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Global Asymptotic Behavior of Some Quadratic Rational Second-Order Difference Equations

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Abstract

The aim of this paper is to describe global behavior of three special cases of the following difference equation

$$x_{n+1} = \frac{\alpha x_n^2 + \beta x_n x_{n-1} + \gamma x_{n-1}}{A x_n^2 + B x_n x_{n-1} + C x_{n-1}},\ n \in \mathbb{N}_0$$

that have two members in the numerator and one member in the denominator with positive initial conditions and positive parameters. Two of them have a unique equilibrium point in corresponding parametric space and one of them possesses infinite number of equilibrium points of non-hyperbolic type. Each equation exhibits completely different dynamics.

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1 Introduction

We investigate global behavior of the following difference equations:

$$x_{n+1} = \frac{\beta x_n x_{n-1} + \gamma x_{n-1}}{A x_n^2}, \tag{1.1}$$

$$x_{n+1} = \frac{\alpha x_n^2 + \beta x_n x_{n-1}}{C x_{n-1}}, \tag{1.2}$$

$$x_{n+1} = \frac{\alpha x_n^2 + \gamma x_{n-1}}{C x_{n-1}}, \tag{1.3}$$

where the parameters $\alpha, \beta, \gamma, A, B, C$ are positive numbers and the initial conditions x_{-1}, x_0 are arbitrary positive numbers. Equations (1.1), (1.2), and (1.3) are a special cases of

$$x_{n+1} = \frac{\alpha x_n^2 + \beta x_n x_{n-1} + \gamma x_{n-1}}{A x_n^2 + B x_n x_{n-1} + C x_{n-1}}, \quad n \in \mathbb{N}_0$$

and

$$x_{n+1} = \frac{A x_n^2 + B x_n x_{n-1} + C x_{n-1}^2 + D x_n + E x_{n-1} + F}{a x_n^2 + b x_n x_{n-1} + c x_{n-1}^2 + d x_n + e x_{n-1} + f}, \quad n \in \mathbb{N}_0. \tag{1.4}$$

Some special cases of (1.4) have been considered in the series of papers [1,2,6,7]. Some special second-order quadratic fractional difference equations have appeared in analysis of competitive and anti-competitive systems of linear rational difference equations in the plane, see [3–5, 10, 11]. Describing the global dynamics of (1.4) is a formidable task as this equation contains as a special cases many equations with complicated dynamics, such as the linear rational difference equation

$$x_{n+1} = \frac{D x_n + E x_{n-1} + F}{d x_n + e x_{n-1} + f}, \quad n \in \mathbb{N}_0.$$

2 Preliminaries

We now state two well-known results that we use in our paper.

Theorem 2.1 (Linearized Stability Theorem)**.** *Let I be some interval of real numbers and let*

$$f : I \times I \to I$$

be a continuously differentiable function, where

$$x_{n+1} = f(x_n, x_{n-1}). \tag{2.1}$$

(a) *If both roots of the quadratic equation*

$$\lambda^2 - p\lambda - q = 0 \tag{2.2}$$

lie in the open unit disc $|\lambda| < 1$, *then the equilibrium* $\bar{x}$ *of* (2.1) *is locally asymptotically stable.*

(b) *If at least one of the roots of* (2.2) *has absolute value greater than one, then the equilibrium* $\bar{x}$ *of equation* (2.1) *is unstable.*

(c) *A necessary and sufficient condition for both roots of* (2.2) *to lie in the open unit disc* $|\lambda| < 1$, *is*

$$|p| < 1 - q < 2. \tag{2.3}$$

In this case, the locally asymptotically stable equilibrium $\bar{x}$ *is also called a sink.*

(d) *A necessary and sufficient condition for both roots of* (2.2) *to have absolute value greater than one is*

$$|q| > 1 \textit{ and } |p| < |1 - q|\,. \tag{2.4}$$

In this case $\bar{x}$ *is a repeller.*

(e) *A necessary and sufficient condition for one root of* (2.2) *to have absolute value greater than one and for the other to have absolute value less than one is*

$$p^2 + 4q > 0 \textit{ and } |p| > |1 - q|\,. \tag{2.5}$$

In this case, the unstable equilibrium $\bar{x}$ *is called a saddle point.*

(f) *A necessary and sufficient condition for a root of* (2.2) *to have absolute value equal to one is*

$$|p| = |1 - q| \tag{2.6}$$

or

$$q = -1 \textit{ and } |p| \leq 2. \tag{2.7}$$

In this case, the equilibrium $\bar{x}$ *is called a nonhyperbolic point.*

The following result was established in [8].

