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Asymptotic Behaviour of Linear Fractional Nabla Difference Equations

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Abstract

The main aim of this article is to establish sufficient conditions on asymptotic behaviour of perturbed linear fractional nabla difference equations using the variation of constants formula. We provide a few examples to illustrate the applicability of established results.

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1 Introduction

Discrete fractional calculus is the unified theory of arbitrary order sums and differences. Looking into the literature of fractional differences, we find two approaches: one using the Δ-point of view (called the fractional delta difference), another using the ∇-perspective (called the fractional nabla difference). In this article, we confine ourselves to the second approach.

The concept of fractional nabla difference traces back to the work of Gray and Zhang [12]. They introduced a new definition of fractional nabla difference which has overcome the drawbacks of earlier definitions given by Granger and Joyeux [11] and Hosking [13]. Later, Anastassiou [3] and Atici and Eloe [4] slightly modified Gray's definition. In the past one decade, several authors gave valuable contributions to the development of the theory of fractional nabla differences. For a detailed introduction to this topic, we refer the recent monograph [10].

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Unlike in the integer case, study of asymptotic behaviour of a perturbed linear fractional nabla difference equation (LFNDE) is quite complicated. Čermák et al. [7] gave a precise asymptotic description of the two-term unperturbed LFNDE

$$\begin{aligned} \big(\nabla^{\alpha}_{\rho(0)}u\big)(t) &= \lambda u(t), \quad t \in \mathbb{N}_1, \\ u(0) &> 0, \end{aligned} \tag{1.1}$$

where $u : \mathbb{N}_0 \to \mathbb{R}$, $0 < \alpha < 1$ and $\lambda \in \mathbb{R}$. Recently, Jia et al. [16] investigated the asymptotic behaviour of the unperturbed LFNDE

$$\begin{aligned} \big(\nabla^{\alpha}_{\rho(0)}u\big)(t) &= a(t)u(t), \quad t \in \mathbb{N}_1, \\ u(0) &> 0, \end{aligned} \tag{1.2}$$

where $u : \mathbb{N}_0 \to \mathbb{R}$, $a : \mathbb{N}_0 \to \mathbb{R}$ and $0 < \alpha < 1$. In this article, we establish sufficient conditions on asymptotic behaviour of (1.1) and (1.2) with perturbations.

The present article is organized as follows: Section 2 contains preliminaries on fractional nabla calculus. In Section 3, we discuss asymptotic behaviour of perturbed LFNDEs. Few examples are provided in Section 4 to demonstrate the applicability of established results.

2 Preliminaries

Throughout this article, we use the following notations, definitions and known results of fractional nabla calculus [10]: Denote the set of all real numbers by $\mathbb{R}$. Define $\mathbb{N}_a = \{a, a+1, a+2, \cdots\}$ and $\mathbb{N}_a^b = \{a, a+1, a+2, \cdots, b\}$ for any a, $b \in \mathbb{R}$ such that $a < b$. Assume that empty sums and products are taken to be 0 and 1, respectively.

Definition 2.1. For any $t \in \mathbb{R} \setminus \{\cdots, -2, -1, 0\}$, the gamma function is defined by

$$\Gamma(t) = \int_0^\infty e^{-s}s^{t-1}ds, \quad t > 0,$$
$$\Gamma(t+1) = t\Gamma(t).$$

Definition 2.2 (Rising Factorial Function)**.** For any $\alpha \in \mathbb{R}$ and $t \in \mathbb{R} \setminus \{\cdots, -2, -1, 0\}$ such that $(t+\alpha) \in \mathbb{R} \setminus \{\cdots, -2, -1, 0\}$, the rising factorial function is defined by

$$t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}, \quad 0^{\overline{\alpha}} = 0.$$

Definition 2.3. Let $\alpha \in \mathbb{R}$ and choose $N \in \mathbb{N}_1$ such that $N - 1 < \alpha < N$.

1. (Fractional Nabla Sum) The α^{th}-order nabla sum of u is given by

$$\big(\nabla^{-\alpha}_{\rho(0)}u\big)(t) = \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t}(t-\rho(s))^{\overline{\alpha-1}}u(s), \quad t \in \mathbb{N}_0,$$

where $\rho(s) = s - 1$.

