International Journal of Difference Equations
ISSN 0973-6069, Volume 11, Number 2, pp. 179–202 (2016)
http://campus.mst.edu/ijde

On a Second-Order Rational Difference Equation with a Quadratic Term

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Abstract

We give the boundedness character, local and global stability of solutions of the following second-order rational difference equation with quadratic denominator,

$$x_{n+1} = \frac{\alpha + \gamma x_{n-1}}{Bx_n + Dx_n x_{n-1} + x_{n-1}} \text{ for } n = 0, 1, \ldots,$$

where the coefficients are positive numbers, and the initial conditions x_{-1} and x_0 are nonnegative numbers such that the denominator is nonzero. In particular, we show that in a certain region, the unique equilibrium is globally asymptotically stable, while in another region, the equilibrium is a saddle and there exist prime period-two solutions.

AMS Subject Classifications: 39A10, 39A11, 29A20.
Keywords: Bifurcation point, local stability, global stability, periodic solutions, rational difference equation, rational recurrence relation.

Received May 6, 2016; Accepted May 8, 2016
Communicated by Gerasimos Ladas

1 Introduction

In this paper, we will investigate the behavior of solutions of a second-order rational recurrence relation with a quadratic term. Namely, we will consider the equation,

$$x_{n+1} = \frac{\alpha + \gamma x_{n-1}}{Bx_n + Dx_n x_{n-1} + x_{n-1}} \text{ for } n = 0, 1, \ldots, \tag{1.1}$$

where the coefficients are positive real numbers and the initial conditions, x_{-1}, x_0, are nonnegative real numbers such that the denominator is defined.

Equation (1.1) is a special case of the following more general equation,

$$x_{n+1} = \frac{ax_n^2 + bx_n x_{n-1} + cx_n + dx_{n-1} + ex_{n-1}^2 + f}{Ax_n^2 + Bx_n x_{n-1} + cx_n + Dx_{n-1} + Ex_{n-1}^2 + F}, \text{ for } n = 0, 1, \ldots, \tag{1.2}$$

with nonnegative coefficients and nonnegative initial conditions such that the denominator is positive. Several cases of (1.2) have been studied recently, for several examples consult [12, 15–17].

We will say that a parameter, B, of a difference equation exhibits a *period-doubling bifurcation* at a number B^* if the unique equilibrium is globally asymptotically stable for $B < B^*$, if the unique equilibrium is nonhyperbolic when $B = B^*$, and if the equilibrium is unstable and there exist prime period-two solutions for $B > B^*$. If such a point B^* exists, we call B^* a bifurcation point.

We will show that parameter B in (1.1) exhibits this type of period-doubling bifurcation. In fact, one can determine the behavior of the solutions of (1.1) by considering the value of B with respect to the other fixed parameters. Amazingly, we can find a threshold where the behavior of solutions bifurcates. Thus, we will show that there is a bifurcation point, B^* for parameter B, and which in terms of positive constants α, γ and D has the form:

$$B^* = \frac{\gamma^2 + \gamma\alpha D + 2\alpha + 2\alpha\sqrt{1 + \gamma D}}{\gamma^2}. \tag{1.3}$$

We will now state the main result of this paper.

Theorem 1.1. *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of* (1.1)*. Then, there is a unique positive equilibrium $\bar{x}$ of* (1.1)*, and the following are true.*

1. *If $B < B^*$ then $\bar{x}$ is globally asymptotically stable.*
2. *If $B = B^*$ then $\bar{x}$ is a global attractor and is nonhyperbolic.*
3. *If $B > B^*$ then $\bar{x}$ is a saddle point, and $\{x_n\}_{n=-1}^{\infty}$ converges to a (not necessarily prime) period-two solution, and there exists a unique prime period-two solution.*

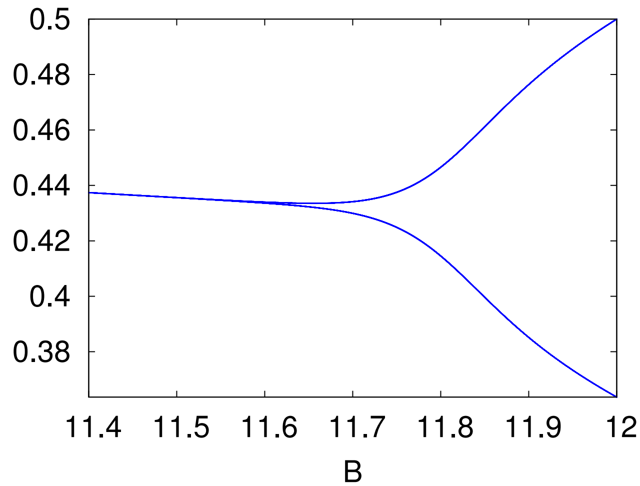

Figure 1.1: Bifurcation diagram showing a period-doubling bifurcation at B^*.

