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On an Inverse Problem for a Quadratic Eigenvalue Problem

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Abstract

This paper is concerned with the eigenvalue problem for the complex symmetric tridiagonal quadratic matrix polynomial (quadratic pencil) and investigates reconstruction of the quadratic pencil from some of its spectral data. It is shown that two appropriately defined (finite) sequences of eigenvalues determine the coefficient matrices in the quadratic matrix polynomial uniquely. In the case of two dimensional matrix coefficients the full solution of the inverse spectral problem is presented including necessary and sufficient conditions for solvability of the inverse problem and a reconstruction procedure.

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1 Introduction

In many applications the underlying equation is a linear second order differential equation with constant coefficients

$$\frac{d^2u(t)}{dt^2} + G\frac{du(t)}{dt} + Ju(t) = 0, \tag{1.1}$$

where G and J are $N \times N$ complex constant matrices, $u(t)$ is an $N \times 1$ vector-function (the desired solution). In the mechanical vibration case the matrices G and J are known as the damping and stiffness matrices, respectively.

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We can look for particular solutions of (1.1) of the form $u(t) = e^{\lambda t}y$, where λ is a complex constant and y is a nonzero constant (independent of t) column vector in $\mathbb{C}^N$. If we substitute $u(t) = e^{\lambda t}y$ into (1.1), then we get the quadratic eigenvalue problem (QEP)

$$Q(\lambda)y = 0, \tag{1.2}$$

where

$$Q(\lambda) = \lambda^2 I + \lambda G + J \tag{1.3}$$

in which I is a unit $N \times N$ matrix.

A complex number λ_0 is said to be an *eigenvalue* of (1.2) (or of the quadratic pencil $Q(\lambda)$) if there exists a nonzero vector $y^{(0)} \in \mathbb{C}^N$ such that $Q(\lambda_0)y^{(0)} = 0$. This vector $y^{(0)}$ is called an *eigenvector* of $Q(\lambda)$, corresponding to the eigenvalue λ_0.

Obviously, a complex number λ_0 is an eigenvalue of $Q(\lambda)$ if and only if $\det Q(\lambda_0) = 0$. Note that $\det Q(\lambda)$ is a monic (leading coefficient unity) polynomial in λ of degree $2N$.

The general theory of differential equations of the type (1.1) is based on the theory of matrix pencils $Q(\lambda) = \lambda^2 I + \lambda G + J$, see [1,5,10,11,13].

Quantities related to the eigenvalues and eigenvectors of the pencil $Q(\lambda)$ are called the spectral characteristics (spectral data) of this pencil. The *inverse spectral problem* for $Q(\lambda)$ is to reconstruct $Q(\lambda)$ (that is, its coefficient matrices J and G) given some of its spectral data.

In the present paper, we consider the following version of the inverse spectral problem (the so-called inverse problem from two spectra) for $Q(\lambda)$. Suppose that the coefficient matrix J in the quadratic pencil (1.3) is a Jacobi matrix (tridiagonal symmetric matrix), while the coefficient matrix G is a diagonal matrix, of the form

$$J = \begin{bmatrix} b_0 & a_0 & 0 & \cdots & 0 & 0 & 0 \\ a_0 & b_1 & a_1 & \cdots & 0 & 0 & 0 \\ 0 & a_1 & b_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & b_{N-3} & a_{N-3} & 0 \\ 0 & 0 & 0 & \cdots & a_{N-3} & b_{N-2} & a_{N-2} \\ 0 & 0 & 0 & \cdots & 0 & a_{N-2} & b_{N-1} \end{bmatrix}, \tag{1.4}$$

$$G = \begin{bmatrix} d_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & d_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & d_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & d_{N-3} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & d_{N-2} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & d_{N-1} \end{bmatrix}, \tag{1.5}$$

in which for each n, a_n, b_n, and d_n are arbitrary complex numbers such that a_n is different from zero:

$$a_n, b_n, d_n \in \mathbb{C},\ a_n \neq 0. \tag{1.6}$$

Let I_1, G_1, and J_1 be the truncated matrices obtained by deleting the last row and last column of the matrices I, G, and J, respectively. The quadratic pencil

$$Q_1(\lambda) = \lambda^2 I_1 + \lambda G_1 + J_1 \tag{1.7}$$

is called the *truncated pencil* (with respect to the pencil $Q(\lambda)$).