Theorem 2.2. *Let* $[a, b]$ *be an interval, and suppose that* $f : [a, b] \times [a, b] \to [a, b]$ *is continuous function that has the following properties:*

a. $f(x, y)$ *is nondecreasing in* $x \in [a, b]$ *for each* $y \in [a, b]$, *and* $f(x, y)$ *is nonincreasing in* $y \in [a, b]$ *for each* $x \in [a, b]$;

b. *All solutions* $(m, M) \in [a, b] \times [a, b]$ *of the system*

$$f(x, y) = x \quad \text{and} \quad f(y, x) = y,$$

satisfy $m = M$.

Then $x_{n+1} = f(x_n, x_{n-1})$, $n = 0, 1, \ldots$ *has a unique equilibrium* $\bar{x} \in [a, b]$ *and every solution of this equation that enters* $[a, b]$ *converges to* $\bar{x}$.

The next result is from [9].

Theorem 2.3. *Let* I_1, I_2 *be intervals in* $\mathbb{R}$ *with endpoints* a_1, a_2 *and* b_1, b_2 *respectively, with* $a_1 < a_2$ *and* $b_1 < b_2$. *Let* T *be a competitive map on* $\mathcal{R} = I_1 \times I_2$. *Let* $\bar{x} \in \mathcal{R}^\circ$. *Suppose that the following hypotheses are satisfied:*

1. $\mathcal{R}^\circ$ *is an invariant set, and* T *is strongly competitive on* $\mathcal{R}^\circ$.
2. *The point* x *is the only fixed point of* T *in* $(Q_1(x) \cup Q_3(x)) \cap \mathcal{R}^\circ$.
3. *The map* T *is continuously differentiable in a neighborhood of* x, *and* x *is a saddle point.*
4. *At least one of the following statements is true.*
   a) T *has no prime period two orbits in* $(Q_1(x) \cup Q_3(x)) \cap \mathcal{R}^\circ$.
   b) *det* $J_T(x) > 0$ *and* $T(x) = x$ *only for* $x = \bar{x}$.

Then the following statements are true.

(i.) *The stable set* $W^s(x)$ *is connected and it is the graph of a continuous increasing curve with endpoints in* $\partial\mathcal{R}$. $\mathcal{R}^\circ$ *is divided by the closure of* $W^s(x)$ *into two invariant connected regions* $W+$ *("below the stable set"), and* W- *("above the stable set"), where*

$$W+ := \{x \in \mathcal{R} \backslash W^s(x) : \text{there exists } x\prime \in W^s(x) \text{ such that } x\prime \preceq_s x\}$$

$$W- := \{x \in \mathcal{R} \backslash W^s(x) : \text{there exists } x\prime \in W^s(x) \text{ such that } x \preceq_s x\prime\}$$

where $(x^1, y^1) \preceq_s (x^2, y^2)$ *means that* $x^1 \leq x^2$, $y^1 \geq y^2$.

(ii.) *The unstable set* $W^u(x)$ *is connected and it is the graph of a continuous decreasing curve.*

(iii.) *For every* $x \in W+$, $T^n(x)$ *eventually enters the interior of the invariant set* $Q_4(x) \cap \mathcal{R}$, *and for every* $x \in W-$, $T^n(x)$ *eventually enters the interior of the invariant set* $Q_2(x) \cap \mathcal{R}$.

