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Closed-form Second Solution to the Confluent Hypergeometric Difference Equation in the Degenerate Case

William C. Parke and **Leonard C. Maximon**
The George Washington University
Department of Physics
Washington, DC 20052 U.S.A.
wparke@gwu.edu and max@gwu.edu

Abstract

We derive a closed-form second solution to the difference equations for the confluent hypergeometric function in which the usual first parameter is a nonpositive integer and the second parameter is a positive integer, i.e., a degenerate case for which the usual second solution is no longer independent of the first. Our second solution enables us to define a linearly independent associated Laguerre function of the second kind which satisfies the difference equations for the confluent hypergeometric function in each of its two parameters in terms of appropriately normalized polynomials. Explicit expressions for these polynomials are given.

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1 Introduction

An independent second solution to the confluent hypergeometric differential equation when the first parameter is a nonpositive integer and the second parameter is a positive integer is given in the current literature as a logarithmic and Laurent series (see [1, Eq.13.2.28]). We give, in closed form, a second independent solution which satisfies both the corresponding differential equation and the difference equation in each of the parameters. The first solution of the confluent hypergeometric differential equation,

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${}_1F_1(-N; n+1; x)$, is, apart from normalization, the well-known associated Laguerre function $L_N^{(n)}(x)$. The polynomial in our second solution together with the factor x^{-n} and an appropriate normalization defines a linearly independent associated Laguerre function of the second kind which satisfies the difference equations for the confluent hypergeometric function in each of its parameters. For closely related analyses see Area *et al.* [2] and Weixlbaumer [5].

2 Derivation

Consider the differential equation for the confluent hypergeometric function, expressed as:

$$xy'' + (b-x)y' - ay = 0 \tag{2.1}$$

in which $a = -N$, a nonpositive integer, and $b = n+1$, a positive integer. Although the analysis that follows requires $b = n+1$, our interest came from the observation that the two standard solutions, ${}_1F_1(a; b; x)$ and $U(a, b, x)$, are no longer independent when $a = -N$, in which case $U(-N, b, x) = (-1)^N (b)_N \; {}_1F_1(-N; b; x)$ (see DLMF [1, Eqs. 13.2.7, 13.2.10 and 13.2.34]). (The Pochhammer symbol $(b)_N = \Gamma(b+N)/\Gamma(b)$.)

Equation (2.1) has a polynomial solution defined by the confluent hypergeometric function

$$\Phi(N, n, x) \equiv {}_1F_1(-N; n+1; x) = \sum_{k=0}^{N} \frac{(-N)_k}{(n+1)_k k!} x^k, \tag{2.2}$$

which constitutes a first solution of the differential equation in x and a first solution of the difference equation arising from a recurrence relation for the confluent hypergeometric, namely

$$(N+n+1)\Phi(N+1, n, x) - (2N+n+1-x)\Phi(N, n, x) + N\Phi(N-1, n, x) = 0. \tag{2.3}$$

We derive a polynomial solution to this equation that is linearly independent of the function $\Phi(N, n, x)$.

Following Nikiforov and Uvarov [4, §11, p. 97, Eq. (4)], a second linearly independent solution to Eq. (2.1) is given by the extended Cauchy integral:[1]

$$\Psi(N, n, x) = \frac{1}{\rho(x)} \int_0^\infty \frac{\rho(s)\Phi(N, n, s)}{s-x} ds \tag{2.4}$$

in which the weight function $\rho(x) = e^{-x} x^{b-1}$ is, for the differential equation (2.1), a solution of the equation $(x\rho(x))' = (b-x)\rho(x)$.

[1] Second solutions may also be expressed using a reduction-of-order method sometimes called "variation of parameters". We have found that this alternate technique may also be used to get our closed-form second solution. However, in the present case, the extended Cauchy integral method is a little more straightforward.

Lemma 2.1. *The* $\Psi(N,n,x)$ *function as defined in* (2.4) *with* $b=n+1$ *obeys the same difference equation as* $\Phi(N,n,x) \equiv {}_1F_1(-N;n+1;x)$.

