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Further Results on Exponential Stability of Autonomous Discrete Systems

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Abstract

We prove some new characterizations for exponential stability of a periodic semigroup of bounded linear operators acting on a Banach space in terms of boundedness of solutions of periodic discrete Cauchy problems. The approach we use is based on the theory of discrete evolution semigroups acting on a space of almost periodic sequences.

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1 Introduction

Difference equations are the appropriate mathematical representation for discrete processes, which have special importance in many areas. During the last decades, the theory of difference equations has gained a lot of attention by researchers. In fact, these equations are involved in different areas such as biology (the study of competitive species in population dynamics), physics (the study of motions of interacting bodies), control theory, the study of control theory, etc. See, for instance, [1, 11].

In the last period, the asymptotic behaviour of dynamical systems has gained the attention of many researchers who established a lot of interesting and substantial results,

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in both the continuous and the discrete cases. One of the central interests in the asymptotic behaviour of solutions of linear dynamical systems is to find conditions for their solutions to be stable, unstable or exponentially dichotomic. Indeed, stability analysis is a fundamental question widely investigated in the literature. The increase of interest in studying the stability of differential equations is demonstrated by the large number of papers on this subject. We can cite among others [2, 3, 5, 6, 13, 15–17, 21, 23, 26, 30, 32, 35, 36, 38].

One of the powerful and efficient methods in the study of the asymptotic behaviour for differential equations is the spectral theory of the evolution semigroups arisen from those systems, [8, 18, 22]. Indeed, the spectral theory for evolution semigroups is a subject tracing its origins to the results of J. Mather, [20], on hyperbolic dynamical systems and J. Howland, [14], on non-autonomous Cauchy problems. The power of this method in the study of asymptotic behaviour for solutions of evolution equations is well highlighted in a lot of papers and monographs. For further details, we refer to [8, 10, 24, 28, 30] and the references therein.

Recently, in the paper [7], C. Buşe et al. have developed a different approach. Indeed, they constructed a space of X-valued functions and defined an evolution semigroup acting on this space. The evolution semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$ was formally defined on an adequate space by

$$(\mathcal{T}(t)f)(s) := \begin{cases} U(s, s-t)f(s-t), & s \geq t \\ 0, & s < t, \end{cases} \qquad t \geq 0, s \in \mathbb{R}. \tag{1.1}$$

In particular, a uniform boundedness condition on solutions for nonautonomous periodic abstract Cauchy problems was proved in order to characterize the exponential stability for the evolution family generated by this Cauchy problem.

In this paper, motivated by [7, 16], the discrete case for exponential stability shall be studied. We aim to prove new characterizations of uniform exponential stability for autonomous periodic discrete systems.

2 Preliminaries

Let X be a complex Banach space and $\mathcal{L}(X)$ the Banach algebra of all linear and bounded operators acting on X. The norms of vectors in X and of operators in $\mathcal{L}(X)$ will be denoted by $\|\cdot\|$. The set of all integers will be denoted by $\mathbb{Z}$ and $\mathbb{Z}_+$ will stand for the set of non-negative integers.

A family $\mathbf{T} := \{T(j) : j \in \mathbb{Z}_+\}$ of linear bounded operators acting on X is called a discrete semigroup if

- $T(0) = I$, where I denotes the identity operator on Y, and
- $T(j+k) = T(j) \circ T(k)$, for all $j, k \in \mathbb{Z}_+$.

It follows that for all $j \in \mathbb{Z}_+$, we have $T(j) = T(1)^j$. The operator $T(1)$ then is called the algebraic generator of the semigroup $\mathbf{T}$. The infinitesimal generator of the semigroup $\mathbf{T}$ is defined as $G := T(1) - I$.

The growth bound of $\mathbf{T}$ is denoted by $\omega_0(\mathbf{T})$. It is defined by the following inferior bound

$$\omega_0(\mathbf{T}) := \inf\{\omega \in \mathbb{R} \ / \ \exists M_\omega > 0 \ : \ \|T(n)\| \leq M_\omega e^{\omega n}\}.$$

If $\omega_0(\mathbf{T})$ is negative, that is equivalently, there exist positive constants M and ω such that for all $n \in \mathbb{Z}_+$, $\|T(n)\| \leq Me^{-\omega n}$, then the family $\mathbf{T}$ is said to be uniformly exponentially stable.

We say that the semigroup $\mathbf{T}$ is q-periodic, for some integer $q > 0$, if for each $j \in \mathbb{Z}_+$, we have $T(j+q) = T(j)$.

