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A Discrete Approach to Continuous Second-Order Boundary Value Problems via Monotone Iterative Techniques

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Abstract

This article investigates nonlinear, second-order difference equations subject to "right-focal" two-point boundary conditions. The particular interest is in identifying sufficient conditions under which solutions to this problem exist. Furthermore, if solutions to this "discrete" problem do exist, then what, if any, is their relationship to solutions to the "continuous", right-focal analogue involving second-order ordinary differential equations?

We show how the solutions to the discrete problem can be applied to show that the continuous problem does have a solution. The methods used herein involve the construction of sequences of vectors which converge (or have a subsequence that converges) to a solution of the discrete boundary value problem. In some cases these convergent sequences are monotone. The ideas herein do not rely on a knowledge of nonlinear analysis and thus such an approach may be accessible to a wider audience as there is no reliance on: fixed-point theorems; upper and lower solutions; or topological degree.

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1 Introduction

This paper considers the nonlinear, second-order difference equation

$$\frac{\Delta\nabla x_i}{h^2} = f\left(t_i, x_i, \frac{\Delta x_i}{h}\right), \quad i = 1, \ldots, n-1; \tag{1.1}$$

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coupled with the "right-focal" boundary conditions:

$$x_0 = 0, \quad \frac{\nabla x_n}{h} = 0; \tag{1.2}$$

and its connections with the boundary value problem:

$$x'' = f(t, x, x'), \qquad t \in [0, N]; \tag{1.3}$$

$$x(0) = 0, \qquad x'(N) = 0. \tag{1.4}$$

Above, $f : [0, N] \times D \subset [0, N] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous, nonlinear function; $N > 0$ is a constant; the step size is $h = N/n$ with $h \leq N/2$; and the grid points are $t_i = ih$ for $i = 0, \ldots, n$. The differences are given by

$$\Delta x_i := \begin{cases} x_{i+1} - x_i, & \text{for } i = 0, \ldots, n-1, \\ 0, & \text{for } i = n; \end{cases}$$

$$\nabla x_i := \begin{cases} x_i - x_{i-1}, & \text{for } i = 1, \ldots, n, \\ 0, & \text{for } i = 0; \end{cases}$$

$$\Delta\nabla x_i := \begin{cases} x_{i+1} - 2x_i + x_{i-1}, & \text{for } i = 1, \ldots, n-1, \\ 0, & \text{for } i = 0 \text{ or } i = n. \end{cases}$$

Equations (1.1), (1.2) are collectively known as a "discrete", two-point boundary value problem (BVP) with right-focal boundary conditions, while (1.3), (1.4) are collectively known as a "continuous", two-point boundary value problem (BVP) with right-focal boundary conditions.

In this work, the particular interest is on identifying sufficient conditions under which the solutions to the BVP (1.1), (1.2) will exist and how to approximate them. Furthermore, if solutions to the discrete problem do exist, then what, if any, is their relationship to solutions to the continuous problem (1.3), (1.4)?

A number of interesting papers have provided important advances in the above directions. For example, Gaines [3], Lasota [6] and Myjak [7] were early contributors by employing fixed-point approaches including: contractive maps; *a priori* bounds on solutions; and lower and upper solutions. More recently, authors such: as Henderson and Thompson [4,5]; Thompson [11], Thompson and Tisdell [12–14]; and Rachůnková and Tisdell [8, 9, 16] have improved and extended several of the earlier results, mostly via an approach involving topological degree and fixed-point theory.

In contrast to the above, the methods used herein involve the construction of sequences of vectors that converge (or have a subsequence that converges) to a solution of (1.1), (1.2). In some cases these convergent sequences are monotone. This approach is known as the "monotone iterative technique" and based on the "method of successive approximations". The ideas herein do not rely on a knowledge of nonlinear analysis and thus such an approach may be accessible to a wider audience as there is no reliance on: fixed-point theorems; upper and lower solutions; or topological degree.

Particular interest in the "right focal" boundary conditions (1.2) and (1.4) is motivated by both abstract and applied problems. For example, Myjak [7, p. 122] presented an example showing that the techniques in [7] cannot be applied to a BVP with right-focal boundary conditions. Furthermore, right-focal conditions naturally appear in the mathematical description of a number of interesting phenomena, such as in beam analysis [1, Example 4.2].

The two fields of differential equations and difference equations provide a rich and natural framework to mathematically describe dynamical phenomena in continuous time and in discrete time, respectively. Difference equations also find important uses in the numerical approximation of solutions to differential equations. These two important applications of difference equations to modelling and approximation naturally motivate a deeper theoretical study of the subject.

When considered side-by-side and compared, the mathematical theory of the two areas of differential equations and difference equations can exhibit strange connections and interesting distinctions, especially concerning qualitative properties of solutions, as the following motivational examples illustrate.

