International Journal of Difference Equations
ISSN 0973-6069, Volume 11, Number 2, pp. 215–223 (2016)
http://campus.mst.edu/ijde

Intensity of Attractors for Closed Relations on Compact Hausdorff Spaces

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Abstract

We investigate two notions for the intensity of attractors for closed relations on compact Hausdorff spaces. Properties of attractors and corresponding attractor blocks are studied, and we show that certain systems of neighborhoods of relations can be used to describe how strongly an attractor attracts its surroundings.

AMS Subject Classifications: 37B35.
Keywords: Dynamics of closed relations, attractors, intensity, invariant sets.

1 Introduction

In the report [8], R. McGehee established two different notions for the intensity of attractors in the setting of discrete dynamical systems on locally compact metric spaces. We will give a brief description of his ideas and results shortly; the main concept was to capture the behavior of orbits in a neighborhood of an attractor. This was done by two different approaches; one was based on the idea of attractor blocks: subsets of the phase space which map into themselves by the map defining the discrete dynamical system; the other was based on the notion of ε-pseudo orbits (or ε-chains): orbits where after each iterative step we are allowed to make a small (ε-size) error. The reason to establish and investigate such a notion originated from the study of structural stability; McGehee argued that in the setting of computer simulations of discrete dynamical systems the notion of ε-structural stability is more useful than the usual notion of structural stability. Indeed, a model system might be structurally stable even though a topologically different system exists within a distance of 10^{-100} from it; if, however, computer simulations

Received May 3, 2016; Accepted May 13, 2016
Communicated by Mehmet Ünal

introduce a round-off error e.g. of order 2^{-64}, then there is a chance that certain topological objects (attractors) might be missed if they do not persist for some perturbations of a certain size.

In two independent papers (see [11, 13]), R. McGehee and the current author investigated various generalizations of some well-established discrete dynamical systems notions and results for the setting of dynamics of closed relations on compact Hausdorff spaces. If the phase space X is a compact Hausdorff space, a closed relation is a closed subset of $X \times X$ in the product topology; a relation then gives rise to a dynamical system where orbits are generated using this relation. The manifold motivations and supporting claims for this approach to study discrete dynamical systems can be found in the introduction of these papers; we just highlight here that the ease and naturalness of the use of relations give a strong argument that this setting is the right approach for some of the topics connected to long-term behavior of dynamical systems.

The dynamics of iterations have been studied by other authors. McGehee [9] introduced variants of ω-limit sets for studying attractors and repellers for closed relations on compact Hausdorff spaces, and studied the notions of invariance using the given setting; Akin [1] developed a general theory for iterations of relations on compact metric spaces; a theory of entropy for relations was created by Langevin, Walczak and Przyticky [5,6]; Barnsley and Vince [2–4] investigated fractals generated by iterated function systems, i.e. relations; McGehee and Sander [10] gave a new proof for the stable manifold theorem using the setting of relations; Sander [12] introduced a notion for hyperbolicity for noninvertible maps and relations; and Manjunath and Jaeger [7] used closed relations to simplify the understanding of the dynamics of random difference equations.

2 Basic Notions and Results

The basic notions and results we need are introduced in [11]. In what follows, we repeat the necessary definitions, and some of the lemmas and theorems established there. The proofs are omitted; they can be found in [11]. The setting of the dynamics is closed relations on compact Hausdorff spaces.

Definition 2.1. A *relation* on a set X is a subset of $X \times X$.

Definition 2.2. If f is a relation on X and $S \subset X$, then the *image* of S under f is the set

$$f(S) \equiv \{y \in X : \text{ there exists } x \in S \text{ satisfying } (x, y) \in f\}.$$

The following construction gives the inverse image of a set; we note that f^{-1} is not necessarily generated by a relation.

Definition 2.3. If f is a relation on X and $S \subset X$, then the *inverse image* of S is the set

$$f^{-1}(S) \equiv \{x \in X : f(x) \subset S\}.$$

A relation f on a set X therefore can be thought of as a set-valued function on X. To create a dynamical system, we iterate relations. To do this, we must be able to compose them. The following definition is the usual generalization of the definition of composition of maps.