Throughout this paper, for a given operator A acting on the Banach space X, we use the following notations:

- $\rho(A)$ for the resolvent set of A, that is, the set of all complex scalars $z \in \mathbb{C}$ for which the operator $zI - A$ is not invertible
- $\sigma(A) := \mathbb{C} \setminus \rho(A)$ for the spectrum of A, and
- $r(A) := \sup\{|z|; \ z \in \sigma(A)\}$ for the spectral radius of A. It is well-known that $r(A) = \lim_{n\to\infty} \|A^n\|^{1/n}$, see for instance [27].

This limit is finite and satisfies that $r(A) \leq \|A\|$, cf. [31, Chapter V, Theorem 3.5] or [37, XIII.2, Theorem 3].

A further result shows that a discrete semigroup $\mathbf{T} := \{T(j) : \ j \in \mathbb{Z}_+\}$ is uniformly exponentially stable if and only if the spectral radius $r(T(1))$ is less than one.

The literature on semigroups is rich. For more details about semigroups, we can refer the reader to the monographs [8, 12, 21, 25, 33].

Proposition 2.1. *Let* $\mathbf{T} := \{T(n) : \ n \in \mathbb{Z}_+\}$ *be a q-periodic semigroup acting on the Banach space* X*. Then, the following assertions are equivalent:*

1. *The family* $\mathbf{T}$ *is uniformly exponentially stable.*
2. *There are two real constants* $N \geq 1$ *and* $\nu > 0$ *such that for all* $n \geq 0$*, we have* $\|T(n)\| \leq Ne^{-\nu n}$.
3. *The spectral radius of* $T(n)$*,*

$$r(T(n)) := \sup\{|\lambda|; \ \lambda \in \sigma(T(n)\} = \lim_{m\to\infty} \|T(n)^m\|^{1/m},$$

is less than 1.

4. *For each $\mu \in \mathbb{R}$, we have that*

$$\sup_{m\in\mathbb{Z}_+} \Big\| \sum_{k=1}^{m} e^{i\mu k} T^k(n) \Big\| = M(\mu)$$

is finite.

Proof. The direct implications **1** $\Rightarrow$ **2** $\Rightarrow$ **3** $\Rightarrow$ **4** are straightforward. Moreover, the fact that assertion **4** implies **1** is also known and is, for instance, a consequence of the uniform ergodic theorem ([21, Theorems 2.1 and 2.7]). □

We recall, now, the following result which can be seen as a consequence of the main result proved in [34].

Corollary 2.2. *The discrete system $x_{n+1} = T(1)x_n$ is uniformly exponentially stable if and only if for each real parameter $\mu \in \mathbb{R}$ and each q-periodic bounded sequence $(f(n))_n$ such that $f(0) = 0$, the unique solution of the following discrete Cauchy problem*

$$\begin{cases} y(n+1) & = T(1)y(n) + e^{i\mu(n+1)} f(n+1) \\ y(0) & = y_0 \end{cases} \tag{2.1}$$

is bounded.

3 Discrete Evolution Semigroup

In this section, we consider a space of X-valued sequences and define a discrete evolution semigroup acting on it. For this purpose, we need the following spaces

$BUC(\mathbb{Z}, X)$ is the space of all X-valued bounded uniformly continuous functions defined on $\mathbb{Z}$, endowed with the "sup" norm $\|f\|_\infty := \sup_{t\in\mathbb{Z}} \|f(t)\|$.

$P_q(\mathbb{Z}, X)$ is the subspace of $BUC(\mathbb{Z}, X)$ consisting of all functions F such that $F(n+q) = F(n)$ for all $n \in \mathbb{Z}$.

$AP_1(\mathbb{Z}, X)$ is the space of all X-valued sequences defined on $\mathbb{Z}$, which can be represented as $f(n) = \sum_{k=-\infty}^{k=\infty} e^{i\mu_k n} c_k(f)$ for all $n \in \mathbb{Z}$, where $\mu_k \in \mathbb{Z}$, $c_k(f) \in X$ and $\|f\|_1 := \sum_{k=-\infty}^{k=\infty} \|c_k(f)\| < \infty$. The spaces $BUC(\mathbb{Z}_+, X)$, $P_q(\mathbb{Z}_+, X)$ and $AP_1(\mathbb{Z}_+, X)$ may be defined in a similar manner. Further details about the space $AP_1(\mathbb{Z}, X)$ may be found, for example, in the recent monograph [9].

For more references about almost periodic functions, we refer the reader to the monographs [4, 9, 19]. These functions have shown an interesting tool to characterize asymptotic behaviour of dynamical systems, cf. for example [7, 16]. One of the most common properties possessed by the almost periodic functions along with the periodic functions is the property of having a Fourier series, [4, 9, 29].