2. (R-L Fractional Nabla Difference) The α^{th}-order nabla difference of u is given by

$$\big(\nabla^{\alpha}_{\rho(0)}u\big)(t) = \frac{1}{\Gamma(-\alpha)}\sum_{s=0}^{t}(t-\rho(s))^{\overline{-\alpha-1}}u(s), \quad t \in \mathbb{N}_N.$$

Acar et al. [1] introduced the exponential function of fractional nabla calculus and analyzed their properties.

Definition 2.4 (See [1])**.** The exponential function of fractional nabla calculus is defined by

$$\hat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) = (1-\lambda)\sum_{k=0}^{\infty}\frac{\lambda^k (t+1)^{\overline{(k+1)\alpha-1}}}{\Gamma((k+1)\alpha)},$$

where $0 < \alpha < 1$, $|\lambda| < 1$ and $t \in \mathbb{N}_0$.

We note that $\hat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}})$ is the unique solution of (1.1) with $u(0) = 1$, which was derived using the N-transform in [6]. Now, we show that

$$\hat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) > 0 \text{ for all } \lambda \in (-1,1) \text{ and } t \in \mathbb{N}_0.$$

Rearranging the terms of (1.1), we get

$$\begin{aligned} u(t) &= \frac{-1}{(1-\lambda)\Gamma(-\alpha)}\sum_{s=0}^{t-1}(t-\rho(s))^{\overline{-\alpha-1}}u(s) \\ &= \frac{-1}{(1-\lambda)}\sum_{s=0}^{t-1}\frac{\Gamma(t-s-\alpha)}{\Gamma(t-s+1)\Gamma(-\alpha)}u(s) \\ &= \frac{\alpha}{(1-\lambda)\Gamma(1-\alpha)}\sum_{s=0}^{t-1}\frac{\Gamma(t-s-\alpha)}{\Gamma(t-s+1)}u(s). \end{aligned}$$

Since $\Gamma(t-s-\alpha)$, $\Gamma(t-s+1) > 0$ for $0 \le s \le (t-1)$; $\Gamma(1-\alpha) > 0$ for $0 < \alpha < 1$, $(1-\lambda) > 0$ for $-1 < \lambda < 1$ and $u(0) = 1 > 0$, using recursion, we have $u(t) = \hat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) > 0$ for all $t \in \mathbb{N}_0$.

Čermák et al. [7] have obtained the following asymptotic properties of exponential functions.

1. For $\lambda = 0$,
$$\lim_{t\to\infty}\frac{\hat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}})}{t^{-(1-\alpha)}} = \frac{1}{\Gamma(\alpha)}.$$

2. For $-1 < \lambda < 0$,
$$\lim_{t\to\infty}\frac{\hat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}})}{t^{-(1+\alpha)}} = \frac{\alpha(1-\lambda)}{\lambda^2\Gamma(1-\alpha)}.$$

Equivalently, for $-1 < \lambda \le 0$,

$$\hat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) \to 0 \text{ as } t \to \infty.$$

Consequently, using limit comparison test, we make the following observations:

1. Since $\sum_{t=1}^{\infty} \frac{1}{t^{1-\alpha}}$ is a divergent series for $0 < \alpha < 1$, the series $\sum_{t=1}^{\infty} \hat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}})$ diverges for $\lambda = 0$.
2. Since $\sum_{t=1}^{\infty} \frac{1}{t^{1+\alpha}}$ is a convergent series for $0 < \alpha < 1$, the series $\sum_{t=1}^{\infty} \hat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}})$ converges absolutely for $-1 < \lambda < 0$.

Definition 2.5. Let $x, y : \mathbb{N}_1 \to \mathbb{R}$. The convolution product of x and y is defined to be the function

$$\big(x * y\big)(t) = \sum_{s=1}^{t} x(t-s)y(s), \quad t \in \mathbb{N}_1.$$

Lemma 2.6 states the asymptotic property of convolution products which will be useful in establishing main results.