We will prove this theorem in multiple steps. In Section 2 we provide some previously known results for reference. Next, in Section 3 we will state and prove several auxiliary results about (1.1). We will then prove Theorem 1.1 by proving a sequence of lemmas given in Section 4. Finally, we conclude this paper with an open problem and an application of Theorem 1.1 to a system of difference equations in Section 5.

Rational difference equations with nonlinear quadratic terms have been studied in recent years. We give several papers in which the authors study rational difference equations with nonlinear terms. Namely, please consult papers: [1, 2, 7–10, 14, 20, 21, 24, 25, 28–30].

2 Preliminaries

In this section, we state several well-known results.

Theorem 2.1 (See [13, Theorem 1.14]). *Let $f : [a, b] \to [a, b]$ be a continuous function, where a and b are real numbers with $a < b$, and consider the difference equation*

$$x_{n+1} = f(x_n, x_{n-1}), \quad \textit{for } n = 0, 1, \dots . \tag{2.1}$$

Suppose that f satisfies the following two conditions:

1. *$f(x, y)$ is nonincreasing in $x \in [a, b]$ for each fixed $y \in [a, b]$, and $f(x, y)$ is nonincreasing in $y \in [a, b]$ for each fixed $x \in [a, b]$;*
2. *If (m, M) is a solution of the system*

$$m = f(M, M), \quad M = f(m, m)$$

then $m = M$.

Then, there exists exactly one equilibrium of (2.1)*, namely,* $\bar{x}$*. Furthermore, every solution of* (2.1) *converges to* $\bar{x}$*.*

Theorem 2.2 (See [11])**.** *Let*

$$x_{n+1} = f(x_n, x_{n-1}), \quad \textit{for } n = 0, 1, \ldots \tag{2.2}$$

with

1. $f \in C[(0,\infty) \times (0,\infty), (0,\infty)]$*;*
2. $f(u, v)$ *is nonincreasing in* u *and* v *respectively;*
3. $xf(x, x)$ *is nondecreasing in* x*;*
4. *Equation* (2.2) *has a unique positive equilibrium* $\bar{x}$*.*

Then, every positive solution of (2.2) *which is bounded from above and from below by positive constants converges to* $\bar{x}$*.*

Theorem 2.3 (See [5, 6])**.** *Let* I *be an interval of real numbers and let* $f : I \times I \to I$ *be a function* $f(u, v)$ *which decreases in* $u \in I$ *for each fixed* $v \in I$*, and increases in* $v \in I$ *for each fixed* $u \in I$*. Then for every solution* $\{x_n\}_{n=-1}^{\infty}$ *of the difference equation*

$$x_{n+1} = f(x_n, x_{n-1}), \quad \textit{for } n = 0, 1, \ldots$$

the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ *and* $\{x_{2n-1}\}_{n=0}^{\infty}$ *of even and odd terms are eventually monotonic.*

We follow the terminology given in [13] for stability of an equilibrium. Here we restate some of their results for convenience.

Suppose that $f : I \times I \to I$, is a continuous function, which defines the difference equation

$$x_{n+1} = f(x_n, x_{n-1}), \quad \text{for } n = 0, 1 \ldots$$

where $f(u, v)$ is continuously differentiable in some neighborhood of the equilibrium $\bar{x}$. We define the following constants

$$a_1 = -\frac{\partial f}{\partial u}(\bar{x}, \bar{x}), \tag{2.3}$$

$$a_0 = -\frac{\partial f}{\partial v}(\bar{x}, \bar{x}). \tag{2.4}$$

The equation

$$\lambda^2 + a_1\lambda + a_0 = 0 \tag{2.5}$$

is called the characteristic equation.

