\mu^2 I - A)^{-1}\mathbf{g}_{20} \\ \mathbf{K}_{11} = (I - A)^{-1}\mathbf{g}_{11} \\ \mathbf{K}_{02} = (\bar{\mu}^2 I - A)^{-1}\mathbf{g}_{02} \end{array} . \tag{1.6}$$

Let

$$\begin{aligned} \mathbf{G}\left(\mathbf{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix} + \frac{1}{2}(\mathbf{K}_{20}z^2 + 2\mathbf{K}_{11}z\bar{z} + \mathbf{K}_{02}\bar{z}^2)\right) \\ = \frac{1}{2}(\mathbf{g}_{20}z^2 + 2\mathbf{g}_{11}z\bar{z} + \mathbf{g}_{02}\bar{z}^2) \\ + \frac{1}{6}(\mathbf{g}_{30}z^3 + 3\mathbf{g}_{21}z^2\bar{z} + 3\mathbf{g}_{12}z\bar{z}^2 + \mathbf{g}_{03}\bar{z}^3) + O(|z|^4), \end{aligned} \tag{1.7}$$

then

$$a(\lambda_0) = \frac{1}{2}\operatorname{Re}(\mathbf{p}\mathbf{g}_{21}\bar{\mu}).$$

The next result of Murakami [12] gives an approximate formula for the periodic solution.

Corollary 1.2. *Assume $a(\lambda_0) \neq 0$ and $\lambda = \lambda_0 + \eta$, where η is a sufficient small parameter. If $\bar{\mathbf{x}}$ is a fixed point of F, then the invariant curve $\Gamma(\lambda)$ from Theorem 1.1 can be approximated by*

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \approx \bar{\mathbf{x}} + 2\rho_0 \operatorname{Re}\left(\mathbf{q}e^{i\theta}\right) + \rho_0^2\left(\operatorname{Re}\left(\mathbf{K}_{20}e^{2i\theta}\right) + \mathbf{K}_{11}\right),$$

where

$$d = \frac{d}{d\eta}|\mu(\lambda)|\Big|_{\lambda=\lambda_0}, \quad \rho_0 = \sqrt{-\frac{d}{a}\eta}, \quad \theta \in \mathbb{R}.$$

Here "Re" represents the real parts of those complex numbers.

2 Local and Global Stability

The equilibrium solutions of (1.1) is the positive solution of the equation $C\bar{x}^2 + D\bar{x} + F - 1 = 0$, that is

$$\bar{x} = \frac{\sqrt{D^2 + 4C(1-F)} - D}{2C}, \quad 0 < F < 1$$

and the origin $\bar{x}_0 = 0$. The linearized equation associated with (1.1) about the equilibrium point $\bar{x}$ is

$$z_{n+1} = pz_n + qz_{n-1},$$

where

$$p = f_u(\bar{x}, \bar{x}) \text{ and } q = f_v(\bar{x}, \bar{x}).$$

Now the following results hold.

Lemma 2.1. *For the equilibrium point $\bar{x}_0$ the following holds:*

(i) If $F > 1$ the equilibrium point $\bar{x}_0$ is locally asymptotically stable.

(ii) If $F < 1$ the equilibrium point $\bar{x}_0$ is a saddle point.

(ii) If $F = 1$ the equilibrium point $\bar{x}_0$ is nonhyperbolic.

(iv) If $F \geq 1$ the equilibrium point $\bar{x}_0$ is globally asymptotically stable.

The proof of part (iv) follows from the fact that every solution $\{x_n\}$ of (1.1) satisfies

$$x_{n+1} = \frac{x_n}{Cx_{n-1}^2 + Dx_n + F} \leq x_n, \quad n = 0, 1, \ldots$$

which shows that $\{x_n\}$ is nonincreasing sequence and so convergent. Consequently $\lim_{n\to\infty} x_n = 0$. The proofs of parts $(i) - (iii)$ are immediate.

Lemma 2.2. *The positive equilibrium $\bar{x}$ satisfies the following:*

(i) If $F < \frac{1}{2}$ and $C < \frac{2D^2}{(1-2F)^2}$ or $\frac{1}{2} \leq F < 1$, the equilibrium point $\bar{x}$ is locally asymptotically stable.