Denote by $\{\lambda_j\}_{j=1}^{2N}$ and $\{\mu_k\}_{k=1}^{2N-2}$ the (finite) sequences of all the eigenvalues of the quadratic pencils $Q(\lambda)$ and $Q_1(\lambda)$, respectively. Each eigenvalue λ_j is counted according to its multiplicity as the root of the polynomial $\det Q(\lambda)$ and each eigenvalue μ_k is counted according to its multiplicity as the root of the polynomial $\det Q_1(\lambda)$. We have

$$\det Q(\lambda) = \prod_{j=1}^{2N} (\lambda - \lambda_j),\ \ \det Q_1(\lambda) = \prod_{k=1}^{2N-2} (\lambda - \mu_k).$$

The sequences

$$\{\lambda_j\}_{j=1}^{2N} \quad \text{and} \quad \{\mu_k\}_{k=1}^{2N-2} \tag{1.8}$$

are called the *two spectra* of the pencil $Q(\lambda)$.

The inverse problem for two spectra consists in determination of pencil $Q(\lambda)$ (that is, its coefficient matrices J and G) from its two spectra. The following three questions should be answered to get a full solution of the inverse problem:

(a) (*Uniqueness of the solution*) Are the matrices J and G determined uniquely by the two spectra given in (1.8)?

(b) (*Existence of the solution*) To find necessary and sufficient conditions for two given sequences of complex numbers $\{\lambda_j\}_{j=1}^{2N}$ and $\{\mu_k\}_{k=1}^{2N-2}$ to be the two spectra for a quadratic pencil of the form (1.3) with the coefficient matrices J and G of the form (1.4), (1.5) with entries from class (1.6).

(c) (*Construction procedure*) To indicate an algorithm for the construction of the matrices J and G from the two spectra.

The inverse spectral problem about two spectra for a linear pencil of the form $J - \lambda I$ with a real Jacobi matrix J (with nonzero first subdiagonal and superdiagonal elements) was studied earlier by Hochstadt [8] and developed further in [2–4, 6, 7, 9].

In our study, we essentially use the property that the eigenvalue problem (1.2) for a column vector $y = \{y_n\}_{n=0}^{N-1}$ is equivalent to the second-order linear difference equation

$$a_{n-1}y_{n-1} + (\lambda^2 + \lambda d_n + b_n)y_n + a_n y_{n+1} = 0, \tag{1.9}$$

$$n \in \{0, 1, \ldots, N-1\}, \quad a_{-1} = a_{N-1} = 1,$$

for $\{y_n\}_{n=-1}^{N}$, with the boundary conditions

$$y_{-1} = y_N = 0. \tag{1.10}$$

This allows, using techniques from the theory of linear second-order difference equations [12], to develop a thorough analysis of the eigenvalue problem (1.2).

The paper is organized as follows. In Section 2, on the base of difference equation (1.9), two auxiliary lemmas are proved which are used in subsequent Section 3. In Section 3, two uniqueness theorems are established. The settings of these theorems are analogous to those of two theorems of Hochstadt's paper [8] where $G = 0$, but the real-valued Jacobi matrix J with nonzero first subdiagonal and superdiagonal elements is to be constructed. In the first theorem of Section 3, the uniqueness problem in the reconstruction of quadratic pencil (1.3) from two sets of prescribed eigenvalues is considered, where G is a diagonal complex-valued matrix and J is a complex-valued, tridiagonal, symmetric matrix with fixed nonzero first subdiagonal and superdiagonal entries. The theorem claims that the diagonal entries of G and J are uniquely determined from two spectra of $Q(\lambda)$. The second theorem of Section 3 tells us that if the eigenvalues of $Q(\lambda)$ are identical with those of $Q_0(\lambda)$ being a quadratic pencil obtained from $Q(\lambda)$ by putting $G = 0$, then necessarily the matrix G in $Q(\lambda)$ is zero provided that the diagonal elements of G are real or are pure imaginary. In Section 4, the full solution of the inverse spectral problem is presented in the case $N = 2$ of the $N \times N$ matrix pencils $Q(\lambda)$. The aim of this section is to give an illustration for the difficult problem in general case of arbitrary N. Finally, in Section 5, we make some conclusions.

As it is seen, in the present paper, we deal with a special class of quadratic matrix polynomials: the coefficient matrices J and G are not arbitrary $N \times N$ complex matrices; the matrix J is a tridiagonal symmetric matrix with first subdiagonal and superdiagonal elements different from zero and G is a diagonal matrix (see (1.4), (1.5), and (1.6)). Often in applications, it is important to identify conditions on the spectral data which ensure the existence of solutions of the inverse problem belonging to the class of physically realizable solutions, that may form a rather narrow class.