Theorem 2.4 (Linearized stability theorem, see [13]). *Suppose that f is a continuously differentiable function defined on some open neighborhood of $\bar{x}$. Then the following are true:*

1. *If the roots of* (2.5) *have absolute value less than one, then the equilibrium $\bar{x}$ is locally asymptotically stable.*
2. *If at least one of the roots of* (2.5) *has absolute value greater than one, then the equilibrium $\bar{x}$ is unstable.*
3. *If all of the roots of* (2.5) *have absolute value greater than one, then the equilibrium $\bar{x}$ is a source.*

If none of the roots of (2.5) are equal to one in absolute value, then the equilibrium is said to be *hyperbolic*. If at least one of the roots has absolute value equal to one, then $\bar{x}$ is said to be *nonhyperbolic*. If an equilibrium is hyperbolic, and at least one of the roots has absolute value greater than one, and at least one of the roots has absolute value less than one, then $\bar{x}$ is said to be a *saddle point*.

Theorem 2.5 ([See [13, Theorem 1.3]). *Consider the second-degree polynomial equation*

$$\lambda^2 + a_1\lambda + a_0 = 0 \tag{2.6}$$

where a_0 and a_1 are real numbers. Then the following statements are true.

1. *A necessary and sufficient condition for the roots of* (2.6) *to lie within the unit disc $|\lambda| < 1$ is*
$$|a_1| < 1 + a_0 < 2$$
2. *A necessary and sufficient condition for both roots of* (2.6) *to have absolute value greater than one is*
$$|a_0| > 1 \quad \textit{and} \quad |a_1| < |1 + a_0|.$$

We will use the method of full limiting sequences, as developed by Karakostas, see [18, 19].

Theorem 2.6 ([See [13, Theorem 1.8]). *Consider the difference equation*

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}) \tag{2.7}$$

where $f \in C[J^{k+1}, J]$ for some interval J of real numbers and some nonnegative integer k. Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of (2.7). *Set $I = \liminf_{n\to\infty} x_n$ and $S = \limsup_{n\to\infty} x_n$, and suppose that $I, S \in J$. Let $\mathcal{L}_0$ be a limit point of the sequence $\{x_n\}_{n=-k}^{\infty}$. Then the following statements are true.*

1. *There exists a solution* $\{L_n\}_{n=-\infty}^{\infty}$ *of* (2.7)*, called a full limiting sequence of* $\{x_n\}_{n=-k}^{\infty}$*, such that* $L_0 = \mathcal{L}_0$*, and such that for every* $N \in \mathbb{Z}$*,* L_N *is a limit point of* $\{x_n\}_{n=-k}^{\infty}$*. In particular,*

$$I \leq L_n \leq S \text{ for all } N \in \mathbb{Z}.$$

2. *For every* $i_0 \in \mathbb{Z}$*, there exists a subsequence* $\{x_{r_i}\}_{i=0}^{\infty}$ *of the solution* $\{x_n\}_{n=-k}^{\infty}$ *such that*

$$L_N = \lim_{i\to\infty} x_{r_i+N} \text{ for every } N \geq i_0.$$

The following inequality will be useful, and can be found as an exercise in [27].

$$\min\left\{\frac{\alpha_1}{B_1}, \frac{\alpha_2}{B_2}, \ldots, \frac{\alpha_n}{B_n}\right\} \leq \frac{\sum_{k=1}^{n}\alpha_k}{\sum_{k=1}^{n}B_k} \leq \max\left\{\frac{\alpha_1}{B_1}, \frac{\alpha_2}{B_2}, \ldots, \frac{\alpha_n}{B_n}\right\}, \tag{2.8}$$

where $\alpha_1, \ldots, \alpha_n$ are nonnegative real numbers and $B_1, B_2, \ldots, B_n$ are positive real numbers.

3 Several Auxiliary Results on Equation (1.1)

In this section we will prove several auxiliary results concerning (1.1). These results will be useful in proving Theorem 1.1. We begin, by showing that every positive solution of (1.1) is bounded.

Theorem 3.1. *Every positive solution of* (1.1) *is bounded from above and from below by positive constants.*

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of (1.1). We iterate for x_n, and use inequality (2.8) to obtain the following:

$$\begin{aligned}
x_{n+1} &= \frac{\alpha + \gamma x_{n-1}}{Bx_n + Dx_nx_{n-1} + x_{n-1}} \\
&\leq \frac{\alpha Bx_{n-1} + \alpha Dx_{n-1}x_{n-2} + \alpha x_{n-2} + B\gamma x_{n-1}^2 + \gamma x_{n-1}x_{n-2}(Dx_{n-1}+1)}{\alpha Dx_{n-1} + D\gamma x_{n-1}x_{n-2} + B\gamma x_{n-2} + Bx_{n-1}^2 + x_{n-1}x_{n-2}(Dx_{n-1}+1)} \\
&\leq \max\left\{\frac{B}{D}, \frac{\alpha}{\gamma}, \frac{\alpha}{B\gamma}, \gamma\right\}.
\end{aligned}$$