(iv.) *Let* $m \in Q_2(x)$ *and* $M \in Q_4(x)$ *be the endpoints of* $W^u(x)$, *where* $m \preceq_s x \preceq_s M$. *For every* $x \in W-$ *and every* $z \in \mathcal{R}$ *such that* $m \preceq_s z$, *there exists* $m \in \mathbb{N}$ *such that* $T^m(x) \preceq_s z$, *and for every* $x \in W+$ *and every* $z \in \mathcal{R}$ *such that* $z \preceq_s M$, *there exists* $m \in \mathbb{N}$ *such that* $z \preceq_s Tm(x)$.

3 Equation $x_{n+1} = \dfrac{\beta x_n x_{n-1} + \gamma x_{n-1}}{A x_n^2}$

This equation is easily reduced to

$$x_{n+1} = \frac{x_n x_{n-1} + \alpha x_{n-1}}{x_n^2}, \tag{3.1}$$

where the parameter α and initial conditions x_0 and x_{-1} are positive numbers. It is easy to see that (3.1) has a unique positive equilibrium given by

$$\bar{x} = \frac{1 + \sqrt{1 + 4\alpha}}{2}.$$

We notice that at the point $x = 0$ the right-hand side of (3.1) becomes undetermined. We can say that the point $x = 0$ is an isolated singular point in the sense there is no solution of (3.1) that converges to this point. The partial derivatives associated to (3.1) at equilibrium $\bar{x}$ are

$$f'_x = \left. \frac{-xy - 2\alpha y}{x^3} \right|_{\bar{x}} = -\frac{\bar{x} + 2\alpha}{\bar{x}^2} = p$$

and

$$f'_y = \left. \frac{x + \alpha}{x^2} \right|_{\bar{x}} = \frac{\bar{x} + \alpha}{\bar{x}^2} = q.$$

Partial derivatives show that the function f is decreasing in the first variable and increasing in the second variable. The characteristic equation associated to (3.1) at equilibrium is

$$\lambda^2 - \frac{1 + \sqrt{1 + 4\alpha} - 4\alpha}{1 + \sqrt{1 + 4\alpha} + 2\alpha}\lambda - 1 = 0.$$

We prove the following lemma.

Lemma 3.1. *(i) The unique positive equilibrium of* (3.1) *is a saddle point.*

(ii) Equation (3.1) *has no period two solution.*

Proof. (i) It is easy to see that both conditions for saddle point are satisfied. Namely, obviously $p^2 + 4q > 0$. On the other hand, condition $|p| > |1 - q|$ becomes

$$\frac{\bar{x} + 2\alpha}{\bar{x}^2} > \left| \frac{\bar{x}^2 - \bar{x} - \alpha}{\bar{x}^2} \right|,$$

which is equivalent to

$$\frac{\bar{x} + 2\alpha}{\bar{x}^2} > \frac{\bar{x}}{\bar{x}^2}.$$

This implies that

$$\bar{x} + 2\alpha > \bar{x},$$

from which the proof follows.

(ii) Let $\{(\phi, \psi), (\psi, \phi)\}$ is prime period-two solution. Two-periodic solution is a positive solution of the system

$$\begin{cases} -y - \alpha = 0 \\ -xy + y = 0, \end{cases} \tag{3.2}$$

where $\phi + \psi = x$ and $\phi\psi = y$. System (3.2) has a unique solution

$$x = 1 \text{ and } y = -\alpha.$$

Since $y = -\alpha < 0$, (3.1) has no prime period-two solution.

This completes the proof. □

Set $\mathcal{R} = (0, \infty) \times (0, \infty)$. The map T associated to (3.1) is given by

$$T(u, v) = \left(v, \frac{uv + \alpha u}{v^2}\right).$$

This map T is not strongly competitive, but the second iteration of the map T, that is, the map

$$T^2(u, v) = \left(\frac{uv + \alpha u}{v^2}, \frac{v^3(uv + \alpha u + \alpha v^2)}{(uv + \alpha u)^2}\right)$$

is strongly competitive. Namely, the Jacobian matrix of the T^2 is given by

$$J_{T^2} = \begin{pmatrix} \frac{\alpha + v}{v^2} & -\frac{u(2\alpha + v)}{v^3} \\ -\frac{v^3(2\alpha v^2 + u(\alpha + v))}{u^3(\alpha + v)^2} & \frac{v^2(\alpha v^2(3v + 5\alpha) + u(2v^2 + 5\alpha v + 3\alpha^2))}{u^2(\alpha + v)^3} \end{pmatrix}.$$