(ii) If $F < \frac{1}{2}$ and $C > \frac{2D^2}{(1-2F)^2}$, the equilibrium point $\bar{x}$ is a repeller.

(iii) If $F < \frac{1}{2}$ and $C = \frac{2D^2}{(1-2F)^2}$, the equilibrium point $\bar{x}$ is nonhyperbolic.

Proof. One can see that

$$p = f_u(\bar{x}, \bar{x}) = \frac{-D\sqrt{4C(1-F)+D^2} + 2C + D^2}{2C},$$

and

$$q = f_v(\bar{x}, \bar{x}) = -\frac{\left(D - \sqrt{4C(1-F)+D^2}\right)^2}{2C} < 0,$$

$$q-p-1=\frac{3D\left(\sqrt{4C(1-F)+D^2}-D\right)+4C(F-2)}{2C}<0,$$

$$q+p-1=\frac{D\left(\sqrt{4C(1-F)+D^2}-D\right)+4C(F-1)}{2C}<0,$$

$$q+1=\frac{D\sqrt{4C(1-F)+D^2}+2CF-C-D^2}{C}.$$

The rest of the proof follows from [6, Theorem 2.13]. □

Now we give a global asymptotic stability result for the positive equilibrium solution. We will show that local asymptotic stability of the positive equilibrium will also imply its global asymptotic stability in substantial subregion of the parametric space.

Theorem 2.3. *Assume that* $F<1$ *and*

$$C\leq\frac{3D^2}{4(1-F)}. \tag{2.1}$$

Then the positive equilibrium of (1.1) *is globally asymptotically stable.*

Proof. Clearly we can consider solutions of (1.1) which are positive, that is for which $x_0>0$. The substitution $y_n=\dfrac{D}{x_n}$ transforms (1.1) into the equation

$$y_{n+1}=1+\left(F+\frac{C}{D^2y_{n-1}^2}\right)y_n=f(x_n,x_{n-1}),\quad n=0,1,\ldots. \tag{2.2}$$

One can easily show that (2.2) has a unique equilibrium $\bar{y}=\dfrac{D}{\bar{x}}$. We will show that $\bar{y}$ is globally asymptotically stable when $F<1$ and $C\leq\dfrac{3D^2}{4(1-F)}$. Our major tool is global asymptotic stability result in [5], more precisely [5, Theorem 1.4.5]. Now we will check the assumptions of this theorem.

1. Clearly $f(x,y)$ is nondecreasing in x and nonincreasing in y.

2. There exists an interval I such that $f:I\times I\to I$. Indeed $I=\left[\dfrac{1}{1-F}\ ,\ U\right]$, where $U\geq\dfrac{D^2}{(1-F)(D^2-C(1-F))}$. If $x,y\in I$, then

$$f(x,y)=1+\left(F+\frac{C}{D^2y^2}\right)x\geq 1+Fx\geq 1+\frac{F}{1-F}=\frac{1}{1-F}.$$

On the other hand, for any $U \geq \dfrac{D^2}{(1-F)(D^2 - C(1-F))}$, we have

$$f\left(U, \frac{1}{1-F}\right) \leq U.$$

Therefore $f(x, y) \in I$, which shows that I is an invariant interval for f.

Next, consider the system of equations

$$\begin{cases} f(M, m) & = M \\ f(m, M) & = m \end{cases} \Leftrightarrow \begin{cases} M & = 1 + \left(F + \dfrac{C}{D^2 m^2}\right) M \\ m & = 1 + \left(F + \dfrac{C}{D^2 M^2}\right) m, \end{cases}$$

which is equivalent to :

$$\begin{cases} Mm^2(1-F)D^2 - CM & = m^2 D^2 \\ mM^2(1-F)D^2 - Cm & = M^2 D^2, \end{cases}$$

and show that $M = m$. Subtracting the second equation from the first we get:

$$(1-F)D^2(m - M)Mm + C(m - M) = D^2(m - M)(m + M).$$

If $M \neq m$, then we have

$$Mm(1-F)D^2 + C = D^2(M + m) \Leftrightarrow M = \frac{D^2 m - C}{(m(1-F) - 1)D^2},$$

which implies

$$\frac{D^2 m - C}{(m(1-F) - 1)D^2} = \frac{1}{1 - \left(F + \frac{C}{D^2 m^2}\right)}.$$

