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Improved Iterative Oscillation Tests for First-Order Deviating Difference Equations

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Abstract

In this paper, improved oscillation conditions are established for the oscillation of all solutions of first-order difference equations with nonmonotone deviating arguments and nonnegative coefficients. They lead to a procedure that checks for oscillations by iteratively computing $\limsup$ on terms recursively defined on the equation's coefficients and deviating argument. This procedure significantly improves all known oscillation criteria. The results and the improvement achieved over the other known conditions are illustrated by examples, numerically solved in MATLAB.

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1 Introduction

Consider the difference equation with a variable retarded argument of the form

$$\Delta x(n) + p(n)x(\tau(n)) = 0, \quad n \in \mathbb{N}_0 \tag{E}$$

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and the (dual) difference equation with a variable advanced argument of the form

$$\nabla x(n) - q(n)x(\sigma(n)) = 0, \quad n \in \mathbb{N}, \tag{E$'$}$$

where $\mathbb{N}_0$ and $\mathbb{N}$ are the sets of nonnegative integers and positive integers, respectively.

Equations (E) and (E$'$) are studied under the following assumptions: everywhere $(p(n))_{n\geq 0}$ and $(q(n))_{n\geq 1}$ are sequences of nonnegative real numbers, $(\tau(n))_{n\geq 0}$ is a sequence of integers such that

$$\tau(n) \leq n-1, \quad \forall n \in \mathbb{N}_0 \quad \text{and} \quad \lim_{n\to\infty} \tau(n) = \infty \tag{1.1}$$

and $(\sigma(n))_{n\geq 1}$ is a sequence of integers such that

$$\sigma(n) \geq n+1, \quad \forall n \in \mathbb{N}. \tag{1.2}$$

Here, Δ denotes the forward difference operator $\Delta x(n) = x(n+1) - x(n)$ and ∇ corresponds to the backward difference operator $\nabla x(n) = x(n) - x(n-1)$.

Set $w = -\min_{n\geq 0} \tau(n)$ and note that w is a finite positive integer, if (1.1) holds.

By a *solution* of (E), we mean a sequence of real numbers $(x(n))_{n\geq -w}$ which satisfies (E) for all $n \geq 0$. It is clear that, for each choice of real numbers $c_{-w}, c_{-w+1}, \ldots, c_{-1}, c_0$, there exists a unique solution $(x(n))_{n\geq -w}$ of (E) which satisfies the initial conditions $x(-w) = c_{-w}$, $x(-w+1) = c_{-w+1}, \ldots, x(-1) = c_{-1}$, $x(0) = c_0$. When the initial data is given, we can obtain a unique solution to (E) by using the method of steps.

By a solution of (E$'$), we mean a sequence of real numbers $(x(n))_{n\geq 0}$, which satisfies (E$'$) for all $n \geq 1$.

A solution $(x(n))_{n\geq -w}$ (or $(x(n))_{n\geq 0}$) of (E) (or (E$'$)) is called *oscillatory*, if the terms $x(n)$ of the sequence are neither eventually positive nor eventually negative. Otherwise, the solution is said to be *nonoscillatory*. An equation is *oscillatory* if all its solutions oscillate.

In the last few decades, the oscillatory behavior and the existence of positive solutions of difference equations with deviating arguments have been extensively studied, see, for example, papers [1–21] and references cited therein. Most of these papers concern the special case where the arguments are nondecreasing, while a small number of these papers are dealing with the general case where the arguments are not necessarily monotone. See, for example, [1–4, 8, 9, 17] and the references cited therein.

The motivation for considering nonmonotone arguments is not of purely mathematical interest. Several phenomena require the use of nonmonotone arguments since there are always natural disturbances, e.g. noise in communication systems, that affect all the parameters of an equation. Therefore, the monotone arguments, adequate from a mathematical point of view, become nonmonotone almost always. In view of this, an interesting question arising in the case when the arguments $\tau(n)$ and $\sigma(n)$ are nonmonotone, is whether we can establish oscillation criteria that substantially improve on all the known results in the literature.

In the present paper, we achieve this goal by establishing criteria which, up to our knowledge, essentially improve all other known results in the literature.

Throughout this paper, we are going to use the following notation:

$$\sum_{i=k}^{k-1} A(i) = 0 \quad \text{and} \quad \prod_{i=k}^{k-1} A(i) = 1,$$

$$\alpha := \liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} p(j),$$

$$\beta := \liminf_{n\to\infty} \sum_{j=n+1}^{\sigma(n)} q(j),$$

$$D(\omega) := \begin{cases} 0, & \text{if } \omega > 1/e, \\ \dfrac{1-\omega-\sqrt{1-2\omega-\omega^2}}{2}, & \text{if } \omega \in [0, 1/e], \end{cases}$$

$$MD := \limsup_{n\to\infty} \sum_{j=\tau(n)}^{n} p(j), \quad \text{where } \tau(n) \text{ is nondecreasing,}$$

and

$$MA := \limsup_{n\to\infty} \sum_{j=n}^{\rho(n)} q(j), \quad \text{where } \sigma(n) \text{ is nondecreasing.}$$

1.1 Retarded Difference Equations (Chronological Review)

In 2008, Chatzarakis, Koplatadze and Stavroulakis [5, 6] proved that if

$$MD > 1, \tag{1.3}$$

or

$$\alpha > \frac{1}{e}, \tag{1.4}$$

then all solutions of (E) oscillate.

It is obvious that there is a gap between the conditions (1.3) and (1.4) when the limit

$$\lim_{n\to\infty} \sum_{j=\tau(n)}^{n-1} p(j)$$

does not exist. How to fill this gap is an interesting problem which has been investigated by several authors. For example, in 2009, Chatzarakis, Philos and Stavroulakis [7] proved that if

$$MD > 1 - D(\alpha), \tag{1.5}$$

then all solutions of (E) oscillate.

Now let us come to the case considered in the present work, i.e., that the argument $\tau(n)$ is not necessarily monotone. Set

$$h(n) = \max_{0 \leq s \leq n} \tau(s). \tag{1.6}$$

Clearly, the sequence $h(n)$ is nondecreasing with $\tau(n) \leq h(n) \leq n-1$ for all $n \geq 0$.

In 2011, Braverman and Karpuz [3] proved that if

$$\limsup_{n\to\infty} \sum_{j=h(n)}^{n} p(j) \prod_{i=\tau(j)}^{h(n)-1} \frac{1}{1-p(i)} > 1, \tag{1.7}$$

then all solutions of (E) oscillate, while, in 2014, Stavroulakis [17] improved (1.7) to

$$\limsup_{n\to\infty} \sum_{j=h(n)}^{n} p(j) \prod_{i=\tau(j)}^{h(n)-1} \frac{1}{1-p(i)} > 1 - D(\alpha). \tag{1.8}$$

In 2015, Braverman, Chatzarakis and Stavroulakis [2] proved that if for some $r \in \mathbb{N}$

$$\limsup_{n\to\infty} \sum_{j=h(n)}^{n} p(j) a_r^{-1}\left(h(n), \tau(j)\right) > 1, \tag{1.9}$$

or

$$\limsup_{n\to\infty} \sum_{j=h(n)}^{n} p(j) a_r^{-1}\left(h(n), \tau(j)\right) > 1 - D(\alpha), \tag{1.10}$$

where

$$\begin{aligned} a_1(n,k) &= \prod_{i=k}^{n-1} [1-p(i)] \\ a_{r+1}(n,k) &= \prod_{i=k}^{n-1} \left[1 - p(i) a_r^{-1}(i, \tau(i))\right] \end{aligned},$$

then all solutions of (E) oscillate.

Remark 1.1. Observe that conditions (1.7) and (1.8) are special cases of (1.9) and (1.10), respectively, when $r = 1$.

In 2017, Asteris and Chatzarakis [1], and Chatzarakis and Shaikhet [9] proved that if for some $\ell \in \mathbb{N}$

$$\limsup_{n\to\infty} \sum_{i=h(n)}^{n} p(i) \prod_{j=\tau(i)}^{h(n)-1} \frac{1}{1-p_\ell(j)} > 1 \tag{1.11}$$

or

$$\limsup_{n\to\infty} \sum_{i=h(n)}^{n} p(i) \prod_{j=\tau(i)}^{h(n)-1} \frac{1}{1-p_\ell(j)} > 1 - D(\alpha), \tag{1.12}$$

where $p_0(n) = p(n)$ and

$$p_\ell(n) = p(n) \left[1 + \sum_{i=\tau(n)}^{n-1} p(i) \prod_{j=\tau(i)}^{h(n)-1} \frac{1}{1-p_{\ell-1}(j)} \right],$$

then all solutions of (E) oscillate.