Consider the linear initial value problem (IVP):

$$x' = -2tx, \qquad t \geq 1; \tag{1.5}$$

$$x(1) = 1; \tag{1.6}$$

and its discrete analogue

$$\frac{\Delta x_i}{h} = -2t_i x_i, \qquad i = 1, 2, 3, \ldots; \tag{1.7}$$

$$x_1 = 1. \tag{1.8}$$

It is easy to see that the solution $x(t) = e^{1-t^2}$ to the continuous IVP (1.5), (1.6) does not oscillate for $x \geq 1$. However if $h > 1/2$, then all solutions to the discrete IVP (1.7), (1.8) do oscillate at every point in the sense that $x_i x_{i+1} < 0$ for $i = 1, 2, 3, \ldots$, which may be verified by rewriting (1.7) as $x_{i+1} = x_i[1 - 2ht_i]$.

Consider the linear BVP

$$x'' = -2x, \qquad t \in [0, 4]; \tag{1.9}$$

$$x(0) = 0, \qquad x(4) = 1; \tag{1.10}$$

and its discrete analogue

$$\frac{\Delta\nabla x_i}{h^2} = -2x_i, \qquad i = 1, \ldots, n-1; \tag{1.11}$$

$$x_0 = 0, \qquad x_n = 1. \tag{1.12}$$

We see that the continuous BVP (1.11), (1.12) has a solution of the form $x(t) = [\sec(4\sqrt{2})]\sin(\sqrt{2}t)$. However for $h = 1$ the discrete BVP (1.11), (1.12) has no solution, for in this case (1.11) becomes $x_{i+1} = -x_{i-1}$ for $i = 1, \ldots, n-1$ and so we obtain $0 = x_0 = x_2 = x_4$ which contradicts the boundary conditions.

This paper is organised in the following manner. Section 2 contains some basic results that will needed throughout the paper.

In Section 3 some results are formulated that ensure the existence of solutions to (1.1) subject to (1.2). Furthermore, the ideas yield a computational procedure for approximating these solutions.

Finally, Section 4 presents a connection between solutions to the discrete problem and solutions to the continuous problem.

2 Preliminaries

In this section some basic results are provided that will be used in the main section, keeping the paper somewhat selfcontained.

Let b and c be positive constants. Consider the set

$$R_{b,c} := \{(t,u,v) \in [0,N] \times \mathbb{R}^2 : \ t \in [0,N], \ |u| \leq b, \ |v| \leq c\}. \tag{2.1}$$

Since we will make the assumption that f is continuous on $R_{b,c}$, we can always choose a constant $M > 0$ such that

$$M \geq \max_{(t,u,v)\in R_{b,c}} |f(t,u,v)|. \tag{2.2}$$

A solution to (1.1) is a vector $\tilde{x} = \{x_i\}_{i=0}^{n} \in \mathbb{R}^{n+1}$ that satisfies (1.1) for each $i = 1, \ldots, n-1$ and whose "graph" lies in $R_{b,c}$, that is,

$$(t_i, x_i, \nabla x_i/h) \in R_{b,c}, \quad \text{for } i = 1, \ldots, n-1.$$

A solution to (1.1) is a continuously twice-differentiable function $x : [0,N] \to \mathbb{R}$, that is, $x \in C^2([0,T])$, that satisfies (1.1) for each $t \in [0,N]$ The following well-known result reduces the study of BVPs to the study of equivalent integral/summation equations.

Lemma 2.1. *Let $f : R_{b,c} \to \mathbb{R}$ be continuous. The discrete BVP* (1.1)*,* (1.2) *has the equivalent summation equation representation*

$$x_i = h\sum_{j=1}^{n-1} G(t_i,t_j) f\left(t_j, x_j, \frac{\nabla x_j}{h}\right), \quad i = 0, \ldots, n, \tag{2.3}$$

where

$$G(t_i,t_j) := \begin{cases} -t_j, & \text{for } 1 \leq j \leq i-1 \leq n-1; \\ -t_i, & \text{for } 1 \leq i \leq j \leq n-1. \end{cases} \tag{2.4}$$

Similarly, the continuous BVP (1.3), (1.4) *has the equivalent integral equation representation*

$$x(t) = \int_0^N G(t,s) f(s, x(s), x'(s))\, ds, \quad t \in [0, N]. \tag{2.5}$$

Proof. Both (2.3) and (2.5) are well known, for example, see [1, pp. 23–24] and can be verified directly. □

The following result establishes some important properties of G in (2.4), known as Green's function.