Proof. Using Eq. (2.4), we have

$$
\begin{aligned}
&(N+n+1)\Psi(N+1,n,x) - (2N+n+1-x)\Psi(N,n,x) + N\Psi(N-1,n,x) \\
&= x^{-n}e^{x}\int_0^\infty ds\frac{e^{s}s^{n}}{s-x}\times \\
&\big[(N+n+1)\Phi(N+1,n,s) - (2N+n+1-x)\Phi(N,n,s) + N\Phi(N-1,n,s)\big]\,ds \\
&= x^{-n}e^{x}\int_0^\infty ds\frac{e^{s}s^{n}}{s-x}\times \\
&\Big[(N+n+1)\Phi(N+1,n,s) - (2N+n+1-s)\Phi(N,n,s) + N\Phi(N-1,n,s) \\
&\qquad -(s-x)\Phi(N,n,s)\Big] \\
&= -x^{-n}e^{x}\int_0^\infty e^{-s}s^{n}\Phi(N,n,s)\,ds
\end{aligned}
\tag{2.5}
$$

in view of (2.3). By substituting (2.2) in the last integral in (2.5), we will have

$$
\begin{aligned}
\int_0^\infty e^{-s}s^{n}\Phi(N,n,s)\,ds &= \int_0^\infty e^{-s}s^{n}\sum_{k=0}^{N}\frac{(-N)_k s^k}{(n+1)_k k!}ds \\
&= \sum_{k=0}^{N}\frac{(-N)_k}{(n+1)_k k!}\int_0^\infty e^{-s}s^{n+k}\,ds \\
&= \sum_{k=0}^{N}\frac{(-N)_k\Gamma(n+k+1)}{(n+1)_k k!} \\
&= n!\sum_{k=0}^{N}(-1)^k\binom{N}{k} \\
&= n!\,(1-1)^N = 0
\end{aligned}
\tag{2.6}
$$

for $N=1,2,\ldots$. Thus, the function $\Psi(N,n,x)$ satisfies the difference equation (2.3), i.e.,

$$
(N+n+1)\Psi(N+1,n,x) - (2N+n+1-x)\Psi(N,n,x) + N\Psi(N-1,n,x) = 0 \tag{2.7}
$$

which proves the lemma. □

Theorem 2.2. *The second solution* $\Psi(N,n,x)$ *as defined in* (2.4) *with* $b=n+1$ *can be expressed as a polynomial of order* $N+n-1$ *times* $x^{-n}e^{x}$ *minus the first solution* $\Phi(N,n,x)$ *times the exponential-integral function* $Ei(x)$.

Proof. Write (2.4) in the form

$$
\begin{aligned}
\Psi(N,n,x) &= x^{-n}e^{x}\int_0^\infty \frac{e^{-s}}{s-x}\left[s^{n}\Phi(N,n,s) - x^{n}\Phi(N,n,x)\right]\,ds \\
&\quad + \Phi(N,n,x)\int_{-x}^\infty \frac{e^{-s}}{s}ds\,.
\end{aligned}
\tag{2.8}
$$

Inserting the polynomial expression (2.2) for Φ, the first of the two integrals in Eq. (2.8) is

$$\sum_{k=0}^{N} \frac{(-N)_k}{(n+1)_k k!} \int_0^\infty \frac{e^{-s}}{(s-x)} \left[s^{n+k} - x^{n+k}\right] ds$$
$$= \sum_{k=0}^{N} \frac{(-N)_k}{(n+1)_k k!} \sum_{m=0}^{n+k-1} (n+k-1-m)!\, x^m \tag{2.9}$$

while in the last term of Eq. (2.8) the exponential-integral function

$$\mathrm{Ei}\,(x) = -\int_{-x}^{\infty} \frac{e^{-s}}{s}\, ds$$

appears. There results[2]

$$\Psi(N,n,x) = \frac{n!}{(N+n)!} P(N,n,x) \frac{e^x}{x^n} - \Phi\,(N,n,x)\,\mathrm{Ei}(x) \tag{2.10}$$

where the polynomial $P\,(N,n,x)$ is

Definition 2.3.

$$P\,(N,n,x) = \frac{(N+n)!}{n!} \sum_{k=0}^{N} \sum_{m=0}^{n+k-1} \frac{(-N)_k (n+k-1-m)!}{(n+1)_k\; k!} x^m\,. \tag{2.11}$$

The expressions in Eqs. (2.10) and (2.11) constitute an explicit closed-form second solution to the confluent hypergeometric difference equations and differential equation in the degenerate case. As such, they prove Theorem (2.2). □

Note: The normalization of the polynomial $P(N,n,x)$ has been chosen to make the coefficient of x^{N+n-1} be $(-1)^N$. It then turns out that all the coefficients of the powers of x are integers.