Recently, Chatzarakis, Purnaras and Stavroulakis [8] proved that if for some $\ell \in \mathbb{N}$

$$\limsup_{n\to\infty} \sum_{i=h(n)}^{n} p(i) \prod_{j=\tau(i)}^{h(n)-1} \frac{1}{1-P_\ell(j)} > 1, \tag{1.13}$$

or

$$\limsup_{n\to\infty} \sum_{i=h(n)}^{n} p(i) \prod_{j=\tau(i)}^{h(n)-1} \frac{1}{1-P_\ell(j)} > 1 - D(\alpha), \tag{1.14}$$

or

$$\limsup_{n\to\infty} \sum_{i=h(n)}^{n} p(i) \prod_{j=\tau(i)}^{n} \frac{1}{1-P_\ell(j)} > \frac{1}{D(\alpha)}, \tag{1.15}$$

where $P_0(n) = p(n)$ and

$$P_\ell(n) = p(n) \left[1 + \sum_{i=\tau(n)}^{n-1} p(i) \exp\left(\sum_{j=\tau(i)}^{n-1} p(j) \prod_{m=\tau(j)}^{j-1} \frac{1}{1-P_{\ell-1}(m)} \right) \right],$$

then all solutions of (E) are oscillatory.

Lately, Chatzarakis and Jadlovská [4] improved (1.13), (1.14) and (1.15) to

$$\limsup_{n\to\infty} \sum_{i=h(n)}^{n} p(i) \exp\left(\sum_{j=\tau(i)}^{h(n)-1} p(j) \prod_{m=\tau(j)}^{j-1} \frac{1}{1-\widetilde{P}_\ell(m)} \right) > 1, \tag{1.16}$$

$$\limsup_{n\to\infty} \sum_{i=h(n)}^{n} p(i) \exp\left(\sum_{j=\tau(i)}^{h(n)-1} p(j) \prod_{m=\tau(j)}^{j-1} \frac{1}{1-\widetilde{P}_\ell(m)} \right) > 1 - D(\alpha), \tag{1.17}$$

and

$$\limsup_{n\to\infty} \sum_{i=h(n)}^{n} p(i) \exp\left(\sum_{j=\tau(i)}^{n} p(j) \prod_{m=\tau(j)}^{j-1} \frac{1}{1-\widetilde{P}_\ell(m)} \right) > \frac{1}{D(\alpha)} - 1, \tag{1.18}$$

respectively, where

$$\widetilde{P}_\ell(n) = p(n)\left[1 + \sum_{i=\tau(n)}^{n-1} p(i)\exp\left(\sum_{j=\tau(i)}^{n-1} p(j)\prod_{m=\tau(j)}^{j-1}\frac{1}{1-\widetilde{P}_{\ell-1}(m)}\right)\right]$$

with $\widetilde{P}_0(n) = \lambda_0 p(n)$ and λ_0 is the smaller root of the transcendental equation $\lambda = e^{\alpha\lambda}$.

1.2 Advanced Difference Equations (Chronological Review)

In 2012, Chatzarakis and Stavroulakis [10] proved that if

$$MA > 1, \tag{1.19}$$

or

$$MA > 1 - \left(1 - \sqrt{1-\beta}\right)^2, \tag{1.20}$$

then all solutions of (E′) oscillate.

Now we come to the case that the argument $\sigma(n)$ is not necessarily monotone. Set

$$\rho(n) = \min_{s\geq n}\sigma(s). \tag{1.21}$$

Clearly, the sequence $\rho(n)$ is nondecreasing with $\sigma(n) \geq \rho(n) \geq n+1$ for all $n \geq 1$.

In 2015, Braverman, Chatzarakis and Stavroulakis [2] proved that if for some $r \in \mathbb{N}$

$$\limsup_{n\to\infty}\sum_{j=n}^{\rho(n)} q(j)b_r^{-1}(\rho(n),\sigma(j)) > 1, \tag{1.22}$$

or

$$\limsup_{n\to\infty}\sum_{j=n}^{\rho(n)} q(j)b_r^{-1}(\rho(n),\sigma(j)) > 1 - D(\beta), \tag{1.23}$$

where

$$\begin{aligned} b_1(n,k) &= \prod_{i=n+1}^{k}\left[1-q(i)\right] \\ b_{r+1}(n,k) &= \prod_{i=n+1}^{k}\left[1-q(i)b_r^{-1}(i,\sigma(i))\right] \end{aligned} \tag{1.24}$$

then all solutions of (E′) oscillate.

In 2017, Asteris and Chatzarakis [1], and Chatzarakis and Shaikhet [9] proved that if for some $\ell \in \mathbb{N}$

$$\limsup_{n\to\infty}\sum_{i=n}^{\rho(n)} q(i)\prod_{j=\rho(n)+1}^{\sigma(i)}\frac{1}{1-q_\ell(j)} > 1, \tag{1.25}$$

or

$$\limsup_{n\to\infty}\sum_{i=n}^{\rho(n)} q(i) \prod_{j=\rho(n)+1}^{\sigma(i)} \frac{1}{1-q_\ell(j)} > 1 - D(\beta), \tag{1.26}$$

where $q_0(n) = q(n)$ and

$$q_\ell(n) = q(n)\left[1 + \sum_{i=n+1}^{\rho(n)} q(i) \prod_{j=\rho(n)+1}^{\sigma(i)} \frac{1}{1-q_{\ell-1}(j)}\right],$$

then all solutions of (E′) oscillate.

Lately, Chatzarakis and Jadlovská [4] proved that if for some $\ell \in \mathbb{N}$

$$\limsup_{n\to\infty}\sum_{i=n}^{\rho(n)} q(i)\exp\left(\sum_{j=\rho(n)+1}^{\sigma(i)} q(j) \prod_{m=j+1}^{\sigma(j)} \frac{1}{1-\widetilde{Q}_\ell(m)}\right) > 1, \tag{1.27}$$

or

$$\limsup_{n\to\infty}\sum_{i=n}^{\rho(n)} q(i)\exp\left(\sum_{j=\rho(n)+1}^{\sigma(i)} q(j) \prod_{m=j+1}^{\sigma(j)} \frac{1}{1-\widetilde{Q}_\ell(m)}\right) > 1 - D\left(\beta\right), \tag{1.28}$$

or

$$\limsup_{n\to\infty}\sum_{i=n}^{\rho(n)} q(i)\exp\left(\sum_{j=n}^{\sigma(i)} q(j) \prod_{m=j+1}^{\sigma(j)} \frac{1}{1-\widetilde{Q}_\ell(m)}\right) > \frac{1}{D(\beta)} - 1, \tag{1.29}$$

where

$$\widetilde{Q}_\ell(n) = q(n)\left[1 + \sum_{i=n+1}^{\sigma(n)} q(i)\exp\left(\sum_{j=n+1}^{\sigma(i)} q(j) \prod_{m=j+1}^{\sigma(j)} \frac{1}{1-\widetilde{Q}_{\ell-1}(m)}\right)\right]$$

with $\widetilde{Q}_0(n) = \lambda_0 q(n)$ and λ_0 is the smaller root of the transcendental equation $\lambda = e^{\beta\lambda}$, then all solutions of (E′) are oscillatory.

2 Main Results

2.1 Retarded Difference Equations

We further study (E) and derive new sufficient oscillation condition, involving $\limsup$, which essentially improve all the previous results.

The following simple result is cited to explain why we can consider only the case

$$p(n) < \frac{1}{1 + \lambda_0 \sum\limits_{i=\tau(n)}^{n-1} p(i)}, \qquad \forall n \geq 0, \tag{2.1}$$

where $\lambda_0 > 1$ is the smaller root of the transcendental equation $\lambda = e^{\alpha\lambda}$ with $0 < \alpha \leq 1/e$.

Theorem 2.1. *Assume that there exists a subsequence $\theta(n)$, $n \in \mathbb{N}$ of positive integers such that*

$$p(\theta(n)) \geq \frac{1}{1+\lambda_0 \sum\limits_{i=\tau(\theta(n))}^{\theta(n)-1} p(\theta(i))}, \qquad \forall n \in \mathbb{N}. \tag{2.2}$$

Then all solutions of (E) *are oscillatory.*

Proof. Assume, for the sake of contradiction, that $(x(n))_{n\geq -w}$ is a nonoscillatory solution of (E). Then it is either eventually positive or eventually negative. As $(-x(n))_{n\geq -w}$ is also a solution of (E), we may restrict ourselves only to the case where $x(n) > 0$ for all large n. Let $n_1 \geq -w$ be an integer such that $x(n) > 0$ for all $n \geq n_1$. Then, there exists $n_2 \geq n_1$ such that $x(\tau(n)) > 0$, $\forall n \geq n_2$. In view of this, (E) becomes

$$\Delta x(n) = -p(n)x(\tau(n)) \leq 0, \quad \forall n \geq n_2,$$

which means that the sequence $(x(n))$ is eventually nonincreasing.