Thus, we define $M = \max\left\{\frac{B}{D}, \frac{\alpha}{\gamma}, \frac{\alpha}{B\gamma}, \gamma\right\}$, which is an upper bound on x_n. Further, we use (2.8) to obtain the following lower bound.

$$\begin{aligned} x_{n+1} &= \frac{\alpha + \gamma x_{n-1}}{Bx_n + Dx_n x_{n-1} + x_{n-1}} \\ &\geq \frac{\alpha + \gamma x_{n-1}}{BM + DMx_{n-1} + x_{n-1}} \\ &= \frac{\alpha + \gamma x_{n-1}}{BM + (DM+1)x_{n-1}} \\ &\geq \min\left\{\frac{\alpha}{BM}, \frac{\gamma}{1+DM}\right\} = m. \end{aligned}$$

Thus, there exists positive constants, m, M such that $m \leq x_n \leq M$ for all $n \geq 1$, which concludes the proof. □

We will use the following cubic polynomial, which will be used to prove several results. Let us define the cubic polynomial h, by

$$h(x) = Dx^3 + (B+1)x^2 - \gamma x - \alpha. \tag{3.1}$$

Theorem 3.2. *Equation* (1.1) *has a unique positive equilibrium.*

Proof. Consider the equation,

$$h(x) = 0, \tag{3.2}$$

where the function h is defined in (3.1). By the well-known Decartes' rule of signs, the cubic equation (3.2) has a unique positive root. We define $\bar{x}$ as the unique positive solution of (3.2). Thus,

$$0 = D\bar{x}^3 + (B+1)\bar{x}^2 - \gamma\bar{x} - \alpha, \tag{3.3}$$

$$\bar{x} = \frac{\alpha + \gamma\bar{x}}{B\bar{x} + D\bar{x}^2 + \bar{x}}. \tag{3.4}$$

Hence, $\bar{x}$ is the positive equilibrium of (1.1), and $\bar{x}$ is unique. □

In the remainder of this section, we will provide several results concerning the local stability of (1.1). We show that the unique equilibrium is locally asymptotically stable when $B < B^*$. Before we state and prove Theorem 3.5, we will give some helpful lemmas, the proofs of which follow by direct computation.

Lemma 3.3. *Let* $f(u, v) = \dfrac{\alpha + \gamma v}{Bu + Duv + v}$, *then*

$$\frac{\partial f}{\partial u} = \frac{-(B + Dv)(\alpha + \gamma v)}{(Bu + Duv + v)^2} \quad \text{and} \quad \frac{\partial f}{\partial v} = \frac{(B\gamma - \alpha D)u - \alpha}{(Bu + Duv + v)^2}. \tag{3.5}$$

Lemma 3.4. *Let constants a_1 and a_0 be as defined in* (2.3) *and* (2.4)*. Then,*

$$a_1 = \frac{(\alpha+\gamma\bar{x})(B+D\bar{x})}{((B+1)\bar{x}+D\bar{x}^2)^2} \quad \textit{and} \quad a_0 = \frac{\alpha-(B\gamma-\alpha D)\bar{x}}{((B+1)\bar{x}+D\bar{x}^2)^2}. \tag{3.6}$$

Further, the characteristic equation of (1.1)*, $\lambda^2 + a_1\lambda + a_0 = 0$, reduces to*

$$\lambda^2 + \left(\frac{(\alpha+\gamma\bar{x})(B+D\bar{x})}{((B+1)\bar{x}+D\bar{x}^2)^2}\right)\lambda + \left(\frac{\alpha-(B\gamma-\alpha D)\bar{x}}{((B+1)\bar{x}+D\bar{x}^2)^2}\right) = 0 \tag{3.7}$$

We are now ready for our theorem on local stability.

Theorem 3.5. *Let $\bar{x}$ be the unique positive equilibrium of* (1.1)*, and B^* be defined in* (1.3)*. Then, the following statements are true.*

1. *If $B < B^*$, then $\bar{x}$ is* locally asymptotically stable.
2. *If $B = B^*$, then $\bar{x}$ is* nonhyperbolic. *Further, one of the roots of the characteristic polynomial is -1, while the other lies inside the unit disk.*
3. *If $B > B^*$, then $\bar{x}$ is* unstable. *Further, it is a* saddle point.

We will prove Theorem 3.5 by using the following lemmas.