Lemma 2.6 (See [21])**.** *Let $x : \mathbb{N}_1 \to \mathbb{R}^+$, $y : \mathbb{N}_1 \to \mathbb{R}$ and $0 < c < \infty$. If $\lim_{t\to\infty} x(t) = c$ and $\sum_{t=1}^{\infty} y(t)$ converges absolutely, then*

$$\big(x * y\big)(t) \sim x(t)\Big(\sum_{t=1}^{\infty} y(t)\Big) \ (t \to \infty).$$

3 Main Results

First, we begin with the perturbed LFNDE

$$\begin{aligned} \big(\nabla^{\alpha}_{\rho(0)} v\big)(t) &= \lambda v(t) + f(t), \quad t \in \mathbb{N}_1, \\ u(0) &> 0, \end{aligned} \tag{3.1}$$

where v, $f : \mathbb{N}_0 \to \mathbb{R}$, $0 < \alpha < 1$ and $\lambda \in \mathbb{R}$. The following results provide sufficient conditions on $f(t)$ so that all solutions of (3.1) tend to zero as $t \to \infty$ provided all solutions of (1.1) tend to zero as $t \to \infty$.

Consider (1.1) with $-1 < \lambda \le 0$. The solution of (1.1) is of the form

$$u(t, 0, u(0)) = u(0)\hat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}), \quad t \in \mathbb{N}_0.$$

Clearly, all solutions of (1.1) tend to zero as $t \to \infty$.

Theorem 3.1. *Consider* (3.1) *with* $-1 < \lambda \leq 0$*. All solutions of* (3.1) *tend to zero as* $t \to \infty$ *provided*

$$\sum_{t=1}^{\infty} f(t) \text{ converges absolutely.} \tag{3.2}$$

Proof. Using variation of constants formula [6], the solution of (3.1) is given by

$$\begin{aligned} v(t,0,u(0)) &= u(t,0,u(0)) + \frac{1}{(1-\lambda)}\Big(\hat{e}_{\alpha,\alpha}(\lambda,t^{\overline{\alpha}}) * f(t)\Big) \\ &= u(0)\hat{e}_{\alpha,\alpha}(\lambda,t^{\overline{\alpha}}) - \frac{\varepsilon}{(1-\lambda)}\sum_{s=1}^{t} f(s) + \frac{1}{(1-\lambda)}\big(\theta * f\big)(t), \end{aligned}$$

where $\theta(t) = \hat{e}_{\alpha,\alpha}(\lambda,t^{\overline{\alpha}}) + \varepsilon$ for some $\varepsilon > 0$. Consider

$$\begin{aligned} &\lim_{t\to\infty} v(t,0,u(0)) \\ &= u(0)\lim_{t\to\infty}\hat{e}_{\alpha,\alpha}(\lambda,t^{\overline{\alpha}}) - \frac{\varepsilon}{(1-\lambda)}\lim_{t\to\infty}\Big(\sum_{s=1}^{t} f(s)\Big) + \frac{1}{(1-\lambda)}\lim_{t\to\infty}\big(\theta * f\big)(t) \\ &= 0 - \frac{\varepsilon}{(1-\lambda)}\Big(\sum_{s=1}^{\infty} f(s)\Big) + \frac{1}{(1-\lambda)}\Big(\sum_{t=1}^{\infty} f(t)\Big)\lim_{t\to\infty}\theta(t) = 0. \end{aligned}$$

This concludes the proof. □

Theorem 3.2. *Consider* (3.1) *with* $-1 < \lambda < 0$ *and* $f : \mathbb{N}_0 \to \mathbb{R}^+$*. All solutions of* (3.1) *tend to zero as* $t \to \infty$ *provided*

$$\lim_{t\to\infty} f(t) = 0.$$

Proof. Using variation of constants formula [6], the solution of (3.1) is given by