Taking into account the fact that (2.2) holds, equation (E) gives

$$\begin{aligned} x(\theta(n)+1) &= x(\theta(n)) - p(\theta(n))x(\tau(\theta(n))) \\ &\leq \frac{1}{1+\lambda_0 \sum\limits_{i=\tau(\theta(n))}^{\theta(n)-1} p(\theta(i))} x(\theta(n)) - p(\theta(n))x(\tau(\theta(n))) \\ &\leq \frac{1}{1+\lambda_0 \sum\limits_{i=\tau(\theta(n))}^{\theta(n)-1} p(\theta(i))} x(\theta(n)) - x(\theta(n))p(\theta(n)) \\ &= x(\theta(n)) \left(\frac{1}{1+\lambda_0 \sum\limits_{i=\tau(\theta(n))}^{\theta(n)-1} p(\theta(i))} - p(\theta(n)) \right) \leq 0, \end{aligned}$$

for all $\theta(n) \geq n_2$, where $\theta(n) \to \infty$ as $n \to \infty$. This contradicts to the assumption that $x(n) > 0$ for all $n \geq n_2$. □

The proofs of our main results are essentially based on the following lemmas.

Lemma 2.2 (See [9]). *Assume that* (1.1) *holds and $h(n)$ is defined by* (1.6). *If $\alpha > 0$ then*

$$\liminf_{n\to\infty} \sum_{j=h(n)}^{n-1} p(j) = \liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} p(j) = \alpha. \tag{2.3}$$

Lemma 2.3 (See [7]). *Assume that*(1.1) *holds,* $h(n)$ *is defined by* (1.6)*,* $0 < \alpha \leq 1/e$ *and* $x(n)$ *is an eventually positive solution of* (E)*. Then*

$$\liminf_{n\to\infty} \frac{x(n+1)}{x(h(n))} \geq D(\alpha). \tag{2.4}$$

Lemma 2.4 (See [4]). *Assume that* (1.1) *holds,* $h(n)$ *is defined by* (1.6)*,* $0 < \alpha \leq 1/e$ *and* $x(n)$ *is an eventually positive solution of* (E)*. Then*

$$\liminf_{n\to\infty} \frac{x(h(n))}{x(n)} \geq \lambda_0, \tag{2.5}$$

where λ_0 *is the smaller root of the transcendental equation* $\lambda = e^{\alpha\lambda}$*.*

Theorem 2.5. *Assume that* (1.1) *and* (2.1) *hold, and* $h(n)$ *is defined by* (1.6)*. If for some* $\ell \in \mathbb{N}$

$$\limsup_{n\to\infty} \sum_{i=h(n)}^{n} p(i) \exp\left(\sum_{j=\tau(i)}^{h(n)-1} p(j) \prod_{m=\tau(j)}^{j-1} \frac{1}{1-R_\ell(m)}\right) > 1, \tag{2.6}$$

where

$$R_\ell(n) = p(n)\left[1 + \sum_{i=\tau(n)}^{n-1} p(i) \exp\left(\sum_{j=\tau(i)}^{n-1} p(j) \prod_{m=\tau(j)}^{j-1} \frac{1}{1-R_{\ell-1}(m)}\right)\right] \tag{2.7}$$

with $R_0(n) = p(n)\left[1 + \lambda_0 \sum_{i=\tau(n)}^{n-1} p(i)\right]$ *and* λ_0 *is the smaller root of the transcendental equation* $\lambda = e^{\alpha\lambda}$*, then all solutions of* (E) *are oscillatory.*

Proof. Assume, for the sake of contradiction, that $(x(n))_{n\geq -w}$ is a nonoscillatory solution of (E). Then it is either eventually positive or eventually negative. As $(-x(n))_{n\geq -w}$ is also a solution of (E), we may restrict ourselves only to the case where $x(n) > 0$ for all large n. Let $n_1 \geq -w$ be an integer such that $x(n) > 0$ for all $n \geq n_1$. Then, there exists $n_2 \geq n_1$ such that $x(\tau(n)) > 0$, $\forall n \geq n_2$. In view of this, (E) becomes

$$\Delta x(n) = -p(n)x(\tau(n)) \leq 0, \quad \forall n \geq n_2,$$

which means that the sequence $(x(n))$ is eventually nonincreasing.

Taking this into account along the fact that $\tau(n) \leq h(n)$, (E) implies

$$\Delta x(n) + p(n)x(h(n)) \leq 0, \quad n \geq n_2. \tag{2.8}$$

Observe that (2.5) implies that for each $\epsilon > 0$ there exists $n(\epsilon)$ such that

$$\frac{x(h(n))}{x(n)} > \lambda_0 - \epsilon, \quad \text{for all } n \geq n(\epsilon) \geq n_2. \tag{2.9}$$

Summing up (E) from $\tau(n)$ to $n-1$, we have

$$x(n) - x(\tau(n)) + \sum_{i=\tau(n)}^{n-1} p(i) x\left(\tau(i)\right) = 0,$$

which, in view of $\tau(i) \leq h(i)$, gives

$$x(n) - x(\tau(n)) + \sum_{i=\tau(n)}^{n-1} p(i) x\left(h(i)\right) \leq 0. \tag{2.10}$$

Combining the inequalities (2.9) and (2.10) we obtain

$$x(n) - x(\tau(n)) + (\lambda_0 - \epsilon) \sum_{i=\tau(n)}^{n-1} p(i) x\left(i\right) \leq 0,$$

or

$$x(n) - x(\tau(n)) + (\lambda_0 - \epsilon)\, x(n) \sum_{i=\tau(n)}^{n-1} p(i) \leq 0. \tag{2.11}$$

Multiplying the last inequality by $p(n)$, we get

$$p(n)x(n) - p(n)x(\tau(n)) + (\lambda_0 - \epsilon)\, p(n)x(n) \sum_{i=\tau(n)}^{n-1} p(i) \leq 0,$$

which, in view of (E), becomes

$$\Delta x(n) + p(n)x(n) + (\lambda_0 - \epsilon)\, p(n)x(n) \sum_{i=\tau(n)}^{n-1} p(i) \leq 0,$$

or

$$\Delta x(n) + p(n) \left[1 + (\lambda_0 - \epsilon) \sum_{i=\tau(n)}^{n-1} p(i) \right] x(n) < 0.$$

Thus

$$\Delta x(n) + R_0(n, \epsilon) x(n) \leq 0, \tag{2.12}$$

where

$$R_0(n, \epsilon) = p(n) \left[1 + (\lambda_0 - \epsilon) \sum_{i=\tau(n)}^{n-1} p(i) \right].$$

Applying the discrete Grönwall inequality in (2.12), we obtain

$$x(k) > x(n) \prod_{i=k}^{n-1} \frac{1}{1 - R_0(i, \epsilon)}, \quad \text{for all } n \geq n(\epsilon). \tag{2.13}$$

Dividing (E) by $x(n)$ and summing up from k to $n - 1$, we take

$$\sum_{j=k}^{n-1} \frac{\Delta x(j)}{x(j)} = -\sum_{j=k}^{n-1} p(j) \frac{x(\tau(j))}{x(j)}. \tag{2.14}$$

Also, since $e^x \geq x + 1$, $x > 0$ we have

$$\begin{aligned} \sum_{j=k}^{n-1} \frac{\Delta x(j)}{x(j)} &= \sum_{j=k}^{n-1} \left(\frac{x(j+1)}{x(j)} - 1 \right) \\ &= \sum_{j=k}^{n-1} \left[\exp\left(\ln \frac{x(j+1)}{x(j)} \right) - 1 \right] \\ &\geq \sum_{j=k}^{n-1} \left[\ln \frac{x(j+1)}{x(j)} + 1 - 1 \right] \\ &= \sum_{j=k}^{n-1} \ln \frac{x(j+1)}{x(j)} = \ln \frac{x(n)}{x(k)}, \end{aligned}$$

or

$$\sum_{j=k}^{n-1} \frac{\Delta x(j)}{x(j)} \geq \ln \frac{x(n)}{x(k)}. \tag{2.15}$$