Theorem 2.4. *Both* $\Phi(N,n,x)$ *and*

Definition 2.5.

$$\overline{\Psi}(N,n,x) \equiv \frac{n!}{(N+n)!} P(N,n,x) \tag{2.12}$$

[2] In considering homogeneous second-order hypergeometric differential equations whose first solution is a polynomial, Nikiforov and Uvarov [4, §11, p. 97, Eq. (5)] have shown that the second solution can be separated into a polynomial part and a "logarithmic" part. The simplest example of this separation is provided by the Legendre polynomial: $Q_n(x) = -W_{n-1}(x) + \frac{1}{2} P_n(x) \ln((1+x)/(1-x))$, in which $W_{n-1}(x)$ is a polynomial of order $n-1$ (see Erdélyi [3, §3.6.2, Eqs. (24) and (26)]). The expression given above in (2.10) illustrates this same separation, derived as in the analysis in [4, §11, p. 97, Eqs. (4) and (5)] but with the important modification of attaching the factor x^n in $\rho(x)$ to the solution y_n to produce (2.8).

satisfy the difference equation (2.3).

Proof. Both $\Phi(N,n,x)$ and $\Psi(N,n,x)$ satisfy (2.1) and (2.7), with $a = -N$ and $b = n+1$. Being that the difference equation (2.7) is linear and homogeneous, factors independent of N, such as e^x and $Ei(x)$, do not change the equality. □

Theorem 2.6. *The polynomial solutions of the difference equation* (2.3)*,* $\overline{\Psi}(N,n,x)$ *and* $\Phi(N,n,x)$*, are linearly independent.*

Proof. Multiplying (2.3) by $\overline{\Psi}(N,n,x)$ and (2.7) written for $\overline{\Psi}$ by $\Phi(N,n,x)$ and subtracting, we have

$$(N+n+1)\mathcal{C}(N+1) = N\mathcal{C}(N) \tag{2.13}$$

where $\mathcal{C}$ is the Casoratian:

$$\mathcal{C}(N) = \Phi(N,n,x)\overline{\Psi}(N-1,n,x) - \overline{\Psi}(N,n,x)\Phi(N-1,n,x)\ .$$

From (2.13)

$$\mathcal{C}(N+1) = \frac{N}{N+n+1}\mathcal{C}(N)$$

from which

$$\mathcal{C}(N+1) = \prod_{k=1}^{N}\Big(\frac{k}{k+n+1}\Big)\mathcal{C}(1)\ .$$

From (2.2) and (2.12),

$$\begin{aligned}
\Phi(0,n,x) &= 1\\
\Phi(1,n,x) &= 1 - \frac{x}{n+1}\\
\overline{\Psi}(0,n,x) &= \sum_{m=0}^{n-1}(n-1-m)!\,x^m\\
\overline{\Psi}(1,n,x) &= \int_0^\infty \frac{e^{-s}}{s-x}\Big[s^n\Big(1-\frac{s}{n+1}\Big) - x^n\Big(1-\frac{x}{n+1}\Big)\Big]\\
&= \sum_{m=0}^{n-1}(n-1-m)!\,x^m - \frac{1}{n+1}\sum_{m=0}^{n}(n-m)!\,x^m\\
&= \sum_{m=0}^{n-1}(n-1-m)!\,x^m - \frac{1}{n+1}\sum_{m=-1}^{n-1}(n-1-m)!\,x^{m+1}\\
&= \Big[1-\frac{x}{n+1}\Big]\sum_{m=0}^{n-1}(n-1-m)!\,x^m - \frac{n!}{n+1}\ .
\end{aligned}$$

We then have

$$\mathcal{C}(1) = \Phi(1,n,x)\overline{\Psi}(0,n,x) - \overline{\Psi}(1,n,x)\Phi(0,n,x) = \frac{n!}{n+1}$$

from which

$$\mathcal{C}(N+1) = \frac{N!(n!)^2}{(N+n+1)!} ,$$

thus proving that the polynomials $\overline{\Psi}(N,n,x)$ and $\Phi(N,n,x)$ are linearly independent solutions of the difference equation (2.3). □

3 Simplifying the Solution

Lemma 3.1. *The expression for the polynomials* (2.11) *may be considerably simplified.*