Lemma 3.6. *Let $\bar{x}$ be the unique positive equilibrium of* (1.1)*, and B^* be defined in* (1.3)*. Then the following are true.*

1. *If $B < B^*$, then*
$$(B\gamma-\alpha D-\gamma)\bar{x}-2\alpha < 0.$$
2. *If $B = B^*$, then*
$$\bar{x} = \frac{2\alpha}{B\gamma-\alpha D-\gamma}.$$
3. *If $B > B^*$, then*
$$(B\gamma-\alpha D-\gamma)\bar{x}-2\alpha > 0.$$

Proof. If $B \le 1 + \dfrac{\alpha D}{\gamma}$, then $B\gamma-\alpha D-\gamma \le 0$, then it must be the case that $B < B^*$. Clearly,

$$B\gamma\bar{x}-\alpha D\bar{x}-\gamma\bar{x}-2\alpha = (B\gamma-\alpha D-\gamma)\bar{x}-2\alpha < 0.$$

Thus, we suppose that $B > 1 + \dfrac{\alpha D}{\gamma}$. We define the linear function

$$g(z) = (B\gamma-\alpha D-\gamma)z-2\alpha.$$

It is clear that g is a line with positive slope and negative y-intercept. The z-intercept of this line occurs at the point

$$\hat{z} = \frac{2\alpha}{B\gamma-\alpha D-\gamma}.$$

We will now show that:

1. if $B < B^*$, then $\bar{x} < \hat{z}$;
2. if $B = B^*$, then $\bar{x} = \hat{z}$;
3. if $B > B^*$, then $\bar{x} > \hat{z}$.

We will use the function h defined in (3.1). Recall that $h(x)$ is a cubic equation, which has a unique positive root at $\bar{x}$. Thus, it suffices to show that:

1. if $B < B^*$, then $h(\hat{z}) > 0$;
2. if $B = B^*$, then $h(\hat{z}) = 0$;
3. if $B > B^*$, then $h(\hat{z}) < 0$.

Observe that

$$\begin{aligned}h(\hat{z}) &= D\left(\tfrac{2\alpha}{B\gamma-\alpha D-\gamma}\right)^3 + (B+1)\left(\tfrac{2\alpha}{B\gamma-\alpha D-\gamma}\right)^2 - \gamma\left(\tfrac{2\alpha}{B\gamma-\alpha D-\gamma}\right) - \alpha \\ &= \frac{-\alpha(B\gamma-\alpha D+\gamma)\left(\gamma^2 B^2 - (2\gamma^2+2\gamma\alpha D+4\alpha)B\right)}{(B\gamma-\alpha D-\gamma)^3} \\ &\quad + \frac{-\alpha(B\gamma-\alpha D+\gamma)(\gamma^2+4\alpha+\alpha^2D^2+2\alpha\gamma D)}{(B\gamma-\alpha D-\gamma)^3}.\end{aligned}$$

We know that $\alpha > 0$ and $B\gamma-\alpha D+\gamma > B\gamma-\alpha D-\gamma > 0$. Hence, we must determine the sign of

$$\gamma^2B^2 - (2\gamma^2+2\gamma\alpha D+4\alpha)B + (\gamma^2+4\alpha+\alpha^2D^2+2\alpha\gamma D).$$

Let us analyze the quadratic equation

$$\gamma^2z^2 - (2\gamma^2+2\gamma\alpha D+4\alpha)z + (\gamma^2+4\alpha+\alpha^2D^2+2\alpha\gamma D) = 0. \tag{3.8}$$

By applying the quadratic formula, we see that the solutions are

$$z = \frac{\gamma^2+\gamma\alpha D+2\alpha \pm 2\alpha\sqrt{1+\gamma D}}{\gamma^2}.$$

Notice that $\dfrac{\gamma^2+\gamma\alpha D+2\alpha - 2\alpha\sqrt{1+\gamma D}}{\gamma^2} < \dfrac{\gamma^2+\gamma\alpha D}{\gamma^2} = 1 + \dfrac{\alpha D}{\gamma}$. Amazingly, $\dfrac{\gamma^2+\gamma\alpha D+2\alpha + 2\alpha\sqrt{1+\gamma D}}{\gamma^2} = B^*$. Hence, if $1 + \dfrac{\alpha D}{\gamma} < B < B^*$ the expression in (3.8) is negative, hence $h(\hat{z}) > 0$; if $B = B^*$, then the expression in (3.8) is zero, hence $h(\hat{z}) = 0$; and if $B > B^*$ then the expression in (3.8) is positive, hence $h(\hat{z}) < 0$. $\square$