The determinant associated to the Jacobian of the map T^2 is

$$\det J_{T^2} = \frac{\alpha v^2 + u(v + \alpha)}{u^2(v + \alpha)}, \text{ for all } (u, v) \in \mathcal{R},$$

which is always positive.

The following theorem describes global behavior of the solution of (3.1).

Theorem 3.2. *There is a global stable manifold $W^s(\bar{x}, \bar{x})$ which is continuous increasing curve with endpoints in boundary of $\mathcal{R}$, divides the first quadrant such that the following holds:*

i) Every initial point (u_0, v_0) *in* $W^s(\bar{x}, \bar{x})$ *is attracted to* $(\bar{x}, \bar{x})$.

ii) If $(u_0, v_0) \in W^+(\bar{x}, \bar{x})$ *(the region below* $W^s(\bar{x}, \bar{x})$*), then the subsequence of even-indexed terms* $\{(u_{2n}, v_{2n})\}$ *tends to zero and the subsequence of odd-indexed terms* $\{(u_{2n+1}, v_{2n+1})\}$ *tends to infinity.*

iii) If $(u_0, v_0) \in W^-(\bar{x}, \bar{x})$ *(the region above* $W^s(\bar{x}, \bar{x})$*), then the subsequence of even-indexed terms* $\{(u_{2n}, v_{2n})\}$ *tends to infinity and the subsequence of odd-indexed terms* $\{(u_{2n+1}, v_{2n+1})\}$ *tends to zero.*

Proof. Lemma 3.1 implies that there exists a unique equilibrium point $(\bar{x}, \bar{x})$ which is a saddle point. $\mathcal{R}$ is an invariant set. We have that $\det J_{T^2} > 0$. It is easy to see that $T(x, y) = (\bar{x}, \bar{x})$ only for $(x, y) = (\bar{x}, \bar{x})$. Since the map T is anti-competitive and T^2 is strongly competitive, we have that all conditions of Theorem 2.3 are satisfied from which the proof follows. □

4 Equation $x_{n+1} = \dfrac{\alpha x_n^2 + \beta x_n x_{n-1}}{C x_{n-1}}$

This equation is easily reduced to

$$x_{n+1} = \frac{\alpha x_n^2 + \beta x_n x_{n-1}}{x_{n-1}}, \tag{4.1}$$

where the parameters α and β and initial conditions x_0 and x_{-1} are positive numbers. Notice that the point $\bar{x} = 0$ is singular point of (4.1), since the right-hand side of (4.1) becomes undetermined of the form $\frac{0}{0}$.

We prove the following lemma.

Lemma 4.1. *If* $\alpha + \beta = 1$, *then* (4.1) *has infinitely many equilibrium points of non-hyperbolic type.*

Proof. The equilibrium is solution of the following equation $\bar{x} = \dfrac{\alpha \bar{x}^2 + \beta \bar{x}^2}{\bar{x}}$, where we assume that $\bar{x} \neq 0$. This implies $\bar{x}^2(1 - \alpha - \beta) = 0$. By using assumption, we have that $\alpha + \beta = 1$ for all values of $\bar{x}$. Now, set $f(x, y) = \dfrac{\alpha x + \beta x y}{y}$. Then, $\dfrac{\partial f}{\partial x} = \dfrac{2\alpha x}{y} + \beta$, and $\dfrac{\partial f}{\partial y} = -\dfrac{\alpha x^2}{y^2}$. This implies that the function $f$is increasing in the first variable and decreasing in the second variable. Denote $p = \dfrac{\partial f}{\partial x}|_{\bar{x}} = 2\alpha + \beta$, and $q = \dfrac{\partial f}{\partial y}|_{\bar{x}} = -\alpha$, then characteristic equation associated to (4.1) is given by