Definition 2.4. If f and g are relations on X, then the *composition* of g with f is the relation $g \circ f$ on X defined by

$$g \circ f \equiv \{(x, z) \in X \times X : \text{there exists } y \in X \text{ such that } (x, y) \in f \text{ and } (y, z) \in g\}.$$

If X is a topological space, then notions of open and closed subsets of $X \times X$ are extended to relations, using the product topology on $X \times X$.

Definition 2.5. If X is a topological space, then a relation f on X is called *open* if and only if f is an open subset of $X \times X$. It is called *closed* if and only if f is a closed subset of $X \times X$.

Recall that we call a subset U of a topological space a neighborhood of V if U contains an open set which contains V. The notation $\mathfrak{N}(V)$ is used to denote the set of all neighborhoods of the set V. Similarly, $\mathfrak{N}^o(V)$ will denote the set of open neighborhoods and $\overline{\mathfrak{N}}(V)$ will denote the set of closed neighborhoods. Note the convenient fact that a neighborhood of a relation is again a relation.

The next lemmas show how images and neighborhoods behave in a topological space.

Lemma 2.6. *If f is an open relation on a topological space X and $S \subset X$, then $f(S)$ is open in X.*

Lemma 2.7. *If f is a closed relation on a compact Hausdorff space X and $K \subset X$ is compact, then $f(K)$ is closed.*

Lemma 2.8. *If f is a relation on a topological space X, K is a closed subset of X and $U \in \mathfrak{N}^o(f(K))$, then there exists $\phi \in \mathfrak{N}^o(f)$ such that $\phi(K) \subset U$.*

To define the nth iterate of a relation it is convenient to introduce first the diagonal (or identity) relation defined as

$$\iota \equiv \{(x, x) \in X \times X : x \in X\}.$$

Then the iterations of the relation can be defined the usual way.

Definition 2.9. If f is a relation on a set X and n is a nonnegative integer, then the relation f^n is defined inductively by

$$f^0 = \iota \text{ and } f^n = f \circ f^{n-1} \text{ for } n = 1, 2, 3, \ldots.$$

One of the most studied property of dynamical systems is limit behavior. For the iterations of relations, these are investigated through the following constructions, due to R. McGehee [11].

Definition 2.10. If f is a relation on a set X, then the *limit relation* of f is

$$f^\infty \equiv \bigcap_{n\geq 0} \bigcup_{k\geq n} f^k.$$

Definition 2.11. If f is a relation on a topological space X, then the *Conley relation* of f is

$$f^\Omega \equiv \bigcap_{\phi\in\overline{\mathfrak{N}}(f)} \phi^\infty.$$

We state some of the basic properties of these constructions.

Theorem 2.12. *If f and g are closed relations on a compact Hausdorff space X, then the following properties hold.*

(a) *If $g \subset f$, then $g^\infty \subset f^\infty$ and $g^\Omega \subset f^\Omega$.*

(b) $f^\infty \subset f^\Omega$.

Theorem 2.13. *If f is a closed relation on a compact Hausdorff space X, then f^Ω is closed.*

Theorem 2.14. *If f is a closed relation on a compact Hausdorff space X, then the following inclusions hold.*

(a) $f \circ f^\infty \subset f^\infty$, $f^\infty \circ f \subset f^\infty$ *and* $f^\infty \circ f^\infty \subset f^\infty$.

(b) $f \circ f^\Omega = f^\Omega$, $f^\Omega \circ f = f^\Omega$ *and* $f^\Omega \circ f^\Omega = f^\Omega$.

The following corollary is an immediate consequence of Lemma 2.7 and Theorem 2.13.

Corollary 2.15. *If f is a closed relation on a compact Hausdorff space X and K is a closed subset of X, then $f^\Omega(K)$ is closed.*

3 Attractors and Attractor Blocks for Closed Relations

The concept of attraction is widely used in dynamical systems theory. In this section, we define the notion of attractor for closed relations, following again [11]. We repeat some of the results proved there as well; we want to illustrate the fact that this notion is really a generalization of the well-known definitions of attractor for maps. Proofs are omitted; they can be found in [11].