2 Some Auxiliary Facts

Denote by $\{P_n(\lambda)\}_{n=-1}^{N}$ the unique solution of (1.9) satisfying the initial conditions

$$P_{-1}(\lambda) = 0, \quad P_0(\lambda) = 1. \tag{2.1}$$

For each $n \geq 0$, $P_n(\lambda)$ is a polynomial of degree $2n$. These polynomials can be found recurrently from (1.9) using initial conditions in (2.1). The leading term of the polynomial $P_n(\lambda)$ has the form

$$P_n(\lambda) = \frac{(-1)^n}{a_0 a_1 \cdots a_{n-1}} \lambda^{2n} + \dots .$$

The solution $\{P_n(\lambda)\}_{n=-1}^{N}$ satisfies the boundary condition $y_{-1} = 0$ given in (1.10). Therefore from the boundary condition $y_N = 0$ given also in (1.10), we get that the roots of the polynomial $P_N(\lambda)$ are eigenvalues of the pencil $Q(\lambda)$. The following lemma states a stronger result.

Lemma 2.1. *The equalities*

$$\det Q(\lambda) = (-1)^N a_0 a_1 \cdots a_{N-2} P_N(\lambda), \tag{2.2}$$

$$\det Q_1(\lambda) = (-1)^{N-1} a_0 a_1 \cdots a_{N-2} P_{N-1}(\lambda) \tag{2.3}$$

hold, so that the eigenvalues and their multiplicities of the pencils $Q(\lambda)$ and $Q_1(\lambda)$ coincide with the roots and their multiplicities of the polynomials $P_N(\lambda)$ and $P_{N-1}(\lambda)$, respectively.

Proof. To prove (2.2) and (2.3), let us set, for each $n \in \{1, 2, \ldots, N\}$,

$$X_n(\lambda) = \begin{bmatrix} x_0(\lambda) & a_0 & 0 & \cdots & 0 & 0 & 0 \\ a_0 & x_1(\lambda) & a_1 & \cdots & 0 & 0 & 0 \\ 0 & a_1 & x_2(\lambda) & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & x_{n-3}(\lambda) & a_{n-3} & 0 \\ 0 & 0 & 0 & \cdots & a_{n-3} & x_{n-2}(\lambda) & a_{n-2} \\ 0 & 0 & 0 & \cdots & 0 & a_{n-2} & x_{n-1}(\lambda) \end{bmatrix}, \tag{2.4}$$

where

$$x_k(\lambda) = \lambda^2 + \lambda d_k + b_k \quad (k = 0, 1, \ldots, N-1), \tag{2.5}$$

and put

$$\Delta_n(\lambda) = \det X_n(\lambda). \tag{2.6}$$

By expanding the determinant $\Delta_{n+1}(\lambda)$ by the elements of the last row, it is not difficult to show that

$$\Delta_{n+1}(\lambda) = (\lambda^2 + \lambda d_n + b_n)\Delta_n(\lambda) - a_{n-1}^2 \Delta_{n-1}(\lambda), \quad n = 0, 1, 2, \ldots, \tag{2.7}$$

$$\Delta_{-1}(\lambda) = 0, \quad \Delta_0(\lambda) = 1. \tag{2.8}$$

Dividing (2.7) by the product $a_0 \cdots a_{n-1}$, we find that the sequence

$$z_{-1} = 0, \; z_0 = 1, \; z_n = (-1)^n (a_0 \cdots a_{n-1})^{-1} \Delta_n(\lambda), \quad n = 1, 2, \ldots,$$

satisfies (1.9) and initial conditions (2.1). Then $z_n = P_n(\lambda)$, $n = 0, 1, \ldots$, by uniqueness of the solution, and hence we have (2.2), (2.3) because $X_N(\lambda) = Q(\lambda)$, $X_{N-1}(\lambda) = Q_1(\lambda)$, and $a_{N-1} = 1$. □

Lemma 2.2. *Under the conditions that the coefficient matrices J and G are of the form* (1.4)*,* (1.5) *with* (1.6)*, the pencils $Q(\lambda)$ and $Q_1(\lambda)$ have no common eigenvalues, that is, $\lambda_j \neq \mu_k$ for all values of j and k.*

Proof. Together with the solution $\{P_n(\lambda)\}_{n=-1}^{N}$, we introduce by $\{R_n(\lambda)\}_{n=-1}^{N}$ the second solution of (1.9) satisfying the initial conditions

$$R_{-1}(\lambda) = -1, \quad R_0(\lambda) = 0. \tag{2.9}$$

For each $n \geq 1$, $R_n(\lambda)$ is a polynomial of degree $2n-2$.