Lemma 3.7. *Let a_0 be the constant defined in Lemma 3.4. Then,*

$$1 + a_0 < 2.$$

Proof. It suffices to show that $a_0 < 1$. Consider the following:

$$\begin{aligned}
a_0 &< 1 \\
\frac{\alpha - (B\gamma - \alpha D)\bar{x}}{((B+1)\bar{x} + D\bar{x}^2)^2} &< 1 \\
\alpha - (B\gamma - \alpha D)\bar{x} &< ((B+1)\bar{x} + D\bar{x}^2)^2 \\
\alpha - (B\gamma - \alpha D)\bar{x} &< (B+1)^2\bar{x}^2 + 2(B+1)D\bar{x}^3 + D^2\bar{x}^4. \qquad (3.9)
\end{aligned}$$

By (3.1), we obtain the following identity.

$$\alpha = (B+1)\bar{x}^2 + D\bar{x}^3 - \gamma\bar{x}. \qquad (3.10)$$

We use (3.10) to simplify (3.9) and obtain the following inequality.

$$-(1 + B + D)\gamma\bar{x} < B(B+1)\bar{x}^2 + BD\bar{x}^3 \qquad (3.11)$$

where the left-hand side of (3.11) is negative while the right-hand side is positive. Thus $1 + a_0 < 2$ is always satisfied. □

Observe that $a_1 > 0$ and thus $|a_1| = a_1$, where a_1 is the constant given in Lemma 3.4.

Lemma 3.8. *Let a_0 and a_1 be the constants defined in Lemma 3.4. The following statements are true:*

1. *if $B < B^*$, then $a_1 < 1 + a_0$;*
2. *if $B = B^*$, then $a_1 = 1 + a_0$;*
3. *if $B > B^*$, then $a_1 > 1 + a_0$.*

Proof. We will prove the first statement, and the proofs of the other two follow similarly. Assume that $B < B^*$. By Lemma 3.6, we see that

$$B\gamma\bar{x} - \alpha D\bar{x} - \gamma\bar{x} - 2\alpha < 0. \qquad (3.12)$$

Thus,

$$\begin{aligned}
B\gamma\bar{x} - \alpha D\bar{x} &< \gamma\bar{x} + 2\alpha \\
\alpha B + 2B\gamma\bar{x} - \alpha D\bar{x} &< B\gamma\bar{x} + \gamma\bar{x} + \alpha B + 2\alpha \\
\alpha B + 2B\gamma\bar{x} - \alpha D\bar{x} &< (B+1)^2\bar{x}^2 - (B+1)^2\bar{x}^2 \\
&\quad + (B+1)\gamma\bar{x} + (B+1)\alpha + \alpha.
\end{aligned}$$

We use (3.1) to obtain the identity

$$D\bar{x}^3 = -(B+1)\bar{x}^2 + \gamma\bar{x} + \alpha$$

which we use to obtain the inequality

$$\alpha B+2B\gamma\bar{x}+D^2\bar{x}^4+(B+1)D\bar{x}^3-\alpha D\bar{x}<(B+1)^2\bar{x}^2+2(B+1)D\bar{x}^3+D^2\bar{x}^4+\alpha.$$

We will also use the identity

$$\gamma\bar{x}=D\bar{x}^3+(B+1)\bar{x}^2-\alpha$$

which also follows from (3.1). Thus,

$$\begin{aligned}\alpha B+\alpha D\bar{x}+B\gamma\bar{x}+D\gamma\bar{x}^2&<(B+1)\bar{x}^2+2(B+1)D\bar{x}^3+D^2\bar{x}^4\\&\quad+\alpha-B\gamma\bar{x}+\alpha D\bar{x}\\(\alpha+\gamma\bar{x})(B+D\bar{x})&<\left((B+1)\bar{x}+D\bar{x}^2\right)^2+\alpha-(B\gamma-\alpha D)\bar{x}\\\frac{(\alpha+\gamma\bar{x})(B+D\bar{x})}{\left((B+1)\bar{x}+D\bar{x}^2\right)^2}&<1+\frac{\alpha-(B\gamma-\alpha D)\bar{x}}{\left((B+1)\bar{x}+D\bar{x}^2\right)^2}\\a_1&<1+a_0,\end{aligned}$$

which concludes the proof. □

Proof of Theorem 3.5. If $B<B^*$ then from Lemma 3.7 and Lemma 3.8 it follows that $|a_1|<1+a_0<2$, and so by Theorem 2.5 it follows that both roots of the characteristic polynomial lie inside the unit disc, and by Theorem 2.4, the equilibrium $\bar{x}$ is locally asymptotically stable.