$$\begin{aligned} v(t,0,u(0)) &= u(t,0,u(0)) + \frac{1}{(1-\lambda)}\Big(\hat{e}_{\alpha,\alpha}(\lambda,t^{\overline{\alpha}}) * f(t)\Big) \\ &= u(0)\hat{e}_{\alpha,\alpha}(\lambda,t^{\overline{\alpha}}) - \frac{\delta}{(1-\lambda)}\sum_{s=1}^{t}\hat{e}_{\alpha,\alpha}(\lambda,s^{\overline{\alpha}}) \\ &\quad + \frac{1}{(1-\lambda)}\Big(\hat{e}_{\alpha,\alpha}(\lambda,t^{\overline{\alpha}}) * g(t)\Big), \end{aligned}$$

where $g(t) = f(t) + \delta$ for some $\delta > 0$. Consider

$$\begin{aligned}\lim_{t\to\infty} v(t,0,u(0)) &= u(0)\lim_{t\to\infty}\hat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) - \frac{\delta}{(1-\lambda)}\lim_{t\to\infty}\Big(\sum_{s=1}^{t}\hat{e}_{\alpha,\alpha}(\lambda, s^{\overline{\alpha}})\Big)\\ &\quad + \frac{1}{(1-\lambda)}\lim_{t\to\infty}\Big(\hat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) * g(t)\Big)\\ &= 0 - \frac{\delta}{(1-\lambda)}\Big(\sum_{s=1}^{\infty}\hat{e}_{\alpha,\alpha}(\lambda, s^{\overline{\alpha}})\Big)\\ &\quad + \frac{1}{(1-\lambda)}\Big(\sum_{t=1}^{\infty}\hat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}})\Big)\lim_{t\to\infty} g(t) = 0.\end{aligned}$$

This concludes the proof. □

Next, we shall consider the LFNDE

$$\begin{aligned}\big(\nabla^{\alpha}_{\rho(0)}v\big)(t) &= a(t)v(t) + f(t), \quad t \in \mathbb{N}_1,\\ u(0) &> 0,\end{aligned} \tag{3.3}$$

which can be regarded as a perturbation of (1.2).

Theorem 3.3. *The solution* $v(t,0,u(0))$ *of the perturbed problem* (3.3) *in terms of the solution* $u(t,0,u(0))$ *of the unperturbed problem* (1.2) *is given by*

$$v(t,0,u(0)) = u(t,0,u(0)) + \sum_{s=1}^{t} u(t,\rho(s),f(s)).$$

Proof. The solution $v(t,0,u(0))$ of (3.3) can be written as

$$v(t,0,u(0)) = u(t,0,u(0)) + p(t),$$

for a particular solution $p(t)$ of (3.3). We show that

$$p(t) = \sum_{s=1}^{t} u(t,\rho(s),f(s)), \quad p(0) = 0.$$

Clearly, $p(0) = 0$. We use the method of verification to show that $p(t)$ is a solution of

$$\big(\nabla^{\alpha}_{\rho(0)}p\big)(t) = a(t)p(t) + f(t), \quad t \in \mathbb{N}_1,$$

or equivalently,

$$p(t) = \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\big[a(s)p(s) + f(s)\big], \quad t \in \mathbb{N}_1.$$

Consider

$$
\begin{aligned}
p(t) &= \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\big[a(s)p(s)+f(s)\big] \\
&= \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}a(s)\Big[\sum_{r=1}^{s}u(s,\rho(r),f(r))\Big] \\
&\quad +\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}f(s) \\
&= \sum_{r=1}^{t}\Big[\frac{1}{\Gamma(\alpha)}\sum_{s=r}^{t}(t-\rho(s))^{\overline{\alpha-1}}a(s)u(s,\rho(r),f(r))\Big] \\
&\quad +\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}f(s) \\
&= \sum_{r=1}^{t}\nabla_{\rho(r)}^{-\alpha}\big[a(t)u(t,\rho(r),f(r))\big]+\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}f(s) \\
&= \sum_{r=1}^{t}\nabla_{\rho(r)}^{-\alpha}\big[\big(\nabla_{\rho(r-1)}^{\alpha}u\big)(t,\rho(r),f(r))\big]+\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}f(s) \\
&= \sum_{r=1}^{t}\Big[u(t,\rho(r),f(r))-\frac{1}{\Gamma(\alpha)}(t-\rho(r))^{\overline{\alpha-1}}f(r)\Big] \\
&\quad +\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}f(s) \\
&= \sum_{r=1}^{t}u(t,\rho(r),f(r))=p(t).
\end{aligned}
$$

This concludes the proof. □

The following result provides sufficient condition on $f(t)$ so that all solutions of (3.3) tend to zero as $t\to\infty$ provided all solutions of (1.2) tend to zero as $t\to\infty$.