Multiply the first of the equations

$$a_{n-1}P_{n-1}(\lambda) + (\lambda^2 + \lambda d_n + b_n)P_n(\lambda) + a_n P_{n+1}(\lambda) = 0,$$

$$a_{n-1}R_{n-1}(\lambda) + (\lambda^2 + \lambda d_n + b_n)R_n(\lambda) + a_n R_{n+1}(\lambda) = 0,$$
$$n \in \{0, 1, \ldots, N-1\},\ a_{-1} = a_{N-1} = 1,$$

by $R_n(\lambda)$ and the second by $P_n(\lambda)$ and subtract the second result from the first one to get

$$a_{n-1}[P_{n-1}(\lambda)R_n(\lambda) - P_n(\lambda)R_{n-1}(\lambda)]$$
$$= a_n[P_n(\lambda)R_{n+1}(\lambda) - P_{n+1}(\lambda)R_n(\lambda)], \quad n \in \{0, 1, \ldots, N-1\}.$$

This means that the expression (Casoratian or Wronskian of the solutions $P_n(\lambda)$ and $R_n(\lambda)$)

$$a_n[P_n(\lambda)R_{n+1}(\lambda) - P_{n+1}(\lambda)R_n(\lambda)]$$

does not depend on $n \in \{-1, 0, 1, \ldots, N-1\}$. On the other hand, the value of this expression at $n = -1$ is equal to 1 by (2.1), (2.9), and $a_{-1} = 1$. Therefore

$$a_n[P_n(\lambda)R_{n+1}(\lambda) - P_{n+1}(\lambda)R_n(\lambda)] = 1 \quad \text{for all} \quad n \in \{-1, 0, 1, \ldots, N-1\}.$$

Putting, in particular, $n = N-1$ and using $a_{N-1} = 1$, we get

$$P_{N-1}(\lambda)R_N(\lambda) - P_N(\lambda)R_{N-1}(\lambda) = 1. \tag{2.10}$$

Suppose now that λ_0 is a common eigenvalue of the pencils $Q(\lambda)$ and $Q_1(\lambda)$. Then by (2.2) and (2.3), we have $P_N(\lambda_0) = P_{N-1}(\lambda_0) = 0$. But this is impossible by (2.10). This contradiction proves the lemma. □

3 Uniqueness Theorems

In this section, we will establish two uniqueness theorems for the inverse spectral problem.

Theorem 3.1. *If the off-diagonal elements $a_0, a_1, \ldots, a_{N-2}$ of the matrix J are fixed, then the two spectra of $Q(\lambda)$ given in* (1.8) *uniquely determine the diagonal elements $b_0, b_1, \ldots, b_{N-1}$ of the matrix J and the diagonal elements $d_0, d_1, \ldots, d_{N-1}$ of G.*

Proof. Suppose that along with the quadratic pencil $Q(\lambda)$ given in (1.3), we have another pencil

$$\widetilde{Q}(\lambda) = \lambda^2 I + \lambda \widetilde{G} + \widetilde{J},$$

where I is a unit $N \times N$ matrix, $\widetilde{G} = diag(\widetilde{d}_0, \widetilde{d}_1, \ldots, \widetilde{d}_{N-1})$ is a diagonal $N \times N$ matrix with the diagonal elements $\widetilde{d}_0, \widetilde{d}_1, \ldots, \widetilde{d}_{N-1}$ in $\mathbb{C}$, and $\widetilde{J}$ is a complex $N \times N$ Jacobi matrix of the form (1.4) in which all a_n are the same as in J but b_n are replaced by $\widetilde{b}_n$. Let $\widetilde{Q}_1(\lambda)$ be the truncated pencil with respect to $\widetilde{Q}(\lambda)$.

Assume that $Q(\lambda)$ and $\widetilde{Q}(\lambda)$ have the same eigenvalues with the same multiplicities and $Q_1(\lambda)$ and $\widetilde{Q}_1(\lambda)$ have the same eigenvalues with the same multiplicities so that we have

$$\det Q(\lambda) = \det \widetilde{Q}(\lambda) \quad \text{and} \quad \det Q_1(\lambda) = \det \widetilde{Q}_1(\lambda). \tag{3.1}$$

We have to prove that then

$$b_n = \widetilde{b}_n, \quad d_n = \widetilde{d}_n \quad (n = 0, 1, \ldots, N-1).$$