Lemma 2.2. *The function G in* (2.4) *satisfies*

$$\begin{aligned} G &\le 0 & (2.6)\\ \nabla_i G(t_i, t_j) := G(t_i, t_j) - G(t_{i-1}, t_j) &\le 0, \ i = 1, \ldots, n, \ j = 1, \ldots, n-1, & (2.7)\\ h \sum_{j=1}^{n-1} |G(t_i, t_j)| &\le \frac{N^2}{2}, \quad i = 0, \ldots, n, & (2.8)\\ \sum_{j=1}^{n-1} |\nabla_i G(t_i, t_j)| &\le N, \quad i = 1, \ldots, n. & (2.9) \end{aligned}$$

Proof. Although the proof involves simple computation, we provide some details for the benefit of the reader. Inequality (2.6) is immediate from the definition of G. For $i = 0, \ldots, n$, we have

$$\begin{aligned} h \sum_{j=1}^{n-1} |G(t_i, t_j)| &= h \sum_{j=1}^{i-1} t_j + h \sum_{j=i}^{n-1} t_i \\ &= h^2 \sum_{j=1}^{i-1} j + h t_i (n-i) \\ &= h^2 \frac{i}{2}(i-1) + h t_i (n-i) \\ &= h t_i \left[n - \left(\frac{i+1}{2} \right) \right] \\ &\le h t_i [n - i/2]. \\ &= t_i [N - t_i/2] \\ &\le N^2/2. \end{aligned}$$

Similarly, (2.7) and (2.9) follow from

$$\nabla_i G(t_i, t_j) := \begin{cases} 0, & \text{for } 1 \le j \le i-1 \le n-1; \\ -h, & \text{for } 1 \le i \le j \le n-1. \end{cases}$$

Thus, for $i = 1, \dots, n$, we have

$$\begin{aligned}\sum_{j=1}^{n-1} |\nabla_i G(t_i, t_j)| &= h(n-i) \\ &= N - t_i \\ &\leq N.\end{aligned}$$

□

3 Main Results

This section contains the main results on existence and approximation of solutions to (1.1), (1.2).

Theorem 3.1. *Let $f : R_{b,c} \to \mathbb{R}$ be continuous and consider* (1.1)*,* (1.2)*. If*

$$MN^2 \leq 2b, \qquad MN \leq c \tag{3.1}$$

$$f(t,0,0) \leq 0, \qquad \text{for all } t \in [0,N] \tag{3.2}$$

$$f(t,u,v) \geq f(t,y,z), \qquad \text{for all } t \in [0,N],\ u \leq y,\ v \leq z \tag{3.3}$$

then the discrete BVP (1.1)*,* (1.2) *has at least one solution $\tilde{x} \in \mathbb{R}^{n+1}$ whose graph lies $R_{b,c}$.*

Proof. The basic idea of the proof is to define a suitable sequence of vectors that will converge to a vector, with this "limit" vector being a solution of (1.1), (1.2). Consider the summation equation (2.3) that, by Lemma 2.1, is equivalent to (1.1), (1.2) and define the sequence of vectors $\tilde{\phi}^{(k)} := (\phi_0^{(k)}, \dots, \phi_n^{(k)})$ for $k = 0, 1, 2, \dots$ in a recursive fashion via

$$\phi_i^{(0)} = 0, \quad i = 0, \dots, n; \tag{3.4}$$

$$\phi_i^{(k+1)} = h \sum_{j=0}^{n-1} G(t_i, t_j) f\left(t_j, \phi_j^{(k)}, \frac{\nabla \phi_j^{(k)}}{h}\right), \quad i = 0, \dots, n. \tag{3.5}$$

Firstly we show that our sequence of vectors $\tilde{\phi}^{(k)}$ is well defined for $k = 0, 1, \dots$ by showing: each $|\phi_i^{(k)}| \leq b$ for $i = 0, \dots, n$; and each $|\nabla \phi_i^{(k)}/h| \leq c$ for $i = 1, \dots, n$. This means that each $(t_i, \phi_i^{(k)}, \nabla \phi_i^{(k)}/h)$ is in the domain of f for each $i = 1, \dots, n-1$ and $k = 0, 1, \dots$. We use proof by induction.

From the definition of $\phi_i^{(0)}$ it is easy to see that $|\phi_i^{(0)}| \leq b$ for $i = 0, \dots, n$; and $|\nabla \phi_i^{(0)}/h| \leq c$ for $i = 1, \dots, n$. Now assume, for some $k_1 \geq 0$, we have $|\phi_i^{(k_1)}| \leq b$ for

$i = 0, \dots, n$; and $|\nabla\phi_i^{(k_1)}/h| \leq c$ for $i = 1, \dots, n$. From (3.5), we have, for $i = 0, \dots, n$

$$\begin{aligned}\left|\phi_i^{(k_1+1)}\right| &\leq h\sum_{j=0}^{n-1}|G(t_i,t_j)|\left|f\left(t_j,\phi_j^{(k_1)},\frac{\nabla\phi_j^{(k)}}{h}\right)\right| \\ &\leq Mh\sum_{j=0}^{n-1}|G(t_i,t_j)| \\ &\leq MN^2/2 \\ &\leq b\end{aligned}$$

from Lemma 2.2 and (3.1). Similarly, for $i = 1, \dots, n$

$$\begin{aligned}\left|\frac{\nabla\phi_i^{(k_1+1)}}{h}\right| &\leq \sum_{j=0}^{n-1}|\nabla_i G(t_i,t_j)|\left|f\left(t_j,\phi_j^{(k_1)},\frac{\nabla\phi_j^{(k)}}{h}\right)\right| \\ &\leq M\sum_{j=0}^{n-1}|\nabla_i G(t_i,t_j)| \leq MN \leq c\end{aligned}$$

from Lemma 2.2 and (3.1).