$$\lambda^2 - (1 + \alpha)\lambda + \alpha = 0. \tag{4.2}$$

Equation (4.2) has two real roots: $\lambda = 1$ and $\lambda = \alpha$. This implies that equilibria are non-hyperbolic. It is possible to have $\alpha < 1, \quad \alpha > 1, \quad$ and $\quad \alpha = 1$. We notice the following, if $\alpha = 1$, then both real roots are equal to one. But, in this case $\beta = 0$, and (4.1) becomes $x_{n+1} = \dfrac{{x_n}^2}{x_{n-1}}$, which can be solved directly. Namely, if we take logarithm of both sides, then we have $\ln x_{n+1} - \ln x_n = \ln x_n - \ln x_{n-1}$. Now, we substitute $x_n = e^{y_n}$, then we have $y_{n+1} - 2y_n + y_{n-1} = 0$. General solution of this equation is $y_n = c_1 + nc_2$. By using substitution, the general solution of the beginning equation is $x_n = C_1 e^{c_2 n}$, where $C_1 = e^{c_1}$. From general solution, we obtain that if constant $c_2 < 0$, then $x_n \to 0$; if $c_2 > 0$, then $x_n \to \infty$. □

The following theorem describes the global dynamics of (4.1).

Theorem 4.2. *The following statements hold:*

i) If $\alpha + \beta < 1$, then

$$\lim_{n\to\infty} x_n = 0.$$

ii) If $\alpha + \beta > 1$, then

$$\lim_{n\to\infty} x_n = \infty.$$

iii) If $\alpha + \beta = 1$, then

$$\lim_{n\to\infty} x_n = L.$$

Proof. Firstly, from (4.1), we have

$$\begin{aligned}
\frac{x_{n+1}}{x_n} &= \alpha\frac{x_n}{x_{n-1}} + \beta \\
&= \alpha\left(\alpha\frac{x_{n-1}}{x_{n-2}} + \beta\right) + \beta \\
&= \alpha^2\left(\alpha\frac{x_{n-2}}{x_{n-3}} + \beta\right) + \alpha\beta + \beta \\
&= \ldots \\
&= \beta + \alpha\beta + \alpha^2\beta + \ldots + \alpha^{n+1}\beta + \alpha^{n+2}\frac{x_{-1}}{x_0} \\
&= \beta(1 + \alpha + \alpha^2 + \ldots + \alpha^{n+1}) + \alpha^{n+2}\frac{x_{-1}}{x_0} \\
&< \frac{\beta}{1-\alpha} + \alpha^{n+2}\frac{x_{-1}}{x_0}.
\end{aligned}$$

Let

$$u_n = \frac{x_n}{x_{n-1}},$$

than

$$u_{n+1} = \alpha u_n + \beta$$

and

$$u_n = \begin{cases} C\alpha^n + \dfrac{\beta}{1-\alpha}, \ \alpha \neq 1 \\ u_0 + n\beta, \ \alpha = 1, \end{cases}$$

where $C = u_0 - \dfrac{\beta}{1-\alpha}$. Now we consider the following cases.

i) Let $\alpha + \beta < 1$, then $\dfrac{\beta}{1-\alpha} < 1$. Now we have

$$\lim_{n\to\infty} u_n = \frac{\beta}{1-\alpha} < 1,$$

and

$$u_n \leq \frac{\beta}{1-\alpha} + \varepsilon < 1, \text{ for } n \geq N = N(\varepsilon).$$

So,

$$u_n = \frac{x_n}{x_{n-1}} < 1, \text{ for } n \geq N$$

if and only if

$$x_n < x_{n-1}, \text{ for } n \geq N.$$

The sequence $\{x_n\}$ is decreasing for $n \geq N$, therefore

$$\lim_{n\to\infty} x_n = L \geq 0.$$

It must be $L = 0$, otherwise $L > 0$ and (4.1) would have a positive equilibrium, which is not possible.