Thus m satisfies the quadratic equation

$$D^2((1-F)C - D^2)m^2 + CD^2 m - C^2 = 0,$$

with discriminant $\Delta = C^2 D^2 (4C(1-F) - 3D^2)$. Clearly for $C < \dfrac{3D^2}{4(1-F)}$ there are no real solutions and for $C = \dfrac{3D^2}{4(1-F)}$ there is the unique solution $m = \dfrac{2c}{D^2} = \bar{y}$.

Consequently, all conditions of [5, Theorem 1.4.5] are satisfied and every solution of (2.2) which enters the interval I must converge to the unique equilibrium $\bar{y}$.

To show that $\bar{y}$ is globally asymptotically stable, it is sufficient to show that every solution of (2.2) must enter I. Observe that by (2.2)

$$y_{n+1} \geq 1 + \left(F + \frac{C}{D^2 U^2}\right) y_n, \quad n = 0, 1, \ldots$$

and so by the result on difference inequalities, see [10]

$$y_n \geq \frac{1}{1-A} - \varepsilon, \quad A = F + \frac{C}{D^2 U^2}, \quad \varepsilon > 0.$$

Since U can be chosen to be large, there exists $N \geq 0$ such that $y_n \geq \dfrac{1}{1-F}$ for all $n \geq N$. Furthermore, as we can choose $U \geq \dfrac{D^2}{(1-F)(D^2 - C(1-F))}$ to be as large as we wish, we can conclude that every solution of (2.2) must enter and remain in I.

Thus we conclude that the unique positive equilibrium $\bar{x}$ of (1.1) is globally asymptotically stable for $C \leq \dfrac{3D^2}{4(1-F)}$.

□

Based on our simulation, we state the following.

Conjecture 2.4. The equilibrium point $\bar{x}$ of equation (1.1) is globally asymptotically stable if it is locally asymptotically stable.

3 Reduction to Normal Form

In this section, we bring the system that corresponds to (1.1) to the normal form which can be used for computation of relevant coefficients of Neimark–Sacker bifurcation.

Assume that $0 < F < \dfrac{1}{2}$. If we make a change of variable $y_n = x_n - \bar{x}$, then the transformed equation is given by

$$y_{n+1} = \frac{\bar{x} + y_n}{C\left(\bar{x} + y_{n-1}\right)^2 + D\left(\bar{x} + y_n\right) + F} - \bar{x}, \quad n = 0, 1, \ldots. \tag{3.1}$$

Set

$$u_n = y_{n-1} \text{ and } v_n = y_n \text{ for } n = 0, 1, \ldots$$

and write (1.1) in the equivalent form

$$\begin{aligned} u_{n+1} &= v_n \\ v_{n+1} &= \frac{\bar{x} + v_n}{C\left(\bar{x} + u_n\right)^2 + D\left(\bar{x} + v_n\right) + F} - \bar{x}. \end{aligned} \tag{3.2}$$

Let $\mathbf{F}$ be the corresponding map defined by:

$$\mathbf{F}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ \dfrac{\bar{x} + v}{C\left(\bar{x} + u\right)^2 + D\left(\bar{x} + v\right) + F} - \bar{x} \end{pmatrix}. \tag{3.3}$$

Then $\mathbf{F}$ has the unique fixed point $(0,0)$ and the Jacobian matrix of $\mathbf{F}$ at $(0,0)$ is given by

$$\mathrm{Jac}_{\mathbf{F}}(0,0) = \begin{pmatrix} 0 & 1 \\ -\dfrac{2C\bar{x}^2}{\left(C\bar{x}^2 + D\bar{x} + F\right)^2} & \dfrac{C\bar{x}^2 + F}{\left(C\bar{x}^2 + D\bar{x} + F\right)^2} \end{pmatrix}.$$