Thus, by induction, we have $(t_i, \phi_i^{(k)}, \nabla\phi_i^{(k)}/h) \in R_{b,c}$ for each $i = 0, 1 \dots, n$ and $k = 0, 1, \dots$ and so our sequence of vectors $\tilde{\phi}^{(k)}$ is well defined in (3.4), (3.5) for each $k = 0, 1, \dots$. Furthermore, the above shows that for each k, the sequences of vectors $\phi_i^{(k)}$ and $\nabla\phi_i^{(k)}/h$ are uniformly bounded for $i = 0, 1, \dots, n$ and $i = 1, \dots, n$ respectively.

We now show $\tilde{\phi}^{(k+1)} \geq \tilde{\phi}^{(k)}$ for $k = 0, 1, \dots$ and $\nabla\tilde{\phi}^{(k+1)}/h \geq \nabla\tilde{\phi}^{(k)}/h$ for $k = 0, 1, \dots$, where we interpret the inequality between two vectors meaning the same inequality holds between their corresponding components. Once again, we use induction. For $i = 0, \dots, n$ consider

$$\begin{aligned}\phi_i^{(1)} &= h\sum_{j=0}^{n-1}G(t_i,t_j)f\left(t_j,0,0\right) \\ &\geq 0 \\ &= \phi_i^{(0)},\end{aligned}$$

where we have used (2.6) and (3.2). Thus, $\tilde{\phi}^{(1)} \geq \tilde{\phi}^{(0)}$. In a similar fashion, for $i = 1, \dots, n$, we have

$$\begin{aligned}\frac{\nabla\phi_i^{(1)}}{h} &= \sum_{j=0}^{n-1}\nabla_i G(t_i,t_j)f\left(t_j,0,0\right) \\ &\geq 0 \\ &= \frac{\nabla\phi_i^{(0)}}{h}.\end{aligned}$$

Now assume that $\tilde{\phi}^{(k_1)} \geq \tilde{\phi}^{(k_1-1)}$ for some $k_1 \geq 1$, that is, assume $\phi_i^{(k_1)} \geq \phi_i^{(k_1-1)}$ for $i = 0, \ldots, n$. Furthermore, assume $\nabla\tilde{\phi}^{(k_1)}/h \geq \nabla\tilde{\phi}^{(k_1-1)}/h$ for some $k_1 \geq 1$, that is, assume $\nabla\phi_i^{(k_1)}/h \geq \nabla\phi_i^{(k_1-1)}/h$ for $i = 1, \ldots, n$.

For each $i = 0, \ldots, n$, we have

$$\begin{aligned}\phi_i^{(k_1+1)} &= h\sum_{j=0}^{n-1} G(t_i, t_j) f\left(t_j, \phi_j^{(k_1)}, \frac{\nabla\phi_j^{(k_1)}}{h}\right)\\ &\geq h\sum_{j=0}^{n-1} G(t_i, t_j) f\left(t_j, \phi_j^{(k_1-1)}, \frac{\nabla\phi_j^{(k_1-1)}}{h}\right)\\ &= \phi_i^{(k_1)},\end{aligned}$$

where we have used (2.6) and (3.3). Thus, $\tilde{\phi}^{(k+1)} \geq \tilde{\phi}^{(k)}$ for $k = 0, 1, \ldots$. Similarly, for each $i = 1, \ldots, n$, we have

$$\begin{aligned}\frac{\nabla\phi_i^{(k_1+1)}}{h} &= \sum_{j=0}^{n-1} \nabla_i G(t_i, t_j) f\left(t_j, \phi_j^{(k_1)}, \frac{\nabla\phi_j^{(k_1)}}{h}\right)\\ &\geq \sum_{j=0}^{n-1} \nabla_i G(t_i, t_j) f\left(t_j, \phi_j^{(k_1-1)}, \frac{\nabla\phi_j^{(k_1-1)}}{h}\right)\\ &= \frac{\phi_i^{(k_1)}}{h},\end{aligned}$$

where we have used (2.7) and (3.3).

From the above, we conclude that $\tilde{\phi}^{(k)}$ is a uniformly bounded and nondecreasing sequence of vectors and so must converge to a vector $\tilde{\phi}$, that is

$$\lim_{k\to\infty} \tilde{\phi}^{(k)} = \tilde{\phi}$$

for some $\tilde{\phi} \in \mathbb{R}^{n+1}$.

We finally show that the above $\tilde{\phi} = (\phi_0, \ldots, \phi_n) \in \mathbb{R}^{n+1}$ is actually a solution to (1.1), (1.2). Since each $|\phi_i^{(k)}| \leq b$ and each $|\nabla\phi_i^{(k)}/h| \leq c$, we must have each $|\phi_i| \leq b$ and $|\nabla\phi_i/h| \leq c$. Thus $(t_i, \phi_i, \nabla\phi_i/h) \in R_{b,c}$ for $i = 0, \ldots, n$. Furthermore, the continuity of f on $R_{b,c}$ ensures that

$$f\left(t_i, \phi_i^{(k)}, \frac{\nabla\phi_i^{(k)}}{h}\right) \to f\left(t_i, \phi_i. \frac{\nabla\phi_i}{h}\right), \quad \text{as } k \to \infty$$

for each $i = 1, \ldots, n$.