Combining (2.14) and (2.15), we obtain

$$-\sum_{j=k}^{n-1} p(j) \frac{x(\tau(j))}{x(j)} \geq \ln \frac{x(n)}{x(k)},$$

or

$$\ln \frac{x(k)}{x(n)} \geq \sum_{j=k}^{n-1} p(j) \frac{x(\tau(j))}{x(j)}. \tag{2.16}$$

Since $\tau(j) < j$, (2.13) implies

$$x(\tau(j)) > x(j) \prod_{i=\tau(j)}^{j-1} \frac{1}{1 - R_0(i, \epsilon)}. \tag{2.17}$$

In view of (2.17), (2.16) gives

$$\ln \frac{x(k)}{x(n)} > \sum_{j=k}^{n-1} p(j) \prod_{i=\tau(j)}^{j-1} \frac{1}{1-R_0(i,\epsilon)},$$

or

$$x(k) > x(n) \exp\left(\sum_{j=k}^{n-1} p(j) \prod_{i=\tau(j)}^{j-1} \frac{1}{1-R_0(i,\epsilon)}\right). \tag{2.18}$$

Summing up (E) from $\tau(n)$ to $n-1$, we have

$$x(n) - x(\tau(n)) + \sum_{i=\tau(n)}^{n-1} p(i) x\left(\tau(i)\right) = 0. \tag{2.19}$$

Setting $k = \tau\left(i\right)$ in (2.18) implies

$$x(\tau(i)) > x(n) \exp\left(\sum_{j=\tau(i)}^{n-1} p(j) \prod_{m=\tau(j)}^{j-1} \frac{1}{1-R_0(m,\epsilon)}\right), \tag{2.20}$$

so, combining (2.19) and (2.20), we find

$$x(n) \ - \ x(\tau(n)) \ + \ x(n) \sum_{i=\tau(n)}^{n-1} p(i) \exp\left(\sum_{j=\tau(i)}^{n-1} p(j) \prod_{m=\tau(j)}^{j-1} \frac{1}{1-R_0(m,\epsilon)}\right) \ \leq \ 0.$$

Multiplying the last inequality by $p(n)$, we get

$$p(n)x(n) - p(n)x(\tau(n)) + p(n)x(n) \sum_{i=\tau(n)}^{n-1} p(i) \exp\left(\sum_{j=\tau(i)}^{n-1} p(j) \prod_{m=\tau(j)}^{j-1} \frac{1}{1-R_0(m,\epsilon)}\right) < 0,$$

which, in view of (E), becomes

$$\Delta x(n) + p(n)x(n) + p(n)x(n) \sum_{i=\tau(n)}^{n-1} p(i) \exp\left(\sum_{j=\tau(i)}^{n-1} p(j) \prod_{m=\tau(j)}^{j-1} \frac{1}{1-R_0(m,\epsilon)}\right) < 0,$$

i.e.,

$$\Delta x(n) + p(n) \left[1 + \sum_{i=\tau(n)}^{n-1} p(i) \exp\left(\sum_{j=\tau(i)}^{n-1} p(j) \prod_{m=\tau(j)}^{j-1} \frac{1}{1-R_0(m,\epsilon)}\right)\right] x(n) \ < \ 0.$$

Therefore,

$$\Delta x(n) + R_1(n,\epsilon)x(n) < 0, \tag{2.21}$$

where

$$R_1(n,\epsilon) = p(n)\left[1 + \sum_{i=\tau(n)}^{n-1} p(i)\exp\left(\sum_{j=\tau(i)}^{n-1} p(j)\prod_{m=\tau(j)}^{j-1}\frac{1}{1-R_0(m,\epsilon)}\right)\right].$$

Repeating the above argument leads to a new estimate

$$\Delta x(n) + R_2(n,\epsilon)x(n) < 0,$$

where

$$R_2(n,\epsilon) = p(n)\left[1 + \sum_{i=\tau(n)}^{n-1} p(i)\exp\left(\sum_{j=\tau(i)}^{n-1} p(j)\prod_{m=\tau(j)}^{j-1}\frac{1}{1-R_1(m,\epsilon)}\right)\right].$$

Continuing by induction, we get, for sufficiently large n,

$$\Delta x(n) + R_\ell(n,\epsilon)x(n) \le 0, \tag{2.22}$$

where

$$R_\ell(n,\epsilon) = p(n)\left[1 + \sum_{i=\tau(n)}^{n-1} p(i)\exp\left(\sum_{j=\tau(i)}^{n-1} p(j)\prod_{m=\tau(j)}^{j-1}\frac{1}{1-R_{\ell-1}(m,\epsilon)}\right)\right]$$

and

$$x(\tau(i)) > x(h(n))\exp\left(\sum_{j=\tau(i)}^{h(n)-1} p(j)\prod_{m=\tau(j)}^{j-1}\frac{1}{1-R_\ell(m,\epsilon)}\right). \tag{2.23}$$

Summing up (E) from $h(n)$ to n, we have

$$x(n+1) - x(h(n)) + \sum_{i=h(n)}^{n} p(i)x(\tau(i)) = 0. \tag{2.24}$$

Combining (2.23) and (2.24), we have, for all sufficiently large n,

$$\begin{aligned} &x(n+1) - x(h(n)) \\ &\quad + x(h(n))\sum_{i=h(n)}^{n} p(i)\exp\left(\sum_{j=\tau(i)}^{h(n)-1} p(j)\prod_{m=\tau(j)}^{j-1}\frac{1}{1-R_\ell(m,\epsilon)}\right) < 0. \end{aligned} \tag{2.25}$$

The inequality is valid if we omit $x(n+1)>0$ in the left-hand side:

$$-x(h(n)) + x(h(n)) \sum_{i=h(n)}^{n} p(i) \exp\left(\sum_{j=\tau(i)}^{h(n)-1} p(j) \prod_{m=\tau(j)}^{j-1} \frac{1}{1-R_\ell(m,\epsilon)} \right) < 0.$$

Thus, as $x(h(n))>0$, for all sufficiently large n, it holds

$$\sum_{i=h(n)}^{n} p(i) \exp\left(\sum_{j=\tau(i)}^{h(n)-1} p(j) \prod_{m=\tau(j)}^{j-1} \frac{1}{1-R_\ell(m,\epsilon)} \right) < 1$$

for all sufficiently large n, from which, by letting $n\to\infty$, we have

$$\limsup_{n\to\infty} \sum_{i=h(n)}^{n} p(i) \exp\left(\sum_{j=\tau(i)}^{h(n)-1} p(j) \prod_{m=\tau(j)}^{j-1} \frac{1}{1-R_\ell(m,\epsilon)} \right) \le 1.$$

Since ϵ may be taken arbitrarily small, this inequality contradicts (2.6).

The proof of the theorem is complete. □

Theorem 2.6. *Assume that* (1.1) *and* (2.1) *hold,* $h(n)$ *is defined by* (1.6) *and* $0<\alpha\le 1/e$. *If for some* $\ell\in\mathbb{N}$

$$\limsup_{n\to\infty} \sum_{i=h(n)}^{n} p(i) \exp\left(\sum_{j=\tau(i)}^{h(n)-1} p(j) \prod_{m=\tau(j)}^{j-1} \frac{1}{1-R_\ell(m)} \right) > 1-D(\alpha), \tag{2.26}$$

where $R_\ell(n)$ *is defined by* (2.7)*, then all solutions of* (E) *are oscillatory.*

Proof. Assume, for the sake of contradiction, that $(x(n))_{n\ge -w}$ is an eventually positive solution of (E). Then, as in the proof of Theorem 2.1, for sufficiently large n, (2.25) is satisfied, i.e.,

$$x(n+1) - x(h(n)) + x(h(n)) \sum_{i=h(n)}^{n} p(i) \exp\left(\sum_{j=\tau(i)}^{h(n)-1} p(j) \prod_{m=\tau(j)}^{j-1} \frac{1}{1-R_\ell(m,\epsilon)} \right) < 0.$$

Therefore

$$\sum_{i=h(n)}^{n} p(i) \exp\left(\sum_{j=\tau(i)}^{h(n)-1} p(j) \prod_{m=\tau(j)}^{j-1} \frac{1}{1-R_\ell(m,\epsilon)} \right) < 1 - \frac{x(n+1)}{x(h(n))},$$

which gives

$$\limsup_{n\to\infty}\sum_{i=h(n)}^{n}p(i)\exp\left(\sum_{j=\tau(i)}^{h(n)-1}p(j)\prod_{m=\tau(j)}^{j-1}\frac{1}{1-R_\ell(m,\epsilon)}\right)\le 1-\liminf_{n\to\infty}\frac{x(n+1)}{x(h(n))}.$$

By Lemma 2.3, inequality (2.4) holds. So the last inequality leads to

$$\limsup_{n\to\infty}\sum_{i=h(n)}^{n}p(i)\exp\left(\sum_{j=\tau(i)}^{h(n)-1}p(j)\prod_{m=\tau(j)}^{j-1}\frac{1}{1-R_\ell(m,\epsilon)}\right)\le 1-D(\alpha).$$

Since ϵ may be taken arbitrarily small, this inequality contradicts (2.26).