A straightforward calculation shows that

$$\mathbf{F}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\dfrac{2C\bar{x}^2}{\left(C\bar{x}^2 + D\bar{x} + F\right)^2} & \dfrac{C\bar{x}^2 + F}{\left(C\bar{x}^2 + D\bar{x} + F\right)^2} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \mathbf{F}_1\begin{pmatrix} u \\ v \end{pmatrix}, \quad (3.4)$$

where

$$\Delta_1 = \left(F + \bar{x}\left(D + C\bar{x}\right)\right)^2$$

and

$$\mathbf{F}_1\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{v+\bar{x}}{C\left(u+\bar{x}\right)^2 + F + D\left(v+\bar{x}\right)} + \dfrac{\bar{x}\left(2Cu\bar{x} - \Delta_1\right) - v\left(C\bar{x}^2 + F\right)}{\Delta_1} \end{pmatrix}.$$

The eigenvalues of $\mathrm{Jac}_{\mathbf{F}}(0,0)$ are $\mu(C)$ and $\overline{\mu(C)}$, where

$$\mu(C) = \frac{\sqrt{2}\sqrt{\Delta} + 2C + D^2 - D\sqrt{4C(1-F) + D^2}}{4C},$$

where

$$\Delta = 2C^2(8F-7) - 2CD^2(F+2) + D^4 + (6CD - D^3)\sqrt{4C(1-F) + D^2}.$$

One can prove that for $C = C_0 = \dfrac{2D^2}{(1-2F)^2}$, we obtain $|\mu(C_0)| = 1$ and

$$\begin{aligned}
\mu(C_0) &= \frac{1}{4}\left(2F + 1 + i\sqrt{(3-2F)(2F+5)}\right), \\
\mu^2(C_0) &= \frac{1}{8}\left(4F^2 + 4F - 7\right) + \frac{1}{8}i\sqrt{(3-2F)(2F+5)}(2F+1), \\
\mu^3(C_0) &= \frac{1}{16}\left(8F^3 + 12F^2 - 18F - 11\right) \\
&\quad + \frac{1}{16}i\sqrt{(3-2F)(2F+5)}\left(4F^2 + 4F - 3\right), \\
\mu^4(C_0) &= \frac{F^4}{2} + F^3 - \frac{5F^2}{4} \\
&\quad + \frac{1}{32}i\sqrt{(3-2F)(2F+5)}\left(8F^3 + 12F^2 - 10F - 7\right) - \frac{7F}{4} + \frac{17}{32}.
\end{aligned}$$

One can see that $\mu^k(C_0) \neq 1$ for $k = 1, 2, 3, 4$ and

$$|\mu(C)|^2 = \frac{-D\sqrt{4C(1-F)+D^2} - 2CF + 2C + D^2}{C}.$$

Furthermore, we get

$$\frac{d}{dC}|\mu(C)| = \frac{D\sqrt{\frac{-D\sqrt{4C(1-F)+D^2}-2CF+2C+D^2}{C}}}{2C\sqrt{4C(1-F)+D^2}},$$

and

$$\left.\frac{d|\mu(C)|}{dC}\right|_{C=C_0} = \frac{(2F-1)^3}{4D^2(2F-3)} > 0.$$

The eigenvectors corresponding to $\mu(C)$ and $\overline{\mu(C)}$ are $\mathbf{q}(C)$ and $\overline{\mathbf{q}(C)}$, where

$$\mathbf{q} = \mathbf{q}(C_0) = \left(\frac{1}{4}\left(2F - i\sqrt{(3-2F)(2F+5)} + 1\right), 1\right)^T.$$