If we now take limits in (3.5) as $k \to \infty$, then we obtain

$$\phi_i = h\sum_{j=0}^{n-1} G(t_i, t_j) f\left(t_j, \phi_j, \frac{\nabla\phi_j}{h}\right), \quad i = 0, \ldots, n;$$

so that our limit vector $\tilde{\phi}$ is indeed a solution to (1.1), (1.2) in light of Lemma 2.2. □

Remark 3.2. The proof of Theorem 3.1 provides a computational tool for approximating (or obtaining) solutions to (1.1), (1.2).

For discrete BVPs where the right-hand side does not feature $\nabla x_i/h$, that is

$$\frac{\Delta\nabla x_i}{h} = f(t_i, x_i), \quad i = 1, \ldots, n-1 \tag{3.6}$$

with $f : R_b \to R$ and

$$R_b := \{(t, u) \in [0, N] \times \mathbb{R}^2 : \ t \in [0, N], \ |u| \le b\}$$

we have the following corollary to Theorem 3.1.

Corollary 3.3. *Let* $f : R_b \to \mathbb{R}$ *be continuous and consider* (3.6)*,* (1.2)*. If*

$$\begin{aligned} MN^2 &\le 2b, \\ f(t, 0) &\le 0, \qquad \textit{for all } t \in [0, N] \\ f(t, u) &\ge f(t, y), \qquad \textit{for all } t \in [0, N], \ u \le y \end{aligned}$$

then the discrete BVP (3.6)*,* (1.2) *has at least one solution* $\tilde{x} \in \mathbb{R}^{n+1}$ *whose graph lies* R_b*.*

Proof. As the proof is virtually identical to that of Theorem 3.1 it is omitted. □

Example 3.4. Consider the difference equation

$$\frac{\Delta\nabla x_i}{h^2} = \frac{1}{10}\left(-t_i - x_i^3 - \left(\frac{\nabla x_i}{h}\right)^5 - 1\right), \quad i = 1, \ldots, n-1 \tag{3.7}$$

subject to (1.2) with $N = 1$. Choose $b = 1$ and $c = 1$ to form $R_{b,c}$ and then see that $M = 2/5$. It is easy to verify that (3.1) and (3.2) hold. Furthermore, if $f(t, p, q)$ denotes the right-hand side of (3.7), then $f(t, p, q)$ is nonincreasing in both p and q, so that (3.3) holds. Thus, all of the conditions of Theorem 3.1 hold and the existence of solution follows.

Furthermore, this solution may be constructed as the limit of the sequence defined recursively in (3.4), (3.5) for the above f.

Remark 3.5. The proof of Theorem 3.1 can be simplified if only existence of solutions is sought (and not approximation). The bounds

$$|\phi_i^{(k)}| \le b, \quad |\nabla\phi_i^{(k)}/h| \le c \tag{3.8}$$

in the proof of Theorem 3.1 guarantee that the sequence of vectors ϕ_i has at least one convergent subsequence and it can be further shown that any such subsequence does converge to a solution of (1.1), (1.2).

However, as can be seen from the proof of Theorem 3.1, there is more going on. The sequence generated in (3.4), (3.5) is also monotone and so must converge to a solution of (3.4), (3.5) and there is no need to search for a convergent subsequence. This is a distinct computational advantage over moving to subsequences.

A result is now presented where the $\phi_i^{(k)}$ are bounded, but not monotone. However, as we shall see, it is easy to choose a pair of monotone subsequences that converge to a solution from above and from below.

Theorem 3.6. *Let* $f : R_{b,c} \to \mathbb{R}$ *be continuous and consider* (1.1)*,* (1.2)*. If*

$$MN^2 \leq 2b, \qquad MN \leq c \tag{3.9}$$

$$f(t,0,0) \geq 0, \qquad \text{for all } t \in [0,N] \tag{3.10}$$

$$f(t,u,v) \leq f(t,y,z), \qquad \text{for all } t \in [0,N],\ u \leq y,\ v \leq z \tag{3.11}$$

then the discrete BVP (1.1)*,* (1.2) *has at least one solution* $\tilde{x} \in \mathbb{R}^{n+1}$ *whose graph lies* $R_{b,c}$*.*

Proof. The existence part of the proof follows that of Theorem 3.1 and so is omitted, however, we point out a few interesting differences. Define the sequence of vectors $\tilde{\phi}^{(k)}$ as in (3.4), (3.5). It can be shown that $\tilde{\phi}^{(k)}$ is uniformly bounded as in (3.8) and so must have a convergent subsequence and, in turn, that the limit of this subsequence is a solution to (1.1), (1.2). However, $\tilde{\phi}^{(k)}$ is not a monotone sequence.