ii) Let $\alpha + \beta > 1$, then $\dfrac{\beta}{1-\alpha} > 1$. If $\alpha \neq 1$, then

$$u_n \geq \frac{\beta}{1-\alpha} > 1, \text{ for } n \geq N = N(\varepsilon).$$

So,

$$u_n = \frac{x_n}{x_{n-1}} > 1, \text{ for } n \geq N$$

if and only if

$$x_n > x_{n-1}, \text{ for } n \geq N.$$

The sequence $\{x_n\}$ is increasing for $n \geq N$, therefore

$$\lim_{n\to\infty} x_n = L > 0.$$

It must be $L = \infty$, otherwise $L \in (0, \infty)$ and (4.1) would have a positive equilibrium, which is not possible. If $\alpha = 1$, then $\beta > 0$ and

$$u_n = u_0 + n\beta > 1, \text{ for } n \geq N = N(\varepsilon),$$

from which the proof follows.

iii) Let $\alpha+\beta=1$, then $\dfrac{\beta}{1-\alpha}=1$. It holds

$$u_n=\begin{cases} C\alpha^n+1, & \alpha\neq 1\\ u_0, & \alpha=1\end{cases},$$

where $C=u_0-1$. If $\alpha\neq 1$, than $\alpha<1$ and

$$\lim_{n\to\infty}u_n=1.$$

If $\alpha=1$, then

$$u_n=\frac{x_n}{x_{n-1}}=u_0$$

if and only if

$$x_n=u_0x_{n-1}=u_0^n.$$

□

Remark 4.3. We noticed that the point $\bar{x}$ is singular point. However, from Theorem 4.2, we see that for $\alpha+\beta<1$, each solution of (4.1) converges to this point. In this case, the point $\bar{x}$ is singular but not isolated point.

5 Equation $x_{n+1}=\dfrac{\alpha x_n^2+\gamma x_{n-1}}{Cx_{n-1}}$

This equation is easily reduced to

$$x_{n+1}=\frac{\alpha x_n^2+x_{n-1}}{x_{n-1}}, \tag{5.1}$$

where the parameter α and initial conditions x_0 and x_{-1} are positive numbers. Notice that the point $x=0$ is singular point. Assume that $\bar{x}\neq 0$, then from equation $\bar{x}=\dfrac{\bar{x}^2+\bar{x}}{\bar{x}}$, we obtain that for $\alpha\in(0,\ 1)$, (5.1) has a unique positive equilibrium $\bar{x}=\dfrac{1}{1-\alpha}$.

Remark 5.1. It is easy to see that for $\alpha=1$, (5.1) has no equilibrium point.

Proposition 5.2. *(i) The unique positive equilibrium*

$$\bar{x}=\frac{1}{1-\alpha}$$

of (5.1) *is locally asymptotically stable for all values of* $\alpha\in(0,\ 1)$.

(ii) *Equation* (5.1) *has no prime period-two solution.*

Proof. (i) The partial derivatives associated to (5.1) at equilibrium $\bar{x}$ are

$$f'_x = \left.\frac{2\alpha x}{y}\right|_{\bar{x}} = 2\alpha$$

and

$$f'_y = \left.\frac{-\alpha x^2}{y^2}\right|_{\bar{x}} = -\alpha.$$

We see that the map associated to (5.1) is increasing in the first variable and decreasing in the second variable. The characteristic equation associated to (5.1) at equilibrium is

$$\lambda^2 - 2\alpha\lambda + \alpha = 0.$$

Since $\alpha \in (0,\ 1)$, it is easy to see that both roots of the characteristic equation are real less than one.

(ii) This is a straightforward calculation.