Definition 3.1. If f is a closed relation on a compact Hausdorff space X, then $A \subset X$ is called an *attractor* for f if there exists $G \in \overline{\mathfrak{N}}(A)$ such that

$$f^{\Omega}(G) = A.$$

The *basin* of the attractor A (or domain of attraction) is the set

$$\mathrm{B}(A) \equiv (f^{\Omega})^{-1}(A).$$

The following lemmas give some fundamental properties of attractors and their basins.

Lemma 3.2. *If A is an attractor for a closed relation f on a compact Hausdorff space X, then the following properties hold.*

(a) *A is closed.*

(b) $f^{\Omega}(A) = f(A) = A$.

(c) $S \subset \mathrm{B}(A)$ *if and only if* $f^{\Omega}(S) \subset A$.

(d) $A \subset \mathrm{B}(A)$.

(e) $f^{\Omega}(\mathrm{B}(A)) = A$.

(f) $\mathrm{B}(A)$ *is open.*

(g) $f^{-1}(\mathrm{B}(A)) = \mathrm{B}(A)$.

Lemma 3.3. *If A is an attractor for a closed relation f on a compact Hausdorff space X, then for any $V \in \mathfrak{N}(A)$ there exists $g \in \overline{\mathfrak{N}}(f)$ such that $g^{\Omega}(A) \subset V^{o}$.*

Attractor blocks are important as they signify the presence of attractors in an easily verifiable manner.

Definition 3.4. If f is a closed relation on a compact Hausdorff space X, then $B \subset X$ is called an *attractor block* for f if

$$f(\overline{B}) \subset B^{o}.$$

A basic property of an attractor is that it can be surrounded by an attractor block; conversely, if we have an attractor block, (i.e. a set whose image lies strictly inside itself) then inside the attractor block we have an attractor. Note that in the definition of the attractor block we did not use any of the constructions given before: it is strictly a statement about the image of a set under the relation.

The following two theorems give the above mentioned connection between attractors and attractor blocks: every attractor block has an attractor in its interior and every attractor can be surrounded by an attractor block.

Theorem 3.5. *If B is an attractor block for a closed relation f on a compact Hausdorff space X, then $f^{\Omega}(B) \subset B^o$ and $f^{\Omega}(B)$ is an attractor for f.*

Theorem 3.6. *If A is an attractor for a closed relation f on a compact Hausdorff space X and V is a neighborhood of A, then there exists a closed attractor block B for f such that $B \subset V$ and $f^{\Omega}(B) = A$.*

4 Intensity for Maps on Locally Compact Metric Spaces

The following is a brief overview of the definitions of two kinds of intensity from [8]. The results are developed for continuous maps on locally compact metric spaces, and the notion of attractor and attractor block are identical to the ones presented here, for the special case of maps. In this section, let $\phi : X \to X$ be a continuous map on the locally compact metric space X.

McGehee establishes in [8] that it is possible to assign a precise quantity to measure the strength of attraction of an attractor.

The first quantity is called "intensity" and is defined in the following way. Every attractor block B associated with A has the property that the minimum distance from the image of B to its complement does not exceed the intensity of A; i.e. let $\beta(B)$ the minimum distance from $\phi(B)$ to B^c; and then let

$$\nu(A) \equiv \sup\{\beta(B) : B \text{ is an attractor block associated with } A\}.$$

The second definition, called "chain intensity", assigns to an attractor A the supremum over all values of ε such that every ε-pseudo orbit (or ε-chain) starting in A stays in some compact subset of the domain of attraction of A. That is, every ε-pseudo orbit which starts in A and for which ε does not exceed the chain intensity of A remains inside a compact subset of the domain of attraction of A. On the other hand, if ε does exceed the chain intensity of A, then one can find an ε-pseudo orbit starting in A and leaving every compact subset of the domain of attraction of A. This is given in [8] as

$$\mu(A) \equiv \sup\{\varepsilon : P_\varepsilon(A) \subset K \subset \mathrm{B}(A), \text{ where } K \text{ is compact}\};$$

here $P_\varepsilon(A)$ denotes the set of all points on all ε-pseudo orbits of arbitrary length, starting at a point in A. Recall that a sequence $(z_0, z_1, \ldots, z_n)$ of points in X satisfying

$$d(\phi(z_{k-1}), z_k) < \varepsilon \text{ , for } k = 1, 2, \ldots, n,$$

is called an ε-pseudo orbit of length n. McGehee then proceeds to prove that these two definitions give the same value; see [8].