The proof of the theorem is complete. □

Remark 2.7. It is clear that the left-hand sides of both conditions (2.6) and (2.26) are identical, also the right-hand side of condition (2.26) reduces to (2.6) in case that $\alpha = 0$. So it seems that Theorem 2.6 is the same as Theorem 2.5 when $\alpha = 0$. However, one may notice that condition $0 < \alpha \le 1/e$ is required in Theorem 2.6 but not in Theorem 2.5.

Theorem 2.8. *Assume that* (1.1) *and* (2.1) *hold,* $h(n)$ *is defined by* (1.6) *and* $0 < \alpha \le 1/e$*. If for some* $\ell \in \mathbb{N}$

$$\limsup_{n\to\infty}\sum_{i=h(n)}^{n}p(i)\exp\left(\sum_{j=\tau(i)}^{n}p(j)\prod_{m=\tau(j)}^{j-1}\frac{1}{1-\widetilde{P}_\ell(m)}\right)>\frac{1}{D(\alpha)}-1, \tag{2.27}$$

where $R_\ell(n)$ *is defined by* (2.7)*, then all solutions of* (E) *are oscillatory.*

Proof. Assume, for the sake of contradiction, that $(x(n))_{n\ge -w}$ is an eventually solution of (E). Then, as in the proof of Theorem 2.1, for sufficiently large n, (2.23) is satisfied. Therefore

$$x(\tau(i)) > x(n+1)\exp\left(\sum_{j=\tau(i)}^{n}p(j)\prod_{m=\tau(j)}^{j-1}\frac{1}{1-R_\ell(m,\epsilon)}\right). \tag{2.28}$$

Summing up (E) from $h(n)$ to n, we have

$$x(n+1)-x(h(n))+\sum_{i=h(n)}^{n}p(i)x(\tau(i))=0,$$

which, in view of (2.28), gives

$$x(n+1)-x(h(n)) + \sum_{i=h(n)}^{n}p(i)x(n+1)\exp\left(\sum_{j=\tau(i)}^{n}p(j)\prod_{m=\tau(j)}^{j-1}\frac{1}{1-R_\ell(m,\epsilon)}\right)<0,$$

or

$$x(n+1) - x(h(n)) \\ + x(h(n)) \sum_{i=h(n)}^{n} p(i) \frac{x(n+1)}{x(h(n))} \exp\left(\sum_{j=\tau(i)}^{n} p(j) \prod_{m=\tau(j)}^{j-1} \frac{1}{1 - R_\ell(m,\epsilon)} \right) < 0.$$

Thus, for all sufficiently large n it holds

$$\sum_{i=h(n)}^{n} p(i) \exp\left(\sum_{j=\tau(i)}^{n} p(j) \prod_{m=\tau(j)}^{j-1} \frac{1}{1 - R_\ell(m,\epsilon)} \right) < \frac{x(h(n))}{x(n+1)} - 1.$$

Letting $n \to \infty$, we take

$$\limsup_{n\to\infty} \sum_{i=h(n)}^{n} p(i) \exp\left(\sum_{j=\tau(i)}^{n} p(j) \prod_{m=\tau(j)}^{j-1} \frac{1}{1 - R_\ell(m,\epsilon)} \right) \le \limsup_{n\to\infty} \frac{x(h(n))}{x(n+1)} - 1,$$

which, in view of (2.4), gives

$$\limsup_{n\to\infty} \sum_{i=h(n)}^{n} p(i) \exp\left(\sum_{j=\tau(i)}^{n} p(j) \prod_{m=\tau(j)}^{j-1} \frac{1}{1 - R_\ell(m,\epsilon)} \right) \le \frac{1}{D(\alpha)} - 1.$$

Since ϵ may be taken arbitrarily small, this inequality contradicts (2.27).

The proof of the theorem is complete. □

Remark 2.9. If $R_\ell(n,\epsilon) \ge 1$ then (2.22) guarantees that all solutions of (E) are oscillatory. In fact, (2.22) gives

$$\Delta x(n) + x(n) \le 0$$

which means that $x(n+1) \le 0$. This contradicts $x(n) > 0$ for all $n \ge n_1$. Thus, in Theorems 2.5, 2.6 and 2.8 we consider only the case $R_\ell(n) < 1$. Another conclusion, that can be draw from the above, is that if at some point through the iterative process, we get a value of ℓ, for which $R_\ell(n) \ge 1$, then the process terminates, since in any case, all solutions of (E) will be oscillatory. The value of ℓ, that is the number of iterations, obviously, depends on the coefficient $p(n)$ and the form of the nonmonotone argument $\tau(n)$.

2.2 Advanced Difference Equations

Similar oscillation theorems for the (dual) advanced difference equation (E′) can be derived easily. The proof of these theorems are omitted, since they are quite similar to the proofs for a retarded equation.

The following simple result is cited to explain why we can consider only the case

$$q(n) < \frac{1}{1+\lambda_0 \sum\limits_{i=n+1}^{\sigma(n)} q(i)}, \qquad \forall n \in \mathbb{N} \tag{2.29}$$

where $\lambda_0 > 1$ is the smaller root of the transcendental equation $\lambda = e^{\beta\lambda}$ with $0 < \beta \leq 1/e$.

Theorem 2.10. *Assume that there exists a subsequence $\theta(n)$, $n \in \mathbb{N}$ of positive integers such that*

$$q(\theta(n)) \geq \frac{1}{1+\lambda_0 \sum\limits_{i=\theta(n)+1}^{\sigma(\theta(n))} q(\theta(i))}, \qquad \forall n \in \mathbb{N}. \tag{2.30}$$

Then all solutions of (E$'$) *are oscillatory.*

Theorem 2.11. *Assume that* (1.2) *and* (2.29) *hold, and $\rho(n)$ is defined by* (1.21). *If for some $\ell \in \mathbb{N}$*

$$\limsup_{n\to\infty} \sum_{i=n}^{\rho(n)} q(i) \exp\left(\sum_{j=\rho(n)+1}^{\sigma(i)} q(j) \prod_{m=j+1}^{\sigma(j)} \frac{1}{1-S_\ell(m)}\right) > 1, \tag{2.31}$$

where

$$S_\ell(n) = q(n)\left[1 + \sum_{i=n+1}^{\sigma(n)} q(i) \exp\left(\sum_{j=n+1}^{\sigma(i)} q(j) \prod_{m=j+1}^{\sigma(j)} \frac{1}{1-S_{\ell-1}(m)}\right)\right] \tag{2.32}$$

with $S_0(n) = q(n)\left[1+\lambda_0 \sum\limits_{i=n+1}^{\sigma(n)} q(i)\right]$ *and λ_0 is the smaller root of the transcendental equation $\lambda = e^{\beta\lambda}$, then all solutions of* (E$'$) *are oscillatory.*

Theorem 2.12. *Assume that* (1.2) *and* (2.29) *hold, $\rho(n)$ is defined by* (1.21) *and $0 < \beta \leq 1/e$. If for some $\ell \in \mathbb{N}$*

$$\limsup_{n\to\infty} \sum_{i=n}^{\rho(n)} q(i) \exp\left(\sum_{j=\rho(n)+1}^{\sigma(i)} q(j) \prod_{m=j+1}^{\sigma(j)} \frac{1}{1-S_\ell(m)}\right) > 1 - D(\beta), \tag{2.33}$$

where $S_\ell(n)$ is defined by (2.32)*, then all solutions of* (E$'$) *are oscillatory.*

Remark 2.13. It is clear that the left-hand sides of both conditions (2.31) and (2.33) are identical, also the right hand side of condition (2.33) reduces to (2.31) in case that $\beta = 0$. So it seems that Theorem 2.12 is the same as Theorem 2.11 when $\beta = 0$. However, one may notice that condition $0 < \beta \leq 1/e$ is required in Theorem 2.12 but not in Theorem 2.11.