Substituting $C = C_0$ into (3.4), we get

$$\mathbf{F}\begin{pmatrix} u \\ v \end{pmatrix} = A\begin{pmatrix} u \\ v \end{pmatrix} + \mathbf{G}\begin{pmatrix} u \\ v \end{pmatrix}, \tag{3.5}$$

where

$$\mathbf{A} = \left.\mathrm{Jac}_{\mathbf{F}}(0,0)\right|_{C=C_0} = \begin{pmatrix} 0 & 1 \\ -1 & F + \frac{1}{2} \end{pmatrix}$$

and

$$\Delta_2 = \frac{(2F-1)\left(2u(Du - 2F + 1) + (4F^2-1)\,v\right)}{2\left(4F^2(Dv+1) - 4F(D(u+v)+1) + D(2u(Du+1)+v) + 1\right)}$$

and

$$\mathbf{G}\begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} 0 \\ \Delta_2 - \left(F + \frac{1}{2}\right)v + u \end{pmatrix}.$$

Hence, for $C = C_0$, (3.2) is equivalent to

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \mathbf{A}\begin{pmatrix} u_n \\ v_n \end{pmatrix} + \mathbf{G}\begin{pmatrix} u_n \\ v_n \end{pmatrix}. \tag{3.6}$$

Define the basis of $\mathbb{R}^2$ by $\mathbf{\Phi} = (\mathbf{q}, \bar{\mathbf{q}})$. Let

$$\begin{aligned}\begin{pmatrix} u \\ v \end{pmatrix} &= \mathbf{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix} = (\mathbf{q}z + \bar{\mathbf{q}}\bar{z}) \\ &= \begin{pmatrix} \frac{1}{4}\left(2Fz + i\Delta_3(\bar{z} - z) + 2F\bar{z} + z + \bar{z}\right) \\ \bar{z} + z \end{pmatrix}.\end{aligned}$$

Let

$$\Delta_3 = \sqrt{(3-2F)(2F+5)}.$$

By using this, one can see that

$$\begin{aligned}
\mathbf{g}_{20} &= \frac{\partial^2}{\partial z^2}\mathbf{G}\left(\mathbf{\Phi}\begin{pmatrix} z \\ \bar{z}\end{pmatrix}\right)\Big|_{z=0} = \begin{pmatrix} 0 \\ \dfrac{D\left(2F\left(-2F-3i\Delta_3+2\right)+i\Delta_3+11\right)}{8F-4}\end{pmatrix}, \\
\mathbf{g}_{11} &= \frac{\partial^2}{\partial z\partial \bar{z}}\mathbf{G}\left(\mathbf{\Phi}\begin{pmatrix} z \\ \bar{z}\end{pmatrix}\right)\Big|_{z=0} = \begin{pmatrix} 0 \\ D\end{pmatrix}, \\
\mathbf{g}_{02} &= \frac{\partial^2}{\partial \bar{z}^2}\mathbf{G}\left(\mathbf{\Phi}\begin{pmatrix} z \\ \bar{z}\end{pmatrix}\right)\Big|_{z=0} = \begin{pmatrix} 0 \\ \dfrac{D\left(2F\left(-2F+3i\Delta_3+2\right)-i\Delta_3+11\right)}{8F-4}\end{pmatrix}
\end{aligned} . \tag{3.7}$$

and

$$\begin{aligned}
\mathbf{K}_{20} &= (\mu^2 I - A)^{-1}\mathbf{g}_{20} \\
&= \begin{pmatrix} -\dfrac{8\left(-4iDF^2+6DF\Delta_3-D\Delta_3+4iDF+11iD\right)}{(2F-1)\left(4F^2-9\right)\left(-4iF^2+2\Delta_3F+\Delta_3-4iF+7i\right)} \\ \dfrac{\left(\Delta_3-2iF-i\right)^2\left(-4iDF^2+6DF\Delta_3-D\Delta_3+4iDF+11iD\right)}{2(2F-1)\left(4F^2-9\right)\left(-4iF^2+2F\Delta_3+\Delta_3-4iF+7i\right)}\end{pmatrix}, \\
\mathbf{K}_{11} &= (I-A)^{-1}\mathbf{g}_{11} = \begin{pmatrix} \dfrac{2D}{3-2F} \\ \dfrac{2D}{3-2F}\end{pmatrix}, \\
\mathbf{K}_{02} &= (\bar{\mu}^2 I - A)^{-1}\mathbf{g}_{02} = \overline{\mathbf{K}_{20}}.
\end{aligned} \tag{3.8}$$