On the other hand, it can be shown that $\tilde{\phi}^{(k)}$ does possess subsequences that are monotone. For example, induction can be used to show

$$\tilde{\phi}^{(2k)} \geq \tilde{\phi}^{(2k+2)} \tag{3.12}$$

$$\tilde{\phi}^{(2k-1)} \leq \tilde{\phi}^{(2k+1)} \tag{3.13}$$

$$\nabla\tilde{\phi}^{(2k)}/h \geq \nabla\tilde{\phi}^{(2k+2)}/h \tag{3.14}$$

$$\nabla\tilde{\phi}^{(2k-1)}/h \leq \nabla\tilde{\phi}^{(2k+1)}/h \tag{3.15}$$

for $k = 0, 1, \ldots$ by invoking (3.10) and (3.11). The details are omitted for brevity.

In addition, it can be shown by induction that

$$(-1)^k[\tilde{\phi}^{(k)} - \tilde{\phi}^{(k+1)}] \leq 0$$

$$(-1)^k[\nabla\tilde{\phi}^{(k)}/h - \nabla\tilde{\phi}^{(k+1)}/h] \leq 0$$

for each k and so $\tilde{\phi}^{(2k)}$ converges monotonically to a solution from above and $\tilde{\phi}^{(2k-1)}$ converges monotonically to a solution from below. □

Corollary 3.7. *Let* $f : R_b \to \mathbb{R}$ *be continuous and consider* (3.6)*,* (1.2)*. If*

$$MN^2 \leq 2b,$$

$$f(t,0) \geq 0, \qquad \text{for all } t \in [0,N]$$

$$f(t,u) \leq f(t,y), \qquad \text{for all } t \in [0,N],\ u \leq y$$

then the discrete BVP (3.6)*,* (1.2) *has at least one solution* $\tilde{x} \in \mathbb{R}^{n+1}$ *whose graph lies* R_b*.*

Remark 3.8. If it is known that the discrete BVP (1.1), (1.2) has, at most, one solution and the conditions of Theorem 3.6 hold, then the even and odd subsequences in the proof of Theorem 3.6 will converge to the same solution from above and below, respectively.

With uniqueness of solutions in mind, the following result is presented.

Theorem 3.9. *Let* $f : R_{b,c} \to \mathbb{R}$ *be continuous and consider* (1.1)*,* (1.2)*. If*

$$f(t,u,v) < f(t,y,z), \quad \text{for all } t \in [0,N],\ u < y,\ v \leq z \tag{3.16}$$

then the discrete BVP (1.1)*,* (1.2) *has, at most, one solution* $\tilde{x} \in \mathbb{R}^{n+1}$ *whose graph lies* $R_{b,c}$*.*

Proof. Let $\tilde{p}$ and $\tilde{q}$ be two solutions to (1.1), (1.2). We show that $\tilde{p} \equiv \tilde{q}$. Let $\tilde{r} := \tilde{p} - \tilde{q}$ and assume there is a $j \in \{0,\dots,n\}$ such that

$$r_j = \max_{i=0,\dots,n} [p_i - q_i] > 0. \tag{3.17}$$

The first boundary condition in (1.2) ensures $j \neq 0$, while the second boundary condition in (1.2) ensures that if $j = n$, then the maximum r_j also occurs at $j = n-1$. If $j \in \{1,\dots,n-1\}$ then the discrete maximum principle gives

$$r_j > 0, \quad \nabla r_j / h \leq 0$$

and

$$\begin{aligned} 0 &\geq \frac{\Delta\nabla r_j}{h^2} \\ &= \frac{\Delta\nabla p_j}{h^2} - \frac{\Delta\nabla q_j}{h^2} \\ &= f(t_j, p_j, \nabla p_j/h) - f(t_j, q_j, \nabla q_j/h) \\ &> 0, \end{aligned}$$

where we have used (3.16), and we reach a contradiction. Thus $j \notin \{1,\dots,n-1\}$.

Combining the above cases, we see that $r_i \leq 0$ for all $i = 0,\dots,n$.

A similar argument to the above for the case $r_j < 0$ shows that $r_i \geq 0$ for all $i = 0,\dots,n$ and so $r_i = 0$ for all $i = 0,\dots,n$. Thus, $\tilde{p} \equiv \tilde{q}$ and there is, at most, one solution. □

Corollary 3.10. *Let* $f : R_b \to \mathbb{R}$ *be continuous and consider* (3.6)*,* (1.2)*. If*

$$f(t,u) < f(t,y), \quad \text{for all } t \in [0,N],\ u < y$$

then the discrete BVP (3.6)*,* (1.2) *has, at most, one solution* $\tilde{x} \in \mathbb{R}^{n+1}$ *whose graph lies* R_b*.*

4 A Discrete Approach to Differential Equations

In this section, we form a relationship between solutions to the discrete BVP (1.1), (1.2) and solutions to the continuous BVP (1.3), (1.4). which is based on the ideas of Gaines [3]. We formulate a sequence of continuous functions that are based on the solutions to (1.1), (1.2) and furnish some conditions under which they will converge to a function as $h \to 0$, with the function being a solution to (1.3), (1.4). The our approach uses the discrete problem to generate existence results for the continuous problem.