The proof is complete. □

Lemma 5.3. *Assume* $\alpha \geq 1$*. Then every solution to* (5.1) *tends to infinity.*

Proof. Firstly, assume that $\alpha > 1$. The sequence $\{x_n\}$ is not strictly decreasing. Assume the opposite. Since the sequence $\{x_n\}$ is bounded from below

$$x_n \geq 1, \text{ for every } n \in \mathbb{N},$$

we have

$$x_n \to a$$

and

$$a = \alpha a + 1 \Rightarrow a = \frac{1}{1-\alpha} < 0,\ (\alpha > 1).$$

We have a contradiction. So,

$$x_{N+1} \geq x_N, \text{ for some } N \in \mathbb{N}.$$

Therefore,

$$x_{N+2} = \alpha\frac{x_{N+1}^2}{x_N} + 1 \geq x_{N+1} + 1.$$

By using induction, we have,

$$x_n \to \infty, \text{ when } n \to \infty.$$

Now, assume that $\alpha = 1$. Then we have

$$x_{n+1} = \frac{x_n^2 + x_{n-1}}{x_{n-1}}. \tag{5.2}$$

Firstly, we show that there exists an integer n_0 such that

$$x_{n_0+1} \geq x_{n_0}.$$

Assume the opposite, that is, for all integers n, $x_{n+1} < x_n$. Since, the sequence $\{x_n\}$ is bounded from below for all n then $\{x_n\}$ must converge. Say $x_n \to a,$ as $n \to \infty$ Then (5.2) implies $a = a + 1$ which is not possible. Now, we use induction to prove that the sequence $\{x_n\}$ is increasing. Assume that the sequence $\{x_n\}$ is increasing for some $k \geq n_0$, that is, $x_{k+1} > x_k$. Then by using (5.2), we have

$$x_{k+2} = \frac{x_{k+1}}{x_k} x_{k+1} + 1 > x_{k+1} + 1 > x_{k+1},$$

from which follow that the sequence $\{x_n\}$ is increasing for all n. This implies that $x_n \to \infty$ as $n \to \infty$. □

Now, we consider the behaviour of the solution of (5.1) when $\alpha \in (0, 1)$. The following result holds.

Theorem 5.4. *Let* $0 < \alpha < \frac{1}{4}$. *Let*

$$S_2 = \left\{ (x, y) : \frac{\alpha}{y} \left(\alpha \frac{y^2}{x} + 1 \right)^2 + 1 \leq \frac{1 + \sqrt{1 - 4\alpha}}{2\alpha} \right\}$$

be the set of initial conditions. Then every positive solution $\{x_n\}_{n=1}^{\infty}$ *of* (5.1) *that starts in the set* S_2 *has a finite limit.*

Proof. Since $x_{n+1} = \alpha \frac{x_n^2}{x_{n-1}} + 1 \geq 1$, we have that $\frac{1}{x_n} \leq 1$ for all $n \in \mathbb{N}$. Set $f(x, y) = \alpha \frac{x^2}{y} + 1$, then $\frac{\partial f}{\partial x} = 2\alpha \frac{x}{y} > 0$ and $\frac{\partial f}{\partial y} = -\alpha \frac{x^2}{y^2} < 0$. Hence, the function $f(x, y)$ is increasing in the first variable and decreasing in the second variable. It is easy to see that the inequalities $\frac{1}{\alpha} > \frac{1 + \sqrt{1 - 4\alpha}}{2\alpha} > 2 > \frac{1}{1 - \alpha} > 1$ for $0 < \alpha < \frac{1}{4}$ hold.

Take the initial conditions $x_{-1},\ x_0 \in S_2$, we have

$$\begin{aligned}
x_2 &\leq \frac{1+\sqrt{1-4\alpha}}{2\alpha},\\
x_3 &= \alpha\frac{x_2^2}{x_1}+1 = \alpha x_2^2 \underbrace{\frac{1}{x_1}}_{\leq 1} + 1 \leq \alpha\left(\frac{1+\sqrt{1-4\alpha}}{2\alpha}\right)^2 + 1 = \frac{1+\sqrt{1-4\alpha}}{2\alpha},\\
x_4 &= \alpha\frac{x_3^2}{x_2}+1 = \alpha x_3^2 \underbrace{\frac{1}{x_2}}_{\leq 1} + 1 \leq \alpha\left(\frac{1+\sqrt{1-4\alpha}}{2\alpha}\right)^2 + 1 = \frac{1+\sqrt{1-4\alpha}}{2\alpha},\\
&\vdots\\
x_n &\leq \frac{1+\sqrt{1-4\alpha}}{2\alpha} \text{ for all } n \geq 2.
\end{aligned}$$