Theorem 2.14. *Assume that* (1.2) *and* (2.29) *hold,* $\rho(n)$ *is defined by* (1.21) *and* $0 < \beta \le 1/e$. *If for some* $\ell \in \mathbb{N}$

$$\limsup_{n\to\infty} \sum_{i=n}^{\rho(n)} q(i) \exp\left(\sum_{j=n}^{\sigma(i)} q(j) \prod_{m=j+1}^{\sigma(j)} \frac{1}{1-S_\ell(m)} \right) > \frac{1}{D(\beta)} - 1, \tag{2.34}$$

where $S_\ell(n)$ *is defined by* (2.32)*, then all solutions of* (E$'$) *are oscillatory.*

Remark 2.15. Similar comments as those in Remark 2.9, can be made for Theorems 2.11, 2.12 and 2.14, concerning equation (E$'$).

2.3 Difference Inequalities

A slight modification in the proofs of Theorems 2.5–2.8 and 2.11–2.14 leads to the following results about deviating difference inequalities.

Theorem 2.16. *Assume that all conditions of Theorem 2.5 [2.11] or 2.6 [2.12] or 2.8 [2.14] hold. Then*

(i) the retarded [advanced] difference inequality

$$\Delta x(n) + p(n)x(\tau(n)) \le 0,\ n \in \mathbb{N}_0\ \left[\nabla x(n) - q(n)x(\sigma(n)) \ge 0,\ n \in \mathbb{N}\right],$$

has no eventually positive solutions;

(ii) the retarded [advanced] difference inequality

$$\Delta x(n) + p(n)x(\tau(n)) \ge 0,\ n \in \mathbb{N}_0\ \left[\nabla x(n) - q(n)x(\sigma(n)) \le 0,\ n \in \mathbb{N}\right],$$

has no eventually negative solutions.

3 Examples and Comments

In this section, examples illustrate cases when the results of the present paper imply oscillation while previously known results fail. The examples not only illustrate the significance of main results, but also serve to indicate the high degree of improvement, compared to the previous oscillation criteria in the literature. All the calculations were made in Matlab.

Example 3.1. Consider the retarded difference equation

$$\Delta x(n) + \frac{31}{250} x\left(\tau(n)\right) = 0, \quad n \in \mathbb{N}_0, \tag{3.1}$$

with (see Fig. 1, (a))

$$\tau(n) = \begin{cases} n-2, & \text{if } n = 5\mu \\ n-1, & \text{if } n = 5\mu+1 \\ n-5, & \text{if } n = 5\mu+2 \\ n-2, & \text{if } n = 5\mu+3 \\ n-4, & \text{if } n = 5\mu+4 \end{cases}$$

where $\mu \in \mathbb{N}_0$ and $\mathbb{N}_0$ is the set of nonnegative integers.

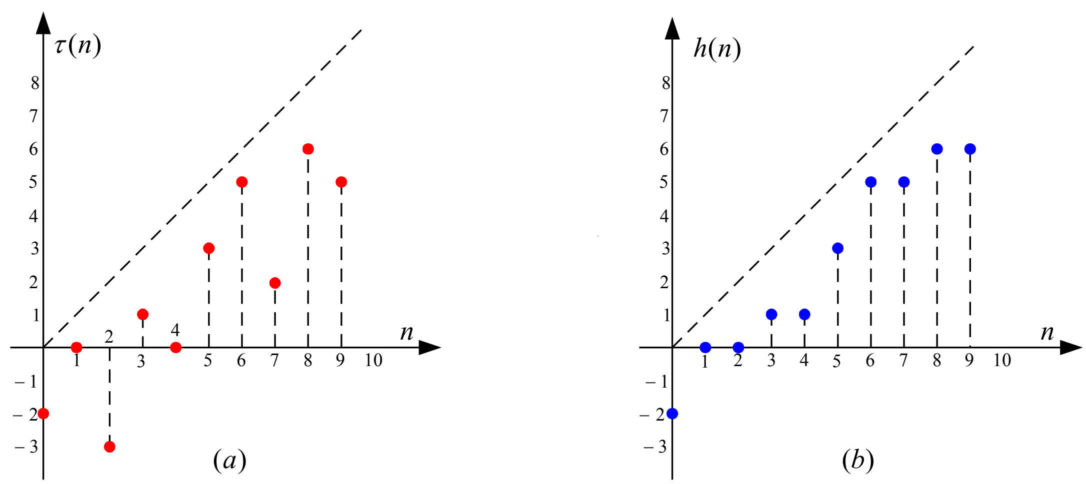

Figure 3.1: The graphs of $\tau(n)$ and $h(n)$

By (1.6), we see (Fig. 3.1, (b)) that

$$h(n) = \begin{cases} n-2, & \text{if } n = 5\mu \\ n-1, & \text{if } n = 5\mu+1 \\ n-2, & \text{if } n = 5\mu+2 \\ n-2, & \text{if } n = 5\mu+3 \\ n-3, & \text{if } n = 5\mu+4 \end{cases}.$$

It is easy to see that

$$\alpha = \liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} p(j) = \liminf_{\mu\to\infty} \sum_{j=5\mu}^{5\mu} \frac{31}{250} = 0.124$$

and therefore, the smaller root of $e^{0.124\lambda} = \lambda$ is $\lambda_0 = 1.15381$.

Clearly,

$$\min_{n\geq 0} \frac{1}{1+\lambda_0 \sum\limits_{i=\tau(n)}^{n-1} p(i)} = \frac{1}{1+\lambda_0 \sum\limits_{i=5\mu-3}^{5\mu+1} p(i)} \simeq 0.583$$

and therefore, (2.1) is satisfied for all $n \geq 0$.

Observe that the function $F_\ell : \mathbb{N}_0 \to \mathbb{R}_+$ defined as

$$F_\ell(n) = \sum_{i=h(n)}^{n} p(i) \exp\left(\sum_{j=\tau(i)}^{h(n)-1} p(j) \prod_{m=\tau(j)}^{j-1} \frac{1}{1 - R_\ell(m)} \right)$$

attains its maximum at $n = 5\mu + 4$, $\mu \in \mathbb{N}_0$, for every $\ell \in \mathbb{N}$. Specifically,

$$F_1(5\mu + 4) = \sum_{i=5\mu+1}^{5\mu+4} p(i) \exp\left(\sum_{j=\tau(i)}^{5\mu} p(j) \prod_{m=\tau(j)}^{j-1} \frac{1}{1 - R_1(m)} \right),$$

where

$$R_1(m) = p(m) \left[1 + \sum_{k=\tau(m)}^{m-1} p(k) \exp\left(\sum_{u=\tau(k)}^{m-1} p(u) \prod_{v=\tau(u)}^{u-1} \frac{1}{1 - R_0(v)} \right) \right]$$

with

$$R_0(v) = p(v) \left[1 + \lambda_0 \sum_{\varphi=\tau(v)}^{v-1} p(\varphi) \right].$$

By using an algorithm on MATLAB software, we obtain

$$F_1(5\mu + 4) \simeq 1.006$$

and therefore

$$\limsup_{n \to \infty} F_1(n) \simeq 1.006 > 1.$$

That is, condition (2.6) of Theorem 2.5 is satisfied for $\ell = 1$. Therefore, all solutions of equation (3.1) are oscillatory.

Observe, however, that

$$MD = \limsup_{\mu \to \infty} \sum_{i=5\mu+1}^{5\mu+4} p(j) = 4 \cdot \frac{31}{250} = 0.496 < 1,$$

$$\alpha = 0.124 < \frac{1}{e},$$

$$0.496 < 1 - D(\alpha) \simeq 0.9911,$$

$$
\begin{aligned}
& \limsup_{n\to\infty} \sum_{j=h(n)}^{n} p(j) \prod_{i=\tau(j)}^{h(n)-1} \frac{1}{1-p(i)} \\
= \quad & \limsup_{\mu\to\infty} \sum_{j=5\mu+1}^{5\mu+4} \frac{31}{250} \prod_{i=\tau(j)}^{5\mu} \frac{1}{1-\frac{31}{250}} \\
= \quad & \frac{31}{250} \cdot \ \limsup_{\mu\to\infty} \left\{ \begin{array}{l} \displaystyle \prod_{i=\tau(5\mu+1)}^{5\mu} \frac{1}{1-\frac{31}{250}} + \prod_{i=\tau(5\mu+2)}^{5\mu} \frac{1}{1-\frac{31}{250}} \\ \displaystyle + \prod_{i=\tau(5\mu+3)}^{5\mu} \frac{1}{1-\frac{31}{250}} + \prod_{i=\tau(5\mu+4)}^{5\mu} \frac{1}{1-\frac{31}{250}} \end{array} \right\} \\
= \quad & \frac{31}{250} \cdot \ \limsup_{\mu\to\infty} \left\{ \begin{array}{l} \displaystyle \prod_{i=5\mu}^{5\mu} \frac{1}{1-\frac{31}{250}} + \prod_{i=5\mu-3}^{5\mu} \frac{1}{1-\frac{31}{250}} \\ \displaystyle + \prod_{i=5\mu+1}^{5\mu} \frac{1}{1-\frac{31}{250}} + \prod_{i=5\mu}^{5\mu} \frac{1}{1-\frac{31}{250}} \end{array} \right\} \\
= \quad & \frac{31}{250} \cdot \ \left\{ \frac{1}{1-\frac{31}{250}} + \left(\frac{1}{1-\frac{31}{250}} \right)^4 + 1 + \frac{1}{1-\frac{31}{250}} \right\} \simeq 0.6177 < 1,
\end{aligned}
$$