Next, we construct a space of X-valued functions and define an evolution semigroup acting on it. For an arbitrary $n \geq 0$, we denote by $\mathcal{A}_n$ the set of all X-valued sequences f defined on $\mathbb{Z}$ for which there exists a function $F \in P_q(\mathbb{Z}, X) \cap AP_1(\mathbb{Z}, X)$ such that $F(n) = 0$, $f = F_{|_{\{n,n+1,\cdots\}}}$ and $f(m) = 0$ for all $m < n$.

We set $\mathcal{A} := \{e^{i\mu} \otimes f \ : \ \mu \in \mathbb{R} \text{ and } f \in \cup_{n\geq 0}\mathcal{A}_n\}$.

Consider the space $E(\mathbb{Z}, X) := \overline{\text{span}}(\mathcal{A})$ which is a closed subspace of $BUC(\mathbb{Z}, X)$ endowed with the "sup" norm. Analogously to the definition given in (1.1), the evolution semigroup $\mathcal{T} := \mathcal{T}(n)\}_{n\geq 0}$ is formally defined on $E(\mathbb{R}, X)$ for $f \in E(\mathbb{R}, X)$ by

$$(\mathcal{T}(n)f)(m) := \begin{cases} T(n)f(m-n), & m \geq n \\ 0, & m < n, \end{cases} \qquad n \geq 0, m \in \mathbb{Z}. \tag{3.1}$$

Proposition 3.1. *The discrete evolution semigroup $\mathcal{T}$ defined in* (3.1) *acts on $E(\mathbb{Z}, X)$.*

Proof. Let $\mu \in \mathbb{R}$ and $f \in \cup_{n\geq 0}\mathcal{A}_n$. Then, there exists $r \geq 0$ and a sequence $(F(n))_n \in P_q(\mathbb{Z}, X) \cap AP_1(\mathbb{Z}, X)$ such that $F(r) = 0$, $f(m) = F(m)$ for all $m \geq r$ and $f(m) = 0$ for all $m < r$.

Therefore, for each $n \geq 0$, and $m \in \mathbb{Z}$, we have

$$(\mathcal{T}(n)f)(m) := \begin{cases} e^{i\mu(m-n)}T(n)F(m-n), & m \geq n+r \\ 0, & 0 \leq m < n+r, \end{cases}$$

The sequence $(G(m))_m$ defined by $G(m) = e^{i\mu(m-n)}T(n)F(m-n)$ is q-periodic and belongs to $AP_1(\mathbb{Z}, X)$. Besides, we have for some $M \geq 1$ and $\omega \in \mathbb{R}$

$$\|G\| \leq \|T(n)\|.\Big\| \sum_{k\in\mathbb{Z}} e^{i\mu_k(m-n)}c_k(F)\Big\| \leq Me^{\omega n}\|F\|_1.$$

Then, $(\mathcal{T}(n)f) \in \mathcal{A}$.

In addition, $\mathcal{T}(n)$ is a linear operator from $E(\mathbb{Z}, X)$ to $BUC(\mathbb{Z}, X)$.

Let $f = \alpha g + \beta h \in \text{span}\,\{\cup_{n\geq 0}\mathcal{A}_n\}$, where $g \in A_r$ and $h \in A_\rho$ with $\rho \leq r$ and α, β are complex scalars. Then, $\mathcal{T}(n)f = \alpha\mathcal{T}(n)(g) + \beta\mathcal{T}(n)(h)$. But $\mathcal{T}(n)(g) \in \mathcal{A}_{n+r}$ and $\mathcal{T}(t)(h) \in \mathcal{A}_{n+\rho}$ and therefore $\mathcal{T}(n)f$ belongs to span $\{\cup_{n\geq 0}\mathcal{A}_n\}$.

Finally, let f in $\overline{\text{span}}\{\cup_{n\geq 0}\mathcal{A}_n\}$. There exists a sequence (f_s) in $\text{span}\{\cup_{n\geq 0}\mathcal{A}_n\}$, such that $\sup\limits_{n\geq 0} \|f_s(n) - f(n)\| \to 0$ as $s \to \infty$.

Thus,

$$\sup_{m\geq 0} \|(\mathcal{T}(n)f_s)(m) - (\mathcal{T}(n)f)(m)\| \leq \widetilde{M}e^{\omega t} \sup_{m\geq n} \|f_s(m-n) - f(m-n)\| \to 0.$$

Therefore, the proof is achieved. □

4 Main Results

The Taylor formula of order one for discrete semigroups may be written as

$$T(j)f - f = \sum_{k=0}^{j-1} T(k)Gf, \ \forall j \in \mathbb{Z}_+ : \ j \geq 1, \ f \in Y. \tag{4.1}$$

Indeed,

$$\sum_{k=0}^{j-1} T(k)Gf = \sum_{k=0}^{j-1} [T(k+1) - T(k)]f = T(j)f - f.$$

Lemma 4.1. *Let $x := (x_n)_{n \in \mathbb{Z}_+}$ and $f := (f_n)_{n \in \mathbb{Z}_+}$ be into $E(\mathbb{Z}, X)$. The following statements are equivalent.*

1. $Gx = -f$
2. $x_n = \sum_{k=0}^{n} T(n-k)f(k)$ *for every* $n \geq 0$.