Along with the solution $\{P_n(\lambda)\}_{n=-1}^{N}$ introduced above in Section 2 for the pencil $Q(\lambda)$, consider also the analogous solution $\{\widetilde{P}_n(\lambda)\}_{n=-1}^{N}$ for the pencil $\widetilde{Q}(\lambda)$. Thus we have

$$a_{n-1}P_{n-1}(\lambda) + (\lambda^2 + \lambda d_n + b_n)P_n(\lambda) + a_n P_{n+1}(\lambda) = 0, \tag{3.2}$$

$$a_{n-1}\widetilde{P}_{n-1}(\lambda) + (\lambda^2 + \lambda \widetilde{d}_n + \widetilde{b}_n)\widetilde{P}_n(\lambda) + a_n \widetilde{P}_{n+1}(\lambda) = 0, \tag{3.3}$$

$$n \in \{0, 1, \ldots, N-1\}, \quad a_{-1} = a_{N-1} = 1,$$

$$P_{-1}(\lambda) = \widetilde{P}_{-1}(\lambda) = 0, \quad P_0(\lambda) = \widetilde{P}_0(\lambda) = 1. \tag{3.4}$$

By Lemma 2.1 and (3.1), we have

$$P_N(\lambda) = \widetilde{P}_N(\lambda) \quad \text{and} \quad P_{N-1}(\lambda) = \widetilde{P}_{N-1}(\lambda). \tag{3.5}$$

Now we multiply (3.2) by $\widetilde{P}_n(\lambda)$ and (3.3) by $P_n(\lambda)$ and subtract the second result from the first one to get

$$a_{n-1}\left[P_{n-1}(\lambda)\widetilde{P}_n(\lambda) - P_n(\lambda)\widetilde{P}_{n-1}(\lambda)\right] - a_n\left[P_n(\lambda)\widetilde{P}_{n+1}(\lambda) - P_{n+1}(\lambda)\widetilde{P}_n(\lambda)\right]$$

$$+ \left[\lambda(d_n - \widetilde{d}_n) + b_n - \widetilde{b}_n\right] P_n(\lambda)\widetilde{P}_n(\lambda) = 0, \quad n \in \{0, 1, \ldots, N-1\}.$$

Summing the last equation for the values $n = 0, 1, \ldots, N-1$ and taking into account that $a_{-1} = a_{N-1} = 1$, we obtain

$$\left[P_{-1}(\lambda)\widetilde{P}_0(\lambda) - P_0(\lambda)\widetilde{P}_{-1}(\lambda)\right] - \left[P_{N-1}(\lambda)\widetilde{P}_N(\lambda) - P_N(\lambda)\widetilde{P}_{N-1}(\lambda)\right]$$

$$+\lambda\sum_{n=0}^{N-1}(d_n-\widetilde{d}_n)P_n(\lambda)\widetilde{P}_n(\lambda)+\sum_{n=0}^{N-1}(b_n-\widetilde{b}_n)P_n(\lambda)\widetilde{P}_n(\lambda)=0. \quad (3.6)$$

Using (3.4) and (3.5), we get that the expressions inside the square brackets in (3.6) vanish and therefore

$$\sum_{n=0}^{N-1}(d_n-\widetilde{d}_n)\lambda P_n(\lambda)\widetilde{P}_n(\lambda)+\sum_{n=0}^{N-1}(b_n-\widetilde{b}_n)P_n(\lambda)\widetilde{P}_n(\lambda)=0. \quad (3.7)$$

The polynomial $\lambda P_n(\lambda)\widetilde{P}_n(\lambda)$ is of degree $4n+1$, and the polynomial $P_n(\lambda)\widetilde{P}_n(\lambda)$ is of degree $4n$. Therefore these polynomials all together obtained for $n=0,1,\ldots,N-1$ are linearly independent (because they are of distinct degrees). Then $d_n-\widetilde{d}_n=0$ and $b_n-\widetilde{b}_n=0$ for all n. The proof is complete. □

Theorem 3.2. *Consider the quadratic pencil $Q(\lambda)$ of Theorem 3.1 and assume that the matrix $G=diag(d_0,d_1,\ldots,d_{N-1})$ is real or pure imaginary. Further, let $Q_0(\lambda)$ be the quadratic pencil obtained from $Q(\lambda)$ by putting $G=0$. Suppose $Q(\lambda)$ and $Q_0(\lambda)$ have the same eigenvalues (with the same multiplicities). Then $d_n=0$ for all $n=0,1,\ldots,N-1$.*

Proof. We use the relationship (2.7) for $n=N-1$ which gives

$$\Delta_N(\lambda)=(\lambda^2+\lambda d_{N-1}+b_{N-1})\Delta_{N-1}(\lambda)-a_{N-2}^2\Delta_{N-2}(\lambda).$$