$$0.6177 < 1 - D(\alpha) \simeq 0.9911,$$

$$\limsup_{n\to\infty} \sum_{i=h(n)}^{n} p(i) \prod_{j=\tau(i)}^{h(n)-1} \frac{1}{1-p_1(j)} \simeq 0.7259,$$

$$0.7259 < 1 - D(\alpha) \simeq 0.9911,$$

$$\limsup_{n\to\infty} \sum_{i=h(n)}^{n} p(i) \prod_{j=\tau(i)}^{h(n)-1} \frac{1}{1-P_1(j)} \simeq 0.8569,$$

$$0.8569 < 1 - D(\alpha) \simeq 0.9911,$$

$$\limsup_{n\to\infty} \sum_{i=h(n)}^{n} p(i) \prod_{j=\tau(i)}^{n} \frac{1}{1-P_1(j)} \simeq 2.7809 < \frac{1}{D(\alpha)} \simeq 112.7906.$$

$$\limsup_{n\to\infty} \sum_{i=h(n)}^{n} p(i) \exp\left(\sum_{j=\tau(i)}^{h(n)-1} p(j) \prod_{m=\tau(j)}^{j-1} \frac{1}{1-\widetilde{P}_1(m)} \right) \simeq 0.9219$$

$$0.9219 < 1 - D(\alpha) \simeq 0.9911,$$

$$\begin{aligned}
&\limsup_{n\to\infty}\sum_{i=h(n)}^{n}p(i)\exp\left(\sum_{j=\tau(i)}^{n}p(j)\prod_{m=\tau(j)}^{j-1}\frac{1}{1-\widetilde{P}_1(m)}\right)\\
\simeq\;& 3.3963<\frac{1}{D(\alpha)}-1\simeq 111.7906.
\end{aligned}$$

That is, none of conditions (1.3), (1.4), (1.5), (1.7)≡(1.9) (for $r=1$), (1.8)≡(1.10) (for $r=1$), (1.11) (for $\ell=1$), (1.12) (for $\ell=1$), (1.13) (for $\ell=1$), (1.14) (for $\ell=1$), (1.15) (for $\ell=1$), (1.16) (for $\ell=1$), (1.17) (for $\ell=1$) and (1.18) (for $\ell=1$) is satisfied.

Notation. It is worth noting that the improvement of condition (2.6) to the corresponding condition (1.3) is significant, approximately 102.82%, if we compare the values on the left-side of these conditions. Also, the improvement compared to conditions (1.7)–(1.9) (for $r=1$), (1.11) (for $\ell=1$), (1.13) (for $\ell=1$) and (1.16) (for $\ell=1$) is very satisfactory, around 62.86%, 38.59%, 17.4% and 9.12%, respectively.

Finally, observe that the conditions (1.9)–(1.10), (1.11)–(1.12), (1.13)–(1.15) and (1.16)–(1.18) do not lead to oscillation for the first iteration. On the contrary, condition (2.6) is satisfied from the first iteration. This means that our condition is better and much faster than (1.9)–(1.10), (1.11)–(1.12), (1.13)–(1.15) and (1.16)–(1.18).

Example 3.2. Consider the advanced difference equation

$$\nabla x(n)-\frac{123}{1000}x\left(\sigma(n)\right)=0,\quad n\in\mathbb{N}, \tag{3.2}$$

with (see Fig. 2, (a))

$$\sigma(n)=\begin{cases} n+1 & \text{if } n=5\mu+1\\ n+6 & \text{if } n=5\mu+2\\ n+2 & \text{if } n=5\mu+3\\ n+3 & \text{if } n=5\mu+4\\ n+1 & \text{if } n=5\mu+5\end{cases},$$

where $\mu\in\mathbb{N}_0$ and $\mathbb{N}_0$ is the set of nonnegative integers.

By (1.21), we see (Fig. 3.2, (b)) that

$$\rho(n)=\begin{cases} n+1, & \text{if } n=5\mu+1\\ n+3, & \text{if } n=5\mu+2\\ n+2, & \text{if } n=5\mu+3\\ n+2, & \text{if } n=5\mu+4\\ n+1, & \text{if } n=5\mu+5\end{cases}.$$

It is easy to see that

$$\beta=\liminf_{n\to\infty}\sum_{j=n+1}^{\sigma(n)}q(j)=\liminf_{\mu\to\infty}\sum_{j=5\mu+2}^{5\mu+2}q(j)=\frac{123}{1000}=0.123$$

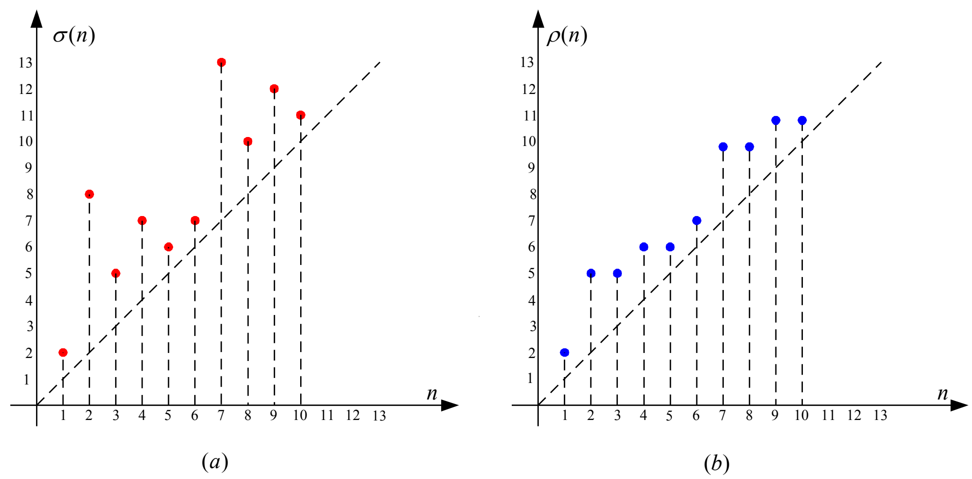

Figure 3.2: The graphs of $\sigma(n)$ and $\rho(n)$

and the smaller root of $e^{0.123\lambda} = \lambda$ is $\lambda_0 = 1.15226$.

Clearly,

$$\min_{n\geq 1} \frac{1}{1+\lambda_0 \sum\limits_{i=n+1}^{\sigma(n)} q(i)} = \frac{1}{1+\lambda_0 \sum\limits_{i=5\mu+3}^{5\mu+8} q(i)} \simeq 0.5404,$$

that is, (2.29) is satisfied for all $n \geq 1$.

Observe that the function $F_\ell : \mathbb{N}_0 \to \mathbb{R}_+$ defined as

$$F_\ell(n) = \sum_{i=n}^{\rho(n)} q(i) \exp\left(\sum_{j=\rho(n)+1}^{\sigma(i)} q(j) \prod_{m=j+1}^{\sigma(j)} \frac{1}{1-S_\ell(m)} \right),$$

attains its maximum at $n = 5\mu + 2$, $\mu \in \mathbb{N}_0$, for every $\ell \in \mathbb{N}$. Specifically,

$$F_1(5\mu+2) = \sum_{i=5\mu+2}^{5\mu+5} q(i) \exp\left(\sum_{j=5\mu+6}^{\sigma(i)} q(j) \prod_{m=j+1}^{\sigma(j)} \frac{1}{1-S_1(m)} \right),$$

where

$$S_1(m) = q(m) \left[1 + \sum_{k=m+1}^{\sigma(m)} q(k) \exp\left(\sum_{u=m+1}^{\sigma(k)} q(u) \prod_{v=u+1}^{\sigma(u)} \frac{1}{1-S_0(v)} \right) \right]$$

with

$$S_0(v) = q(v) \left[1 + \lambda_0 \sum_{\varphi=v+1}^{\sigma(v)} q(\varphi) \right].$$

By using an algorithm on MATLAB software, we obtain

$$F_1(5\mu+2) \simeq 1.004$$

and therefore

$$\limsup_{n\to\infty} F_1(n) \simeq 1.004 > 1.$$

That is, condition (2.31) of Theorem 2.11 is satisfied for $\ell = 1$. Therefore, all solutions of equation (3.2) are oscillatory.