Consider (1.2) with $a(t)\le 0$ for all $t\in\mathbb{N}_0$. Using [5, Lemma 2.2 and Theorem 3.2], the solution of (1.2) is given by

$$u(t,0,u(0))=\phi(t,0)u(0),\quad t\in\mathbb{N}_0,$$

where

$$\phi(t,s)=\frac{1}{\Gamma(\alpha)}\sum_{k=0}^{\infty}E_a^k(t-\rho(s))^{\overline{\alpha-1}},\quad 0\le s\le(t-1),$$
$$\phi(t,t)=1.$$

We show that

$$\phi(t,0) > 0 \text{ for all } t \in \mathbb{N}_0.$$

Let $u(0) = 1$. Rearranging the terms of (1.2), we get

$$\begin{aligned} u(t) &= \frac{-1}{(1-a(t))\Gamma(-\alpha)} \sum_{s=0}^{t-1} (t-\rho(s))^{\overline{-\alpha-1}} u(s) \\ &= \frac{-1}{(1-a(t))} \sum_{s=0}^{t-1} \frac{\Gamma(t-s-\alpha)}{\Gamma(t-s+1)\Gamma(-\alpha)} u(s) \\ &= \frac{\alpha}{(1-a(t))\Gamma(1-\alpha)} \sum_{s=0}^{t-1} \frac{\Gamma(t-s-\alpha)}{\Gamma(t-s+1)} u(s). \end{aligned}$$

Since $\Gamma(t-s-\alpha)$, $\Gamma(t-s+1) > 0$ for $0 \le s \le (t-1)$; $\Gamma(1-\alpha) > 0$ for $0 < \alpha < 1$, $(1-a(t)) > 0$ for $a(t) \le 0$ and $u(0) = 1 > 0$, using recursion, we have $u(t) = \phi(t,0) > 0$ for all $t \in \mathbb{N}_0$. Also, from [16, Theorem B], we have

$$\lim_{t\to\infty} \phi(t,0) = 0.$$

Thus, all solutions of (1.2) tend to zero as $t \to \infty$.

Theorem 3.4. *Consider* (3.3) *with* $a(t) \le 0$ *for all* $t \in \mathbb{N}_0$. *All solutions of* (3.3) *tend to zero as* $t \to \infty$ *provided* $\sum_{t=0}^{\infty} f(\sigma(t))$ *converges absolutely.*

Proof. Using Theorem 3.3, the solution of (3.3) is given by

$$\begin{aligned} v(t,0,u(0)) &= \phi(t,0)u(0) + \sum_{s=1}^{t} \phi(t,\rho(s)) f(s) \\ &= \phi(t,0)u(0) - f(\sigma(t)) + \sum_{s=0}^{t} \phi(t,s) f(\sigma(s)) \\ &= \phi(t,0)u(0) - f(\sigma(t)) - \epsilon \sum_{s=0}^{t} f(\sigma(s)) \\ &\quad + \big(\psi(t) * f(\sigma(t))\big), \quad t \in \mathbb{N}_0, \end{aligned}$$

where $\sigma(t) = t+1$ and $\psi(t) = \phi(t,0) + \epsilon$ for some $\epsilon > 0$. Consider

$$\begin{aligned} &\lim_{t\to\infty} v(t,0,u(0)) \\ = \;& u(0) \lim_{t\to\infty} \phi(t,0) - \lim_{t\to\infty} f(\sigma(t)) - \epsilon \lim_{t\to\infty} \Big(\sum_{s=0}^{t} f(\sigma(s))\Big) + \lim_{t\to\infty} \big(\psi(t) * f(\sigma(t))\big) \\ = \;& 0 - \lim_{t\to\infty} f(\sigma(t)) - \epsilon\Big(\sum_{t=0}^{\infty} f(\sigma(t))\Big) + \Big(\sum_{t=0}^{\infty} f(\sigma(t))\Big) \lim_{t\to\infty} \psi(t) \\ = \;& 0. \end{aligned}$$