Hence we can show inductively that

$$\Delta_N(\lambda)=\lambda^{2N}+A_N\lambda^{2N-1}+B_N\lambda^{2N-2}+\ldots,$$

where

$$A_N=\sum_{k=0}^{N-1}d_k,\quad B_N=\sum_{k,l=0,k>l}^{N-1}d_kd_l+\sum_{k=0}^{N-1}b_k.$$

In order for $Q(\lambda)$ and $Q_0(\lambda)$ to have the same eigenvalues it is clearly necessary that

$$\sum_{k=0}^{N-1}d_k=0,\qquad \sum_{k,l=0,k>l}^{N-1}d_kd_l=0,$$

and by squaring the first of these and using the second, we find that

$$\sum_{k=0}^{N-1}d_k^2=0.$$

Since d_k have to be all real or all pure imaginary, we conclude that all d_k must vanish. □

4 The Inverse Spectral Problem at $N = 2$

In the case $N = 2$, we have the quadratic pencil

$$Q(\lambda) = \begin{bmatrix} \lambda^2 + \lambda d_0 + b_0 & a_0 \\ a_0 & \lambda^2 + \lambda d_1 + b_1 \end{bmatrix} \tag{4.1}$$

with the coefficient numbers

$$a_0, b_0, b_1, d_0, d_1 \in \mathbb{C}, \ a_0 \neq 0. \tag{4.2}$$

The truncated pencil is

$$Q_1(\lambda) = \lambda^2 + \lambda d_0 + b_0. \tag{4.3}$$

Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ be all the eigenvalues (taking into account their multiplicity) of $Q(\lambda)$ and μ_1, μ_2 all the eigenvalues (roots) of $Q_1(\lambda)$.

The inverse problem consists in finding the coefficient numbers of $Q(\lambda)$ indicated in (4.2) from the two (finite) eigenvalue sequences

$$\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} \quad \text{and} \quad \{\mu_1, \mu_2\}. \tag{4.4}$$

From (4.1), we have, by (4.3),

$$\det Q(\lambda) = (\lambda^2 + \lambda d_1 + b_1)Q_1(\lambda) - a_0^2. \tag{4.5}$$

On the other hand, we have

$$\det Q(\lambda) = \prod_{j=1}^{4}(\lambda - \lambda_j) = \lambda^4 + \Lambda_1\lambda^3 + \Lambda_2\lambda^2 + \Lambda_3\lambda + \Lambda_4, \tag{4.6}$$

$$Q_1(\lambda) = \prod_{k=1}^{2}(\lambda - \mu_k) = \lambda^2 + M_1\lambda + M_2, \tag{4.7}$$

where

$$\Lambda_1 = -(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4), \tag{4.8}$$

$$\Lambda_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4, \tag{4.9}$$

$$\Lambda_3 = -(\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_2\lambda_3\lambda_4), \quad \Lambda_4 = \lambda_1\lambda_2\lambda_3\lambda_4, \tag{4.10}$$

$$M_1 = -(\mu_1 + \mu_2), \quad M_2 = \mu_1\mu_2. \tag{4.11}$$

Substituting (4.6) and (4.7) in (4.5), we get

$$\prod_{j=1}^{4}(\lambda - \lambda_j) = (\lambda^2 + \lambda d_1 + b_1)\prod_{k=1}^{2}(\lambda - \mu_k) - a_0^2 \tag{4.12}$$

or

$$\lambda^4 + \Lambda_1\lambda^3 + \Lambda_2\lambda^2 + \Lambda_3\lambda + \Lambda_4$$
$$= (\lambda^2 + \lambda d_1 + b_1)(\lambda^2 + M_1\lambda + M_2) - a_0^2. \tag{4.13}$$

Equation (4.13) can be written as

$$\lambda^4 + \Lambda_1\lambda^3 + \Lambda_2\lambda^2 + \Lambda_3\lambda + \Lambda_4$$
$$= \lambda^4 + (M_1 + d_1)\lambda^3 + (M_2 + d_1M_1 + b_1)\lambda^2$$
$$+(d_1M_2 + b_1M_1)\lambda + b_1M_2 - a_0^2. \tag{4.14}$$