Therefore $1 \leq x_n \leq \dfrac{1+\sqrt{1-4\alpha}}{2\alpha}$ for all $n \geq 2$. Hence, for any $(x_{-1}, x_0) \in S_2$, we have that $f(x_n, x_{n-1}) \in \left[1, \dfrac{1+\sqrt{1-4\alpha}}{2\alpha}\right]$. Since $\alpha \in (0,\ \dfrac{1}{4})$, all solutions $(m, M) \in \left[1, \dfrac{1+\sqrt{1-4\alpha}}{2\alpha}\right] \times \left[1, \dfrac{1+\sqrt{1-4\alpha}}{2\alpha}\right]$ of the system

$$\begin{aligned}
m^2 &= \alpha m + M\\
M^2 &= \alpha M + m.
\end{aligned}$$

satisfy $m = M = \dfrac{1}{1-\alpha}$. Theorem 2.2 implies $\lim_{n\to\infty} x_n = \dfrac{1}{1-\alpha}$. □

Note that the previous statement is true if there exists $N \in \mathbb{N}$ such that $x_N \leq \dfrac{1+\sqrt{1-4\alpha}}{2\alpha}$. Set

$$\begin{aligned}
S_3 &= \left\{(x_{-1}, x_0) \in I : x_2 \geq \frac{1+\sqrt{1-4\alpha}}{2\alpha} \wedge x_3 \leq \frac{1+\sqrt{1-4\alpha}}{2\alpha}\right\},\\
S_4 &= \left\{(x_{-1}, x_0) \in I : x_2, x_3 \geq \frac{1+\sqrt{1-4\alpha}}{2\alpha} \wedge x_4 \leq \frac{1+\sqrt{1-4\alpha}}{2\alpha}\right\},\\
&\vdots\\
S_n &= \left\{(x_{-1}, x_0) \in I : x_2, x_3, \ldots, x_{n-1} \geq \frac{1+\sqrt{1-4\alpha}}{2\alpha} \wedge x_n \leq \frac{1+\sqrt{1-4\alpha}}{2\alpha}\right\},\\
&\vdots
\end{aligned}$$

Obviously, if $(x, y) \in S_2 \Rightarrow x_2 \leq \dfrac{1+\sqrt{1-4\alpha}}{2\alpha} \Rightarrow x_3 \leq \dfrac{1+\sqrt{1-4\alpha}}{2\alpha}$ so $(x, y) \in S_3$ which implies $S_2 \subset S_3$. Continuing on this, one can conclude that the following holds

$S_2 \subset S_3 \subset \ldots \subset S_n \subset \ldots$. Construct a set $\bigcup_{k=2}^{\infty} S_k$. It is clear that the basin of attraction of a unique equilibrium point $\mathcal{B}\left(\frac{1}{1-\alpha}\right) \supset \bigcup_{k=2}^{\infty} S_k$. All this leads to the following theorem.

Theorem 5.5. *Let* $0 < \alpha < \frac{1}{4}$. *The basin of attraction* $\mathcal{B}\left(\frac{1}{1-\alpha}\right)$ *of a unique equilibrium point* $\frac{1}{1-\alpha}$ *of the difference equation* (5.1) *is a superset of* $\bigcup_{k=2}^{\infty} S_k$, *that is,* $\mathcal{B}\left(\frac{1}{1-\alpha}\right) \supset \bigcup_{k=2}^{\infty} S_k$.

By using the previous result and numerical simulations, we believe that the following holds.

Conjecture 5.6. Let $\frac{1}{4} \le \alpha < 1$. Then every positive solution of (5.1) has a finite limit.

Conjecture 5.7. The basin of attraction $\mathcal{B}\left(\frac{1}{1-\alpha}\right)$ of a unique equilibrium point $\frac{1}{1-\alpha}$ is equal to $\bigcup_{k=2}^{\infty} S_k$.

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