Proof. First, use

$$\sum_{k=0}^{N} \sum_{m=0}^{n+k-1} = \sum_{m=0}^{n-1} \sum_{k=0}^{N} + \sum_{m=n}^{N+n-1} \sum_{k=m-n+1}^{N} . \tag{3.1}$$

From the first double sum on the right-hand side we have, writing $(n+k-1-m)! = (n-m)_k(n-m-1)!$,

$$\sum_{m=0}^{n-1} x^m (n-m-1)! \sum_{k=0}^{N} \frac{(-N)_k (n-m)_k}{(n+1)_k k!}$$

in which we can use Gauss' formula ${}_2F_1(a,b;c;1) = \dfrac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$ to write

$$\sum_{k=0}^{N} \frac{(-N)_k (n-m)_k}{(n+1)_k k!} = {}_2F_1(-N, n-m; n+1; 1) = \frac{n!(N+m)!}{(N+n)!m!} .$$

Thus, the first double sum on the right-hand side of (3.1) gives

$$\frac{n!}{(N+n)!} \sum_{m=0}^{n-1} \frac{(N+m)!(n-m-1)!}{m!} x^m .$$

From (2.11) and (3.1) there results

$$\begin{aligned} &P(N,n,x) \\ &= \sum_{m=0}^{n-1} \left[\frac{(N+m)!\,(n-m-1)!}{m!} \right] x^m \\ &- x^n \sum_{m=0}^{N-1} \left[\sum_{k=0}^{N-m-1} \frac{N!}{(N-k-m-1)!} \frac{(N+n)!}{(n+k+m+1)!} \frac{(-1)^k k!}{(k+m+1)!} \right] (-x)^m . \end{aligned}$$

The expression for the coefficients in the inner sum of the second term, which we write as

$$\begin{aligned} &c\,(N,n,m)\\ &=(-1)^{m+1}\sum_{k=0}^{N-m-1}(-1)^k\,\frac{N!}{(N-k-m-1)!}\frac{(N+n)!}{(n+k+m+1)!}\frac{k!}{(k+m+1)!}\\ &=(-1)^{m+1}\,\frac{N!}{(N-m-1)!}\frac{(n+N)!}{(m+n+1)!\,(m+1)!}\\ &\quad\times{}_3F_2\,(1,1,-N+m+1;2+m,2+m+n;1) \end{aligned} \tag{3.2}$$

can also be simplified.

First, we re-express the ${}_3F_2$ hypergeometric polynomial in terms of an integral over an ${}_2F_1$ hypergeometric function (DLMF [1, 16.5.2])

$$\begin{aligned} &{}_3F_2\,(a_1,a_2,c;b,d;z)\\ =\ &\frac{\Gamma\,(d)}{\Gamma\,(c)\,\Gamma\,(d-c)}\int_0^1 t^{c-1}\,(1-t)^{d-c-1}\ {}_2F_1\,(a_1,a_2;b;zt)\,dt\ . \end{aligned}$$

In our case,

$$\begin{aligned} &{}_3F_2\,(-N+m+1,1,1;m+n+2,m+2;1)\\ =\ &(m+1)\int_0^1 (1-t)^m\ {}_2F_1\,(-N+m+1,1;m+n+2;t)\,dt\ . \end{aligned}$$

In turn, the ${}_2F_1$ can be written as an integral (DLMF [1, 15.6.1]), giving

$$\begin{aligned} &{}_3F_2\,(-N+m+1,1,1;m+n+2,m+2;1)\\ =\ &(m+1)\,(m+n+1)\int_0^1 (1-t)^m\left(\int_0^1 (1-s)^{m+n}\,(1-st)^{N-m-1}\,ds\right)\,dt\ . \end{aligned}$$

Now the trick in this simplification is to expand the integrand factor $(1-st)^{N-m-1}$ not in s, but in $1-s\equiv u$:

$$\begin{aligned} &{}_3F_2\,(-N+m+1,1,1;m+n+2,m+2;1)\\ =\ &(m+1)\,(m+n+1)\int_0^1 (1-t)^m\left(\int_0^1 u^{m+n}\,(1-t+tu)^{N-m-1}\,du\right)\,dt\\ =\ &(m+1)\,(m+n+1)\sum_{k=0}^{N-m-1}\binom{N-m-1}{k}\int_0^1 (1-t)^{N-1-k}\,t^k\left(\int_0^1 u^{m+n}u^k du\right)\,dt\\ =\ &(m+1)\,(m+n+1)\sum_{k=0}^{N-m-1}\binom{N-m-1}{k}\frac{1}{m+n+k+1}\int_0^1 t^k\,(1-t)^{N-1-k}\ dt\\ =\ &(m+1)\,(m+n+1)\sum_{k=0}^{N-m-1}\frac{(N-m-1)!}{(N-m-1-k)!k!}\frac{k!\,(N-k-1)!}{N!}\frac{1}{m+n+k+1}\\ =\ &\frac{(m+1)\,(m+n+1)}{N!}\,(N-m-1)!\sum_{k=0}^{N-m-1}\frac{(N-k-1)!}{(N-m-1-k)!}\frac{1}{m+n+k+1} \end{aligned}$$

where we used

$$\int_0^1 t^a (1-t)^b \, dt = \frac{\Gamma(a+1)\,\Gamma(b+1)}{\Gamma(a+b+2)}.$$

We have arrived at an alternate and simpler expression for our coefficients:

$$\begin{aligned} & c(N,n,m) \\ = \; & (-1)^{m+1} \frac{(n+N)!}{(n+m)!m!} \sum_{k=0}^{N-m-1} \frac{(N-1-k)!}{(N-1-k-m)!} \frac{1}{n+m+k+1} \\ = \; & (-1)^{m+1} \frac{(n+N)!}{(n+m)!} \sum_{k=n+m+1}^{N+n} \binom{N+n+m-k}{m} \frac{1}{k} \\ = \; & (-1)^{m+1} \frac{(n+N)!}{(n+m)!m!} \sum_{k=n+m+1}^{N+n} (N+n+1-k)_m \frac{1}{k} \, . \end{aligned} \tag{3.3}$$

This completes the proof. □

As a check, note that if $m = N-1$ (corresponding to the highest power of x in our polynomial $P(N,m,x)$), then

$$c(N,n,N-1) = (-1)^N \; .$$

The case $m = 0$ is particularly simple, and shows that harmonic numbers enter these coefficients.

$$c(N,n,0) = -\frac{(n+N)!}{n!} \sum_{k=n+1}^{N+n} \frac{1}{k} \, .$$

The case $m = 1$ leads to

$$c(N,n,1) = \frac{(n+N)!}{(n+1)!} \left((N+n+1) \sum_{k=n+2}^{N+n} \frac{1}{k} - (N-1) \right) \, .$$

The Pochhammer factor $(N+n+1-k)_m$ in Eq. (3.3) is a polynomial of degree m in the summation variable k. The contribution to $c(N,n,m)$ from this polynomial expanded in powers of k will be a harmonic sum from the k^0 term while the terms for powers of k from 1 to m will, after canceling the denominator k, lead to a polynomial in N and n.

4 Special Cases

For the purpose of verifying computational algorithms, we give here a few explicit cases for the polynomial $P(N,n,x)$:

n	$N=0$
0	0
1	1
2	$x+1$
3	x^2+x+2
4	x^3+x^2+2x+6
5	$x^4+x^3+2x^2+6x+24$
6	$x^5+x^4+2x^3+6x^2+24x+120$

n	$N=1$
0	-1
1	$-x+1$
2	$-x^2+2x+1$
3	$-x^3+3x^2+2x+2$
4	$-x^4+4x^3+3x^2+4x+6$
5	$-x^5+5x^4+4x^3+6x^2+12x+24$
6	$-x^6+6x^5+5x^4+8x^3+18x^2+48x+120$

n	$N=2$
0	$x-3$
1	x^2-5x+2
2	x^3-7x^2+6x+2
3	$x^4-9x^3+12x^2+6x+4$
4	$x^5-11x^4+20x^3+12x^2+12x+12$
5	$x^6-13x^5+30x^4+20x^3+24x^2+36x+48$
6	$x^7-15x^6+42x^5+30x^4+40x^3+72x^2+144x+240$

n	$N=3$
0	$-x^2+8x-11$
1	$-x^3+11x^2-26x+6$
2	$-x^4+14x^3-47x^2+24x+6$
3	$-x^5+17x^4-74x^3+60x^2+24x+12$
4	$-x^6+20x^5-107x^4+120x^3+60x^2+48x+36$
5	$-x^7+23x^6-146x^5+210x^4+120x^3+120x^2+144x+144$
6	$-x^8+26x^7-191x^6+336x^5+210x^4+240x^3+360x^2+576x+720$