5 Intensity of Attractors for Closed Relations

We now return to the setting of closed relations on compact Hausdorff spaces and formulate the topological version of intensity. A compact Hausdorff space is not necessarily metrizable (only if it is second countable); the absence of a metric compels us to define the intensity in a different way. Instead of a real number, we use families of neighborhoods of the relation to describe the strength of attractors.

Let A be an attractor on a compact Hausdorff space X; for the definition of intensity we introduce the following notation.

$$\mathfrak{B}_A \equiv \{B : B \text{ is a closed attractor block associated with } A\}.$$

The first possible definition of intensity utilizes attractor blocks.

Definition 5.1. The *intensity* of the attractor A is the following family of closed neighborhoods of f:

$$\mathfrak{J}(A) \equiv \{\psi \in \overline{\mathfrak{N}}(f) : \text{there exists } B \in \mathfrak{B}_A \text{ such that } \psi(B) \subset B^o\}.$$

The second possible definition of intensity uses the Conley relation.

Definition 5.2. The *chain intensity* of the attractor A is the following family of closed neighborhoods of f:

$$\mathfrak{I}(A) \equiv \{\phi \in \overline{\mathfrak{N}}(f) : \phi^\Omega(A) \subset \mathrm{B}(A)\}.$$

The main result of this note is the next theorem. It asserts that the two families defined above are identical.

Theorem 5.3. $\mathfrak{J}(A) = \mathfrak{I}(A)$.

Proof. Let $\phi \in \mathfrak{I}(A)$. This means $\phi^\Omega(A) \subset \mathrm{B}(A)$. By Lemma 3.3 there exists $\chi \in \overline{\mathfrak{N}}(\phi)$ such that $\chi^\Omega(A) \subset \mathrm{B}(A)$. Let $B \equiv \chi^\Omega(A)$. Then B is closed by Corollary 2.15 and because $\chi \in \overline{\mathfrak{N}}(f)$, there exists an open ξ such that $f \subset \xi \subset \chi$ and this (using Lemma 2.6 and Theorem 2.14) implies that

$$f(B) = f(\chi^\Omega(A)) \subset \xi(\chi^\Omega(A)) \subset \chi(\chi^\Omega(A)) = \chi^\Omega(A) = B$$

which means $f(B) \subset B^o$ and proves that B is a closed attractor block for f. $\chi \in \overline{\mathfrak{N}}(\phi)$, which implies that there exists an open η such that $\phi \subset \eta \subset \chi$ and then

$$\phi(B) \subset \eta(B) \subset \chi(B) = \chi(\chi^\Omega(A)) = \chi^\Omega(A) = B,$$

which implies $\phi(B) \subset B^o$, then $\phi \in \mathfrak{J}(A)$ and we conclude that $\mathfrak{I}(A) \subset \mathfrak{J}(A)$.

Let $\psi \in \mathfrak{J}(A)$. This means there exists $B \in \mathfrak{B}_A$ such that $\psi(B) \subset B^o$. By Lemma 2.8 and the normality of $X \times X$ there exists $\eta \in \overline{\mathfrak{N}}(\psi)$, such that $\eta(B) \subset B^o$. By a simple induction we deduce that $\eta^k(B) \subset B^o$, and this implies that $\eta^\infty(B) \subset B^o$. Then $\psi^\Omega(B) \subset B^o \subset \mathrm{B}(A)$ and we conclude that $\mathfrak{J}(A) \subset \mathfrak{I}(A)$ and this completes the proof. □

6 Possible Extensions

In [8], McGehee also deals with subsystems, i.e. the restriction of maps to positively invariant subsets of the base space. Attractors in the setting of maps restrict to subsystems in a clear and elegant way. The setting of dynamics of relations has its pitfalls, though: the notion of invariance is quite different in this case. The various possibilities might give rise to different results for the iterations of relations. We plan to investigate these generalizations.

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