By using $\mathbf{K}_{20}$, $\mathbf{K}_{11}$ and $\mathbf{K}_{02}$, we have that

$$\begin{aligned}
\mathbf{g}_{21} &= \frac{\partial^3}{\partial z^2\partial \bar{z}}\mathbf{G}\left(\mathbf{\Phi}\begin{pmatrix} z \\ \bar{z}\end{pmatrix}+\frac{1}{2}\mathbf{K}_{20}z^2+\mathbf{K}_{11}z\bar{z}+\frac{1}{2}\mathbf{K}_{02}\bar{z}^2\right)\Big|_{z=0} \\
&= \begin{pmatrix} 0 \\ \dfrac{4D^2\left(2F\left(6F+i\Delta_3-16\right)+3i\Delta_3+1\right)}{(1-2F)^2\left(4F^2-9\right)}\end{pmatrix}.
\end{aligned} \tag{3.9}$$

Next we have that $\mathbf{pA} = \mu\mathbf{p}$ and $\mathbf{pq} = 1$, where

$$\mathbf{p} = \left(\frac{2i}{\Delta_3}, -\frac{(3-2F)(2F+5)-i(2F+1)\Delta_3}{2\left(4F^2+4F-15\right)}\right).$$

One can see that

$$a(C_0) = \frac{1}{2}\operatorname{Re}(\mathbf{p}\mathbf{g}_{21}\bar{\mu}) = \frac{4D^2}{(1-2F)^2(2F-3)} < 0.$$

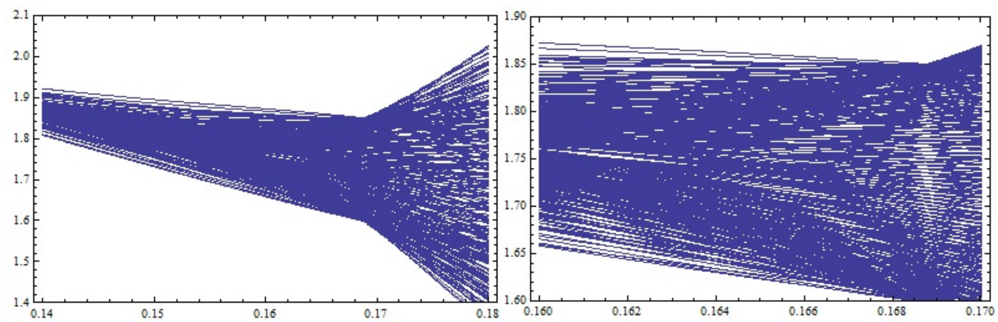

Figure 3.1: Bifurcation diagrams in (C, x) plane for $D = 0.11$, $F = 0.31$ and $C = 0.168$.

Theorem 3.1. *Let* $0 < F < \dfrac{1}{2}$ *and*

$$\bar{x} = \frac{\sqrt{D^2 + 4C(1-F)} - D}{2C}.$$

Then there is a neighborhood U of the equilibrium point $\bar{x}$ and a $\rho > 0$ such that for

$$\left| C - \frac{2D^2}{(1-2F)^2} \right| < \rho$$

and $x_0, x_{-1} \in U$, the ω-limit set of solution of (1.1)*, with initial condition x_0, x_{-1} is equilibrium point $\bar{x}$ if*

$$C < \frac{2D^2}{(1-2F)^2}$$

and belongs to a closed invariant C^1 curve $\Gamma(C)$ encircling the equilibrium point $\bar{x}$ if

$$C > \frac{2D^2}{(1-2F)^2}.$$

Furthermore, $\Gamma(C_0) = 0$ and invariant curve $\Gamma(C)$ can be approximated by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \approx \begin{pmatrix} \bar{x} \\ \bar{x} \end{pmatrix} + 2\rho_0 \operatorname{Re}\left(\mathbf{q} e^{i\theta}\right) + \rho_0^2 \left(\operatorname{Re}\left(\mathbf{K}_{20} e^{2i\theta}\right) + \mathbf{K}_{11} \right),$$

where

$$\rho_0 = \frac{(1-2F)^{3/2} \sqrt{C(1-2F)^2 - 2D^2}}{4D^2}.$$

Proof. The proof follows from above discussion and Theorem 1.1 and Corollary 1.2. See Figure 3.2 for a graphical illustration. □