If $B=B^*$, then Lemma 3.8 implies that $a_1=1+a_0$. Thus, the characteristic polynomial (3.7) reduces to

$$\lambda^2+(1+a_0)\lambda+a_0=0,$$

which factors as

$$(\lambda+1)(\lambda+a_0)=0.$$

This means that one root is $\lambda=-1$, and the other is $\lambda=-a_0$. It remains to show that $|a_0|<1$. Since we have shown that $a_0<1$, it suffices to show that $a_0>-1$. Hence,

$$\begin{aligned}a_0&>-1\\a_1-1&>-1\\a_1&>0.\end{aligned}$$

But, it is clear from the Lemma 3.4 that $a_1>0$, and thus $|a_0|<1$. So we conclude that when $B=B^*$ the characteristic polynomial (2.6) has one root which is negative one, and one root which is inside the unit disc, and thus the unique positive equilibrium $\bar{x}$ is nonhyperbolic.

Finally, if $B>B^*$, it follows from Lemma 3.8 that $a_1>1+a_0$ and thus by Theorem 2.5, at least one of the roots of the characteristic polynomial must lie outside of the unit disk, and thus by Theorem 2.4, the unique positive equilibrium $\bar{x}$ is unstable. However, by Statement (3) of Theorem 2.4, and by Lemma 3.7 we see that it cannot be the case that both roots are outside of the unit disk. Hence, $\bar{x}$ must be a saddle point. □

Existence of Period-Two Solutions

In this next subsection, we will investigate the existence of period-two solutions of (1.1). Amazingly, period-two solutions emerge when $B > B^*$.

Theorem 3.9. *There exist period-two solutions of* (1.1) *if and only if* $B > B^*$.

Just by the definition, we can check that $B^* > 1$. Further, it is clear that if $B > B^*$, then it follows that $\gamma B > \alpha D$.

Proof. Suppose that $..., \phi, \psi, ...$ is a solution of (1.1). Then, we see that

$$\phi = \frac{\alpha + \gamma\phi}{B\psi + D\phi\psi + \phi}, \quad \psi = \frac{\alpha + \gamma\psi}{B\phi + D\phi\psi + \psi}.$$

We clear the denominators to obtain

$$B\phi\psi + D\phi^2\psi + \phi^2 = \alpha + \gamma\phi, \tag{3.13}$$
$$B\phi\psi + D\phi\psi^2 + \psi^2 = \alpha + \gamma\psi. \tag{3.14}$$

Subtract (3.14) from (3.13) to get

$$D\phi\psi(\phi - \psi) + (\phi + \psi)(\phi - \psi) = \gamma(\phi - \psi).$$

Since we assume $\phi \neq \psi$ we simplify to

$$\gamma = D\phi\psi + \phi + \psi, \tag{3.15}$$
$$\phi = \frac{\gamma - \psi}{1 + D\psi}. \tag{3.16}$$

We now substitute (3.16) into (3.14) to get

$$\begin{aligned}
\phi(B\psi + D\psi^2) + \psi^2 &= \alpha_\gamma\psi \\
\left(\frac{\gamma - \psi}{1 + D\psi}\right)(B\psi + D\psi^2) + \psi^2 &= \alpha + \gamma\psi \\
(\gamma - \psi)(B\psi + D\psi^2) + \psi^2 + D\psi^3 &= (\alpha + \gamma\psi)(1 + D\psi) \\
B\gamma\psi + D\gamma\psi^2 - B\psi^2 - D\psi^3 + \psi^2 + D\psi^3 &= \alpha + \alpha D\psi + \gamma\psi + D\gamma\psi^2 \\
(1 - B)\psi^2 + (B\gamma - \alpha D - \gamma)\psi - \alpha &= 0. \qquad (3.17)
\end{aligned}$$

By the symmetry of ϕ, ψ we see that ϕ and ψ must satisfy the quadratic equation:

$$(1 - B)x^2 + (B\gamma - \alpha D - \gamma)x - \alpha = 0. \tag{3.18}$$

By using the quadratic formula, we see that

$$x = \frac{\gamma(1 - B) + \alpha D \pm \sqrt{(B\gamma - \alpha D - \gamma)^2 + 4(1 - B)\alpha}}{2(1 - B)}. \tag{3.19}$$

Thus, we have found the possible solutions for ϕ, ψ. Now, we must try to show that these are actually solutions.