Observe, however, that

$$MA = \limsup_{n\to\infty} \sum_{j=n}^{\rho(n)} q(j) = \limsup_{\mu\to\infty} \sum_{j=5\mu+2}^{5\mu+5} q(j) = 0.492 < 1,$$

$$0.492 < 1 - \left(1 - \sqrt{1-\beta}\right)^2 \simeq 0.996,$$

$$\limsup_{n\to\infty} \sum_{j=n}^{\rho(n)} q(j) b_1^{-1}(\rho(n), \sigma(j)) = \limsup_{\mu\to\infty} \sum_{j=5\mu+2}^{5\mu+5} q(j) b_1^{-1}(5\mu+5, \sigma(j))$$

$$\begin{aligned}
&= \frac{123}{1000} \cdot \limsup_{\mu\to\infty} \left[\begin{array}{c} b_1^{-1}(5\mu+5, \sigma(5\mu+2)) + b_1^{-1}(5\mu+5, \sigma(5\mu+3)) \\ +b_1^{-1}(5\mu+5, \sigma(5\mu+4)) + b_1^{-1}(5\mu+5, \sigma(5\mu+5)) \end{array} \right] \\
&= \frac{123}{1000} \cdot \limsup_{\mu\to\infty} \left[\begin{array}{c} b_1^{-1}(5\mu+5, 5\mu+8) + b_1^{-1}(5\mu+5, 5\mu+5) \\ +b_1^{-1}(5\mu+5, 5\mu+7) + b_1^{-1}(5\mu+5, 5\mu+6) \end{array} \right] \\
&= \frac{123}{1000} \cdot \left[\frac{1}{\left(1-\frac{123}{1000}\right)^4} + \frac{1}{1-\frac{123}{1000}} + \frac{1}{\left(1-\frac{123}{1000}\right)^3} + \frac{1}{\left(1-\frac{123}{1000}\right)^2} \right] \simeq 0.6904 < 1,
\end{aligned}$$

$$0.6904 < 1 - D(\beta) \simeq 0.9913,$$

$$\limsup_{n\to\infty} \sum_{i=n}^{\rho(n)} q(i) \prod_{j=\rho(n)+1}^{\sigma(i)} \frac{1}{1-q_1(j)} \simeq 0.7045 < 1,$$

$$0.7045 < 1 - D(\beta) \simeq 0.9913.$$

$$\limsup_{n\to\infty} \sum_{i=n}^{\rho(n)} q(i) \exp\left(\sum_{j=\rho(n)+1}^{\sigma(i)} q(j) \prod_{m=j+1}^{\sigma(j)} \frac{1}{1-\widetilde{Q}_1(m)} \right) \simeq 0.9165 < 1,$$

$$0.9165 < 1 - D\left(\beta\right) \simeq 0.9913,$$

$$\begin{aligned}
&\limsup_{n\to\infty} \sum_{i=n}^{\rho(n)} q(i) \exp\left(\sum_{j=n}^{\sigma(i)} q(j) \prod_{m=j+1}^{\sigma(j)} \frac{1}{1-\widetilde{Q}_1(m)} \right) \\
\simeq\ & 3.3513 < \frac{1}{D(\beta)} - 1 \simeq 113.7846,
\end{aligned}$$

That is, none of conditions (1.19), (1.20), (1.22) (for $r = 1$), (1.23) (for $r = 1$), (1.25) (for $\ell = 1$), (1.26) (for $\ell = 1$), (1.27) (for $\ell = 1$), (1.28) (for $\ell = 1$) and (1.29) (for $\ell = 1$) is satisfied.

Notation. It is worth noting that the improvement of condition (2.31) to the corresponding condition (1.19) is significant, approximately 104.07%, if we compare the values on the left-side of these conditions. Also, the improvement compared to conditions (1.22) (for $r = 1$), (1.25) (for $\ell = 1$) and (1.27) (for $\ell = 1$) is very satisfactory, around 45.42%, 42.51% and 9.58%, respectively.

Finally, observe that the conditions (1.22)–(1.23), (1.25)–(1.26) and (1.27)–(1.29) do not lead to oscillation for the first iteration. On the contrary, condition (2.31) is satisfied from the first iteration. This means that our condition is better and much faster than (1.22)–(1.23), (1.25)–(1.26) and (1.27)–(1.29).

Remark 3.3. Similarly, one can construct examples, illustrating the other main results, in the paper.

References

[1] P. G. Asteris and G. E. Chatzarakis. New oscillation tests for difference equations with non-monotone arguments. *DCDIS Series A: Mathematical Analysis*, 24:287–302, 2017.

[2] E. Braverman, G. E. Chatzarakis, and I. P. Stavroulakis. Iterative oscillation tests for difference equations with several non-monotone arguments. *J. Difference Equ. Appl.*, 21(9):854–874, 2015.

[3] E. Braverman and B. Karpuz. On oscillation of differential and difference equations with non-monotone delays. *Appl. Math. Comput.*, 218(7):3880–3887, 2011.

[4] G. E. Chatzarakis and I. Jadlovská. Oscillations in deviating difference equations using an iterative technique. *J. Inequal. Appl.*, 2017(1):173, 2017.

[5] G. E. Chatzarakis, R. Koplatadze, and I. P. Stavroulakis. Optimal oscillation criteria for first order difference equations with delay argument. *Pacific J. Math.*, 235(1):15–33, 2008.

[6] G. E. Chatzarakis, R. Koplatadze, and I. P. Stavroulakis. Oscillation criteria of first order linear difference equations with delay argument. *Nonlinear Anal.*, 68(4):994–1005, 2008.

[7] G. E. Chatzarakis, Ch. G. Philos, and I. P. Stavroulakis. Oscillations of first order linear difference equations with general delay argument. *Portugal. Math.*, 66:513–533, 2009.

[8] G. E. Chatzarakis, I. K. Purnaras, and P. I Stavroulakis. Oscillations of deviating difference equations with non-monotone arguments. *J. Difference Equ. Appl.*, 2017 (in press).

[9] G. E. Chatzarakis and L. Shaikhet. Oscillation criteria for difference equations of non-monotone arguments. *Mediterr. J. Math.*, 14(2), 2017.

[10] G. E. Chatzarakis and I. P. Stavroulakis. Oscillations of difference equations with general advanced argument. *Cent. Eur. J. Math.*, 10(2):807–823, 2012.

[11] M.-P. Chen and J. S. Yu. Oscillations of delay difference equations with variable coefficients. In *Proceedings of the First International Conference on Difference Equations (San Antonio, TX, 1994)*, pages 105–114. Gordon and Breach, Luxembourg, 1995.

[12] L. H. Erbe and B. G. Zhang. Oscillation of discrete analogues of delay equations. *Differential Integral Equations*, 2(3):300–309, 1989.

[13] I. Győri and G. Ladas. Linearized oscillations for equations with piecewise constant arguments. *Differential Integral Equations*, 2(2):123–131, 1989.

[14] G. Ladas. Explicit conditions for the oscillation of difference equations. *J. Math. Anal. Appl.*, 153(1):276–287, 1990.

[15] G. Ladas, Ch. G. Philos, and Y. G. Sficas. Sharp conditions for the oscillation of delay difference equations. *J. Appl. Math. Simulation*, 2(2):101–111, 1989.

[16] X. Li and D. Zhu. Oscillation of advanced difference equations with variable coefficients. *Ann. Differential Equations*, 18(3):254–263, 2002.

[17] I. P. Stavroulakis. Oscillation criteria for delay and difference equations with non-monotone arguments. *Appl. Math. Comput.*, 226:661–672, 2014.

[18] X. H. Tang and J. S. Yu. Oscillation of delay difference equation. *Comput. Math. Appl.*, 37(7):11–20, 1999.

[19] X. H. Tang and R. Y. Zhang. New oscillation criteria for delay difference equations. *Comput. Math. Appl.*, 42(10-11):1319–1330, 2001.

[20] W. Yan, Q. Meng, and J. Yan. Oscillation criteria for difference equation of variable delays. *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.*, 13A(Part 2, suppl.):641–647, 2006.

[21] B. G. Zhang and Ch.-J. Tian. Nonexistence and existence of positive solutions for difference equations with unbounded delay. *Comput. Math. Appl.*, 36(1):1–8, 1998.