Therefore, by equating coefficients of the same powers of λ on both sides of (4.14), we get

$$M_1 + d_1 = \Lambda_1, \tag{4.15}$$

$$M_2 + d_1M_1 + b_1 = \Lambda_2, \tag{4.16}$$

$$d_1M_2 + b_1M_1 = \Lambda_3, \tag{4.17}$$

$$b_1M_2 - a_0^2 = \Lambda_4. \tag{4.18}$$

Hence

$$d_1 = \Lambda_1 - M_1, \tag{4.19}$$

$$b_1 = \Lambda_2 - M_2 - (\Lambda_1 - M_1)M_1, \tag{4.20}$$

$$(\Lambda_1 - M_1)M_2 + [\Lambda_2 - M_2 - (\Lambda_1 - M_1)M_1]M_1 = \Lambda_3, \tag{4.21}$$

$$a_0^2 = [\Lambda_2 - M_2 - (\Lambda_1 - M_1)M_1]M_2 - \Lambda_4. \tag{4.22}$$

Next, substituting (4.7) in (4.3), we find that

$$d_0 = M_1 = -(\mu_1 + \mu_2), \quad b_0 = M_2 = \mu_1\mu_2. \tag{4.23}$$

The following theorem gives a full solution of the inverse spectral problem at $N = 2$.

Theorem 4.1. *Let two (finite) sequences of complex numbers in* (4.4) *be given. In order for these sequences to be the two spectra for a quadratic pencil $Q(\lambda)$ of the form* (4.1) *with coefficient numbers belonging to the class* (4.2)*, it is necessary and sufficient that the following conditions are satisfied:*

(i) The two sequences in (4.4) have no common terms, that is, $\lambda_j \neq \mu_k$ for all possible values of j and k.

(ii) It holds true that

$$\prod_{j=1}^{4}(\mu_1-\lambda_j)=\prod_{j=1}^{4}(\mu_2-\lambda_j) \tag{4.24}$$

and if $\mu_1=\mu_2$ then

$$f'(\mu_1)=0, \tag{4.25}$$

where

$$f(\lambda):=\prod_{j=1}^{4}(\lambda-\lambda_j). \tag{4.26}$$

Under the conditions (i) and (ii) the coefficient numbers d_0, b_0, d_1, b_1, and a_0 of the pencil $Q(\lambda)$ for which the sequences in (4.4) are two spectra, are recovered by the formulas

$$d_0=-(\mu_1+\mu_2),\quad b_0=\mu_1\mu_2, \tag{4.27}$$

$$d_1=-(\lambda_1+\lambda_2+\lambda_3+\lambda_4)+\mu_1+\mu_2, \tag{4.28}$$

$$\begin{aligned} b_1 &= \lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_1\lambda_4+\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_3\lambda_4 \\ &-(\mu_1+\mu_2)(\lambda_1+\lambda_2+\lambda_3+\lambda_4)+\mu_1^2+\mu_2^2+\mu_1\mu_2, \end{aligned} \tag{4.29}$$

$$a_0^2=-\prod_{j=1}^{4}(\mu_1-\lambda_j). \tag{4.30}$$

Proof. The necessity of the conditions (i) and (ii) of Theorem 4.1 follows immediately from (4.12). To prove sufficiency suppose that two sequences of complex numbers in (4.4) are given which satisfy the conditions of Theorem 4.1. We construct the numbers d_0, b_0, d_1, b_1, and a_0 by (4.27)–(4.30) and using these numbers, we construct the quadratic pencil $Q(\lambda)$ by (4.1) and its truncation $Q_1(\lambda)$ by (4.3). It follows from (4.27) that μ_1, μ_2 are roots of the quadratic polynomial $Q_1(\lambda)$. It remains to show that $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are roots of the polynomial $\det Q(\lambda)$. For this purpose, we will show that

$$\det Q(\lambda)=f(\lambda), \tag{4.31}$$

where $f(\lambda)$ is defined by (4.26).

Dividing with a remainder the polynomial $f(\lambda)$ by the polynomial $Q_1(\lambda)=(\lambda-\mu_1)(\lambda-\mu_2)$, we can write

$$f(\lambda)=(\lambda^2+\lambda\widetilde{d}_1+\widetilde{b}_1)\prod_{k=1}^{2}(\lambda-\mu_k)+\lambda\alpha+\beta, \tag{4.32}$$

where $\widetilde{d}_1, \widetilde{b}_1, \alpha, \beta$ are some complex numbers. Comparing coefficients of λ^3 and λ^2 on both sides of (4.32), and taking into account (4.28), (4.29), we find that

$$\widetilde{d}_1=d_1,\quad \widetilde{b}_1=b_1.$$

Therefore (4.32) takes the form

$$f(\lambda) = (\lambda^2 + \lambda d_1 + b_1) \prod_{k=1}^{2} (\lambda - \mu_k) + \lambda\alpha + \beta. \tag{4.33}$$

Let us show that $\alpha = 0$ in (4.33). Consider two possible cases.