This concludes the proof. □

Next, we investigate asymptotic behaviour of the perturbed LFNDE

$$\begin{aligned} \big(\nabla^{\alpha}_{\rho(0)} v\big)(t) &= \big(\lambda + f(t)\big) v(t), \quad t \in \mathbb{N}_1, \\ u(0) &> 0. \end{aligned} \tag{3.4}$$

Using [16, Theorem B], we have the following asymptotic result:

Corollary 3.5. *Assume that* $\lambda + f(t) \leq 0$*. Then, the solution of* (3.4) *satisfy*

$$\lim_{t\to\infty} v(t, 0, u(0)) = 0.$$

Finally, we investigate asymptotic behaviour of the perturbed LFNDE

$$\begin{aligned} \big(\nabla^{\alpha}_{\rho(0)} v\big)(t) &= \big(a(t) + f(t)\big) v(t), \quad t \in \mathbb{N}_1, \\ u(0) &> 0. \end{aligned} \tag{3.5}$$

Using [16, Theorem B], we have the following asymptotic result:

Corollary 3.6. *Assume that* $a(t) + f(t) \leq 0$*. Then, the solution of* (3.5) *satisfy*

$$\lim_{t\to\infty} v(t, 0, u(0)) = 0.$$

4 Examples

In this section, we investigate the asymptotic behaviour of a few perturbed LFNDEs using the conditions established in Section 3.

Example 4.1. Consider the initial value problem associated with a discrete fractional nabla relaxation equation

$$\begin{aligned} \big(\nabla^{0.5}_{0} w\big)(t) &= (t+1)^{-1.5}, \quad t \in \mathbb{N}_1, \\ w(0) &= 1. \end{aligned}$$

Solution: Here $\lambda = 0$ and $f(t) = (t+1)^{-1.5} > 0$ for all $t \in \mathbb{N}_0$. Clearly, $\lim_{t\to\infty} f(t) = 0$. Then, from Theorem 3.2, we have $w(t, 0, 1) \to 0$ as $t \to \infty$.

Example 4.2. Consider the initial value problem associated with a discrete fractional nabla relaxation equation

$$\begin{aligned} \big(\nabla^{0.5}_{0} w\big)(t) + (0.5) w(t) &= \left(\frac{1}{2}\right)^{t}, \quad t \in \mathbb{N}_1, \\ w(0) &= 1. \end{aligned}$$

Solution: Here $\alpha = 0.5$, $\lambda = -0.5$ and $f(t) = \left(\frac{1}{2}\right)^t$. Clearly, $-1 < \lambda \leq 0$, $f(t) > 0$ for all $t \in \mathbb{N}_0$ and $\lim_{t\to\infty} f(t) = 0$. Then, from Theorem 3.2, we have $w(t, 0, 1) \to 0$ as $t \to \infty$.

Example 4.3. Consider the initial value problem

$$\begin{aligned}\big(\nabla_0^{0.5} u\big)(t) = -\Big(\frac{1}{2}\Big)^t u(t) + (t+1)^{-1.5}, \quad t \in \mathbb{N}_1,\\ u(0) = 1.\end{aligned}$$

Solution: Here $\alpha = 0.5$, $a(t) = -\left(\frac{1}{2}\right)^t$ and $f(t) = (t+1)^{-1.5}$. Clearly, $a(t) \leq 0$ for all $t \in \mathbb{N}_0$ and $\sum_{t=0}^{\infty} f(t+1) = \sum_{t=0}^{\infty} \frac{1}{(t+2)^{1.5}}$ converges absolutely. Then, from Theorem 3.4, we have $u(t, 0, 1) \to 0$ as $t \to \infty$.

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