The following result involves a bound on the solutions and their backward differences to (1.1), (1.2), with the bounds being independent of h.

We require the following notation. Denote the sequence $n_m \to \infty$ as $m \to \infty$; let $0 < h_m = N/n_m$; and let $t_i^m = ih_m$ for $i = 0, \dots, n$. If (1.1), (1.2) has a solution for $h = h_m$ and $m \geq m_0$ that we denote by

$$\tilde{x}^m := (x_0^m, \dots, x_n^m), \tag{4.1}$$

then we construct the following sequence of continuous functions from (4.1) via linear interpolation to form

$$x^m(t) := x_i^m + \frac{(x_{i+1}^m - x_i^m)}{h_m}(t - t_i^m), \quad t_i^m \leq t \leq t_{i+1}^m; \tag{4.2}$$

for $m \geq m_0$ and $t \in [0, N]$. Note that $x^m(t_i^m) = x_i^m$ for $i = 0, \dots, n$.

Furthermore, define $v_i^m := (x_i^m - x_{i-1}^m)/h$ and similarly construct the sequence of continuous functions v^m on $[0, N]$ by

$$v^m(t) := \begin{cases} v_i^m + \dfrac{v_{i+1}^m - v_i^m}{h_m}(t - t_i^m), & \text{for } t_i^m \leq t \leq t_{i+1}^m; \\ v_1^m, & \text{for } 0 \leq t \leq t_1^m. \end{cases} \tag{4.3}$$

Lemma 4.1. *Let $f : [0, N] \times D \subseteq [0, N] \times \mathbb{R} \to \mathbb{R}$ be continuous and let $R \geq 0$ and $T \geq 0$ be constants. If* (1.1), (1.2) *has a solution for $h \leq h_m$ and $m \geq m_0$ that we denote by $\tilde{x}^m$ with*

$$\max_{i=0,\dots,n} |x_i^m| \leq R, \quad m \geq m_0; \tag{4.4}$$

$$\max_{i=1,\dots,n} \left| \frac{\nabla x_i^m}{h} \right| \leq T, \quad m \geq m_0; \tag{4.5}$$

then (1.3), (1.4) *has a solution $x = x(t)$ that is the limit of a subsequence of* (4.2).

Proof. The proof is similar to that of [3, Lemma 2.4] and so is only sketched.

For $m \geq m_0$ consider the sequence of functions $x^m(t)$ for $t \in [0, 1]$ in (4.2). We show that the sequence of functions x^m is uniformly bounded and equicontinuous on

$[0,1]$. For $t \in [t_i^m, t_{i+1}^m]$ and $m \geq m_0$, we have

$$\begin{aligned} |x^m(t)| &\leq |x_i^m| + \left| \frac{(x_{i+1}^m - x_i^m)}{h_m} \right| |t - t_i^m| \\ &\leq R + TN. \end{aligned}$$

Similar calculations show that v^m is uniformly bounded on $[0,N]$.

For $\beta, \gamma \in [0,N]$ and given $\varepsilon > 0$, consider

$$\begin{aligned} |x^m(\beta) - x^m(\gamma)| &\leq \left| \frac{(x_{i+1}^m - x_i^m)}{h_m} \right| |\beta - \gamma| \\ &\leq T|\beta - \gamma| \\ &< \varepsilon \end{aligned}$$

whenever $|\beta - \gamma| < \delta(\varepsilon) := \varepsilon/T$. Thus, x^m is equicontinuous on $[0,N]$.

A similar argument shows v^m is equicontinuous on $[0,N]$.

The convergence theorem of Arzelà–Ascoli [10, p. 527] guarantees the sequence of continuous functions $x^m = x^m(t)$ has a subsequence $x^{k(m)}(t)$ that converges uniformly to a continuous function $x = x(t)$ for $t \in [0,N]$. That is,

$$\max_{t \in [0,N]} |x^{k(m)}(t) - x(t)| \to 0, \quad \text{as } m \to \infty$$

Similarly, $v^m = v^m(t)$ has a subsequence $v^{k(m)}(t)$ that converges uniformly to a continuous function $y = y(t)$ for $t \in [0,N]$. That is,

$$\max_{t \in [0,N]} |v^{k(m)}(t) - y(t)| \to 0, \quad \text{as } m \to \infty.$$

Furthermore, it can be shown that $x' = y$ on $[0,N]$.