Case 1: $\mu_1 \neq \mu_2$. In this case, we get, from (4.33),

$$f(\mu_1) = \mu_1\alpha + \beta, \quad f(\mu_2) = \mu_2\alpha + \beta. \tag{4.34}$$

Next, we have $f(\mu_1) = f(\mu_2)$ by the condition (4.24). Therefore (4.34) gives $(\mu_1 - \mu_2)\alpha = 0$ and hence $\alpha = 0$.

Case 2: $\mu_1 = \mu_2$. In this case, (4.33) can be written as

$$f(\lambda) = (\lambda^2 + \lambda d_1 + b_1)(\lambda - \mu_1)^2 + \lambda\alpha + \beta.$$

Hence

$$f'(\lambda) = (2\lambda + d_1)(\lambda - \mu_1)^2 + 2(\lambda^2 + \lambda d_1 + b_1)(\lambda - \mu_1) + \alpha.$$

Therefore $f'(\mu_1) = \alpha$ and by the condition (4.25) we get $\alpha = 0$.

Thus, (4.33) takes the form

$$f(\lambda) = (\lambda^2 + \lambda d_1 + b_1) \prod_{k=1}^{2} (\lambda - \mu_k) + \beta.$$

Hence, taking into account (4.24) and (4.30),

$$f(\mu_1) = f(\mu_2) = \beta = -a_0^2,$$

and, finally, we get

$$f(\lambda) = (\lambda^2 + \lambda d_1 + b_1)Q_1(\lambda) - a_0^2.$$

Comparing this with (4.1), we arrive at (4.31). □

It follows that the pencil (4.1) is not uniquely restored from the two spectra. This is connected with the fact that the a_0 is determined from (4.30) up to a sign. Therefore, we get precisely two distinct quadratic pencils possessing the same two spectra. Thus, we can say that the inverse problem with respect to the two spectra is solved uniquely up to sign of the off-diagonal element of the recovered matrix pencil.

Remark 4.2. If we define the numbers $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$ by (4.8)–(4.10), then

$$f(\lambda) = \lambda^4 + \Lambda_1\lambda^3 + \Lambda_2\lambda^2 + \Lambda_3\lambda + \Lambda_4,$$

$$f'(\lambda) = 4\lambda^3 + 3\Lambda_1\lambda^2 + 2\Lambda_2\lambda + \Lambda_3$$

and, therefore, the condition (4.24) stating that $f(\mu_1) = f(\mu_2)$ is equivalent to the condition

$$\mu_1^3 + \mu_1^2\mu_2 + \mu_1\mu_2^2 + \mu_2^3 \\ +\Lambda_1(\mu_1^2 + \mu_1\mu_2 + \mu_2^2) + \Lambda_2(\mu_1 + \mu_2) + \Lambda_3 = 0, \tag{4.35}$$

whereas the condition (4.25) is equivalent to

$$4{\mu_1}^3 + 3\Lambda_1{\mu_1}^2 + 2\Lambda_2\mu_1 + \Lambda_3 = 0. \tag{4.36}$$

Next, defining the numbers M_1, M_2 by (4.11), it is not difficult to show directly that (4.35), (4.36) are equivalent to (4.21), and the formula (4.30) is equivalent to (4.22).

5 Conclusions

This work deals with the two spectra inverse problem for a class of quadratic matrix pencils.

The quadratic pencil under consideration is the $N \times N$ matrix-valued polynomial given by

$$Q(\lambda) = \lambda^2 I + \lambda G + J,$$

where I is the identity matrix in $\mathbb{C}^N$, G is an $N \times N$ diagonal matrix, and J is an $N \times N$ Jacobi matrix.

The first main result establishes that the spectrum of $Q(\lambda)$ and the spectrum of the truncated pencil $Q_1(\lambda)$ (obtained from $Q(\lambda)$ by deleting the last column and row) uniquely determine the matrices G and J when the off-diagonal elements of the Jacobi matrix J are fixed. The second main result asserts that if the sequence of eigenvalues (taking into account multiplicities) of $Q(\lambda)$ coincides with the sequence of eigenvalues of $Q_0(\lambda)$ obtained from $Q(\lambda)$ by putting $G = 0$, then $Q(\lambda) = Q_0(\lambda)$ provided that the coefficient matrix G in $Q(\lambda)$ is real or pure imaginary. Finally, the paper provides necessary and sufficient conditions for two finite sequences to be the spectra of $Q(\lambda)$ and $Q_1(\lambda)$ in the particular case of $N = 2$. In this case a procedure for reconstruction is given.

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