n	$N = 4$
0	$x^3 - 15\,x^2 + 58\,x - 50$
1	$x^4 - 19\,x^3 + 102\,x^2 - 154\,x + 24$
2	$x^5 - 23\,x^4 + 158\,x^3 - 342\,x^2 + 120\,x + 24$
3	$x^6 - 27\,x^5 + 226\,x^4 - 638\,x^3 + 360\,x^2 + 120\,x + 48$
4	$x^7 - 31\,x^6 + 306\,x^5 - 1066\,x^4 + 840\,x^3 + 360\,x^2 + 240\,x + 144$
5	$x^8 - 35\,x^7 + 398\,x^6 - 1650\,x^5 + 1680\,x^4 + 840\,x^3 + 720\,x^2 + 720\,x + 576$
6	$x^9 - 39x^8 + 502x^7 - 2414x^6 + 3024x^5 + 1680x^4 + 1680x^3 + 2160x^2 + 2880x + 2880$

Note: The coefficients of x in the polynomial $P\,(N, n, x)$ up to the power x^{n-1} are all positive and contain relatively simple (factorial) factors, while those for powers x^n up to the highest power x^{N+n-1} have oscillating signs and some may have very high prime number factors, much larger than $N + n - 1$, so that they will not reduce to simple factorials. Rather, the coefficients for powers at and above x^n involve harmonic sums[3] (See Section 3).

5 Associated Laguerre function of the Second Kind

The function $\Phi(N, n, x)$ considered here is, apart from a normalization factor, the well-known associated Laguerre polynomial $L_N^{(n)}(x)$ (DLMF [1, Eq. 18.5.12]). A normalized associated Laguerre polynomial of the first kind can be defined by

Definition 5.1.

$$\overline{L}_N^{(n)}(x) \equiv \frac{N!\,n!}{(N+n)!} L_N^{(n)}(x) = \Phi(N, n, x).$$

We define an independent associated Laguerre function of the second kind by

Definition 5.2.

$$\overline{\overline{L}}_N^{(n)}(x) \equiv \frac{n!}{(N+n)!}\,\frac{1}{x^n} P(N, n, x)$$

with the advantage that both of these Laguerre functions, $\overline{L}_N^{(n)}(x)$ and $\overline{\overline{L}}_N^{(n)}(x)$, satisfy recurrence relations as given by DLMF [1, Eqs. 13.3.1, 13.3.2]:

$$(b-a)\,{}_1F_1(a-1;b;x) + (2a-b+x)\,{}_1F_1(a;b;x) - a\,{}_1F_1(a+1;b;x) = 0 \qquad (5.1)$$

$$b(b-1)\,{}_1F_1(a;b-1;x) + b(1-b-x)\,{}_1F_1(a;b;x) + x(b-a)\,{}_1F_1(a;b+1;x) = 0 \quad (5.2)$$

[3]These properties of the coefficients apply even more generally to the second solution polynomials allied with the full hypergeometric functions ${}_2F_1(-N, b; c; x)$, but we leave the explicit derivation of these polynomials to the especially engaged reader.

where ${}_1F_1(a;b;x)$ is a confluent hypergeometric function with $a = -N$, $b = n+1$.

We have, for either $\overline{L}_N^{(n)}(x)$ or for $\overline{\overline{L}}_N^{(n)}(x)$,

$$(n+1+N)\,\overline{\overline{L}}_{N+1}^{(n)} - (2N+n+1-x)\,\overline{\overline{L}}_N^{(n)} + N\,\overline{\overline{L}}_{N-1}^{(n)} = 0$$

as well as

$$n(n+1)\,\overline{\overline{L}}_N^{(n-1)} - (n+1)(n+x)\,\overline{\overline{L}}_N^{(n)} + x(n+1+N)\,\overline{\overline{L}}_N^{(n+1)} = 0\ .$$

6 Comments

Curiously, the closed-form second solution to the confluent hypergeometric differential equation in the case when the first parameter a in ${}_1F_1(a;b;x)$ takes the value of a nonpositive integer and the second parameter b is an integer greater than zero is not yet found in standard references. The DLMF compilation gives instead a logarithmic and infinite Laurent power series representation. We have checked that the DLMF expression can be reproduced by expanding the right-hand side of Eq. (2.10) into such a series.

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