The continuity of f ensures that the above limit function will be a solution to (1.3), (1.4). □

The next theorem, which is motivated by [3, Theorem 2.5], requires the following notation. If (1.1), (1.2) has a solution $\tilde{x}$ for $0 < h \leq h_0$, then we define the continuous function $x(t, \tilde{x})$ by

$$x(t, \tilde{x}) := x_i + \frac{(x_{i+1} - x_i)}{h}(t - t_i), \quad t_i \leq t \leq t_{i+1}$$

and define the continuous function $v(t, \tilde{x})$ by

$$v(t, \tilde{x}) := \begin{cases} \dfrac{x_i - x_{i-1}}{h} + \dfrac{x_{i+1} - 2x_i + x_{i-1}}{h^2}(t - t_i), & \text{for } t_i \leq t \leq t_{i+1}; \\ \dfrac{x_1 - x_0}{h}, & \text{for } 0 \leq t \leq t_1. \end{cases} \tag{4.6}$$

Theorem 4.2. *Let* $f : [0, N] \times D \subseteq [0, N] \times \mathbb{R} \to \mathbb{R}$ *be continuous and let* $R \geq 0$ *and* $T \geq 0$ *be constants. Assume* (1.1)*,* (1.2) *has a solution for* $h \leq h_0$ *that we denote by* $\tilde{x}$ *with*

$$\max_{i=0,\ldots,n} |x_i| \leq R, \quad \max_{i=0,\ldots,n} \left| \frac{\nabla x_i}{h} \right| \leq T. \tag{4.7}$$

Given any $\varepsilon > 0$ *there exists a* $\delta = \delta(\varepsilon)$ *such that if* $h \leq \delta$*, then* (1.3)*,* (1.4) *has a solution* $x = x(t)$ *with*

$$\max_{t\in[0,N]} |x(t, \tilde{x}) - x(t)| \leq \varepsilon \tag{4.8}$$

$$\max_{t\in[0,N]} |v(t, \tilde{x}) - x'(t)| \leq \varepsilon. \tag{4.9}$$

Proof. Suppose, for some $\varepsilon > 0$, there is a sequence h_m such that $h_m \to 0$ as $m \to \infty$ and for $h = h_m = N/n_m$ (1.1), (1.2) has a solution $\tilde{x}^m$ with every solution $x = x(t)$ to (1.3), (1.4) satisfying at least one of

$$\max_{t\in[0,N]} |x(t, \tilde{x}) - x(t)| > \varepsilon \tag{4.10}$$

$$\max_{t\in[0,N]} |v(t, \tilde{x}) - x'(t)| > \varepsilon. \tag{4.11}$$

By assumption, for m sufficiently large, there is a $R \geq 0$ and $T \geq 0$ such that the solution $\tilde{x}^m$ to (1.1), (1.2) satisfies

$$\max_{i=0,\ldots,n} |x_i^m| \leq R, \quad \max_{i=0,\ldots,n} |v_i^m| \leq T.$$

Thus, the conditions of Lemma 4.1 are satisfied, and so we obtain a subsequence $x^{k(m)}(t)$ of $x^m(t)$ and a subsequence $v^{k(m)}(t)$ of $v^m(t)$ that converge uniformly on $[0, N]$ to a solution x of (1.3), (1.4). Thus, the inequalities (4.10) or (4.11) cannot hold. □

We now relate the above abstract results to the ideas from earlier sections.

Theorem 4.3. *Let the conditions of Theorem 3.1 or Theorem 3.6 hold. Given any* $\varepsilon > 0$ *there is a* $\delta = \delta(\varepsilon)$ *such that if* $h \leq \delta$*, then* (1.3)*,* (1.4) *has a solution* x *that satisfies* (4.8) *and* (4.9)*.*

Proof. We show that the conditions of Theorem 4.2 are satisfied for $R_{b,c} = [0, N] \times D$. The solution $\tilde{x}$ to (1.1), (1.2) guaranteed to exist by Theorem 3.1 satisfies $|x_i| \leq b$ for $i = 0, \ldots, n$ and $|\nabla x_i/h| \leq c$ for $i = 1, \ldots, n$ and so (4.7) holds with $R = b$ and $T = c$.

Thus, all of the conditions of Theorem 4.2 hold and the result follows. □

Example 4.4. The differential equation

$$x'' = e^x \tag{4.12}$$

arises in certain problems from radiation, electrohydrodynamics and a range of other problems involving diffusion [1, p. 113]. We claim that (4.12) has a solution $x = x(t)$ for $x \in [0, 1/2]$ satisfying (1.4).

Let $f(t, p) := e^p$. Choose $b = \ln 2$ to form R_b (with $N = 1/2$) so that $M = 2$. It is easy to see that $f(t, p)$ is nondecreasing in p, in fact, f is strictly increasing in p.

The conditions of Corollary 3.7 are satisfied and so the difference equation

$$\frac{\Delta\nabla x_i}{h^2} = e^{x_i}, \quad i = 1, \ldots, n-1 \tag{4.13}$$

subject to (1.2) has at least solution. Furthermore, this solution may be constructed as the limit of the sequence defined recursively in (3.4), (3.5) and the even subsequence will converge monotonically to the solution from above, while the odd subsequence will converge to the solution from below in view of Remark 3.8.

In addition, the conditions of Lemma 4.1, Theorem 4.2 and Theorem 4.3 hold and so the continuous BVP (4.13), (1.4) does possess a solution and, given any $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon)$ such that if $h \leq \delta$, then x satisfies (4.8) and (4.9).

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