The proof of the above Lemma 4.1 is similar to the proof of [16, Lemma 3].
Now, we shall state the main result of this section.

Theorem 4.2. *Let $\mathbf{T}$ be a q-periodic discrete semigroup acting on the Banach space X and let $\mathcal{T}$ be its associated discrete evolution semigroup defined on $E(\mathbb{Z}, X)$. We denote by G the operator $G := \mathcal{T}(1) - I$, where $\mathcal{T}(1)$ stands for the algebraic generator of $\mathcal{T}$. The following assertions are equivalent:*

1. $\mathbf{T}$ *is uniformly exponentially stable.*
2. $\mathcal{T}$ *is uniformly exponentially stable.*
3. G *is an invertible operator.*
4. *For each sequence $f \in E(\mathbb{Z}, X)$, the sequence*

   $$n \mapsto \sum_{k=0}^{n} T(n-k)f(k)$$

   belongs to $E(\mathbb{Z}, X)$.
5. *For each sequence $f \in E(\mathbb{Z}, X)$, the sequence*

   $$n \mapsto \sum_{k=0}^{n} T(n-k)f(k)$$

   is bounded $\mathbb{Z}_+$.

6. *For each sequence* $f \in P_q(\mathbb{Z}, X)$*, the sequence*

$$n \mapsto \sum_{k=0}^{n} T(n-k)f(k)$$

is bounded on $\mathbb{Z}_+$.

Proof. Let us prove that 1 implies 2. Indeed, since the family $\mathbf{T}$ is uniformly exponentially stable, there exist two constants $N \geq 1$ and $\omega > 0$ such that for any $n \geq 0$, we have

$$\|\mathbf{T}(n)\| \leq Me^{-\omega n}.$$

Then, for each $f \in E(\mathbb{Z}, X)$, we have

$$\|\mathcal{T}(n)f\|_{E(\mathbb{Z},X)} = \sup_{m \geq n} \|T(n)f(m-n)\|,$$

which implies that

$$\|\mathcal{T}(n)f\|_{E(\mathbb{Z},X)} \leq Me^{-\omega n} \sup_{m \geq n} \|T(n)f(m-n)\| = Me^{-\omega n}\|f\|_{E(\mathbb{Z},X)}.$$

Therefore, the family $\mathcal{T}$ is uniformly exponentially stable.

Now, we shall prove that 2 implies 3. Thanks to Proposition 2.1, we know that the evolution semigroup $\mathcal{T}$ is uniformly exponentially stable if and only if the spectral radius $r(\mathcal{T}(1)) < 1$. That means that 1 is not an eigenvalue of the operator $\mathcal{T}(1)$, that is, $1 \notin \rho(\mathcal{T}(1))$ and so that the operator $G = \mathcal{T}(1) - I$ is an invertible one. Let us assume now that the operator G is invertible. Then, according to Lemma 4.1, for every sequence $f \in E(\mathbb{Z}, X)$, there exists $u \in E(\mathbb{Z}, X)$ such that $Gu = -f$. Therefore, by Lemma 4.1, $u(n) = \sum_{k=0}^{n} T(n-k)f(k)$ and by Proposition 3.1, the sequence $\left(\sum_{k=0}^{n} T(n-k)f(k)\right)_n$ belongs to $E(\mathbb{Z}, X)$.

If the statement 4 holds true, since $\left(\sum_{k=0}^{n} T(n-k)f(k)\right)_n$ belongs to $E(\mathbb{Z}, X)$ and $E(\mathbb{Z}, X)$ is included in $BUC(\mathbb{Z}, X)$, we deduce that $\left(\sum_{k=0}^{n} T(n-k)f(k)\right)_n$ is bounded.

The statement 5 implies clearly the statement 6.

Eventually, as a consequence of Proposition 2.2, the statement 6 implies that the family $\mathbf{T}$ is uniformly exponentially stable. This completes the proof. □

In terms of Cauchy problems, the result contained in Theorem 4.2 may be read as follows

Corollary 4.3. *The semigroup* $\mathbf{T}$ *is uniformly exponentially stable if and only if for each forcing sequence* $f \in E(\mathbb{Z}, X)$*, the solution of the Cauchy problem*

$$\begin{cases} x_{n+1} & = T(1)x_n + f(n+1), \; n \in \mathbb{Z}_+ \\ x_0 & = 0 \end{cases}$$

is bounded on $\mathbb{Z}_+$*.*

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