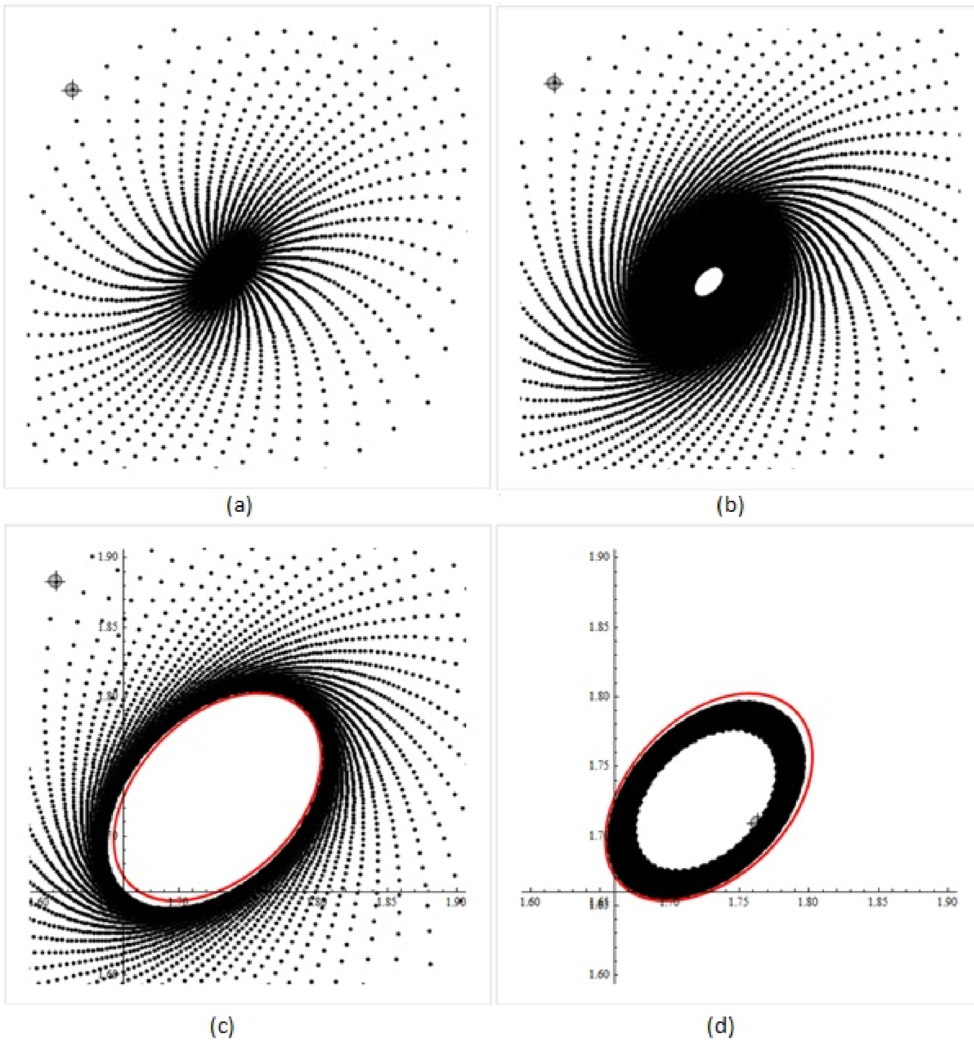

Figure 3.2: (a) Trajectory for $D = 0.11$, $F = 0.31$ and $C = 0.166$, where $C_0 = 0.16759$ b) Trajectory for $D = 0.11$, $F = 0.31$ and $C = C_0 = 0.16759$. (c)-(d) Trajectories and invariant curve (red) for $D = 0.11$, $F = 0.31$ and $C = 0.168$.

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