Without loss of generality, let

$$\begin{aligned}\psi &= \frac{\gamma(1-B)+\alpha D+\sqrt{(B\gamma-\alpha D-\gamma)^2+4(1-B)\alpha}}{2(1-B)}\\ \phi &= \frac{\gamma(1-B)+\alpha D-\sqrt{(B\gamma-\alpha D-\gamma)^2+4(1-B)\alpha}}{2(1-B)}\end{aligned} \tag{3.20}$$

We will substitute both of these in to (3.13). Notice that

$$\phi\psi = \frac{-\alpha}{1-B}.$$

Now consider

$$\frac{\alpha+\gamma\phi}{B\psi+D\phi\psi+\phi}-\phi = \frac{\alpha+\gamma\phi-B\phi\psi-D\phi\psi\phi-\phi^2}{B\psi+D\phi\psi+\phi} \tag{3.21}$$

Let us look at the numerator from the right-hand side of (3.21).

$$\begin{aligned}\alpha+\gamma\phi-B\phi\psi-D\phi^2\psi-\phi^2 &= \alpha+\gamma\phi+\frac{\alpha B}{1-B}+\frac{\alpha D\phi}{1-B}-\phi^2\\ &= \alpha\left(1+\frac{B}{1-B}\right)+\left(\gamma+\frac{\alpha D}{1-B}-\phi\right)\phi\\ &= \frac{\alpha}{1-B}+\left(\frac{\gamma(1-B)+\alpha D-(1-B)\phi}{1-B}\right)\phi\end{aligned}$$

Now, by using (3.20), we obtain:

$$\frac{\gamma(1-B)+\alpha D-(1-B)\phi}{1-B} = \psi.$$

Thus,

$$\begin{aligned}\frac{\gamma(1-B)+\alpha D-(1-B)\phi}{1-B} &= \psi\\ \frac{\alpha}{1-B}+\left(\frac{\gamma(1-B)+\alpha D-(1-B)\phi}{1-B}\right)\phi &= \frac{\alpha}{1-B}+\psi\phi\\ \frac{\alpha}{1-B}+\left(\frac{\gamma(1-B)+\alpha D-(1-B)\phi}{1-B}\right)\phi &= \frac{\alpha}{1-B}+\frac{-\alpha}{1-B}.\end{aligned}$$

Hence,

$$\frac{\alpha+\gamma\phi}{B\psi+D\phi\psi+\phi}-\phi = 0.$$

By symmetry, we see obtain the same condition for ψ. And thus, a prime period-two solutions exist. Since equation (3.15) is symmetric with respect to ϕ and ψ, then ϕ also has to satisfy (3.19). Thus, we will only have the existence of period-two solutions when the discriminant of (3.19) is positive. Hence, we desire a condition that guarantees that

$$(B\gamma - \alpha D - \gamma)^2 + 4(1 - B)\alpha > 0. \tag{3.22}$$

Let us solve the inequality in (3.22) by setting the expression equal to zero. Hence, we solve

$$0 = \gamma^2 B^2 - 2(\gamma^2 + \gamma\alpha D + 2\alpha)B + \gamma^2 + 2\gamma\alpha D + 4\alpha + \alpha^2 D^2. \tag{3.23}$$

Using the quadratic formula, we obtain:

$$B = \frac{(\gamma^2 + \gamma\alpha D + 2\alpha) \pm 2\alpha\sqrt{\gamma D + 1}}{\gamma^2} \tag{3.24}$$

We will analyze the roots in (3.24). Notice that (3.23) has a positive leading coefficient, so it is concave up, and the coefficient of B is negative. Further, there are exactly two positive solutions for (3.23). Since we know that $B > 1$ is a necessary condition for the existence of prime period-two solutions, then we know that period-two solutions will only exist when

$$B > \frac{\gamma^2 + \gamma\alpha D + 2\alpha + 2\alpha\sqrt{1 + \gamma D}}{\gamma^2} = B^*$$

Notice that if $B = B^*$, then the discriminant of (3.18) is zero, and so there would only be one solution of (3.18). Hence there are no prime period-two solutions when $B = B^*$.

If $1 < B < B^*$ then the discriminant of (3.18) is negative, in which case there are no real solutions, hence there can be no period-two solutions for $1 < B < B^*$.

If $B < 1$ then the quadratic function in (3.18) opens upwards, and one of its roots is negative, so there can be at most one positive root. Hence, there are no period-two solutions for $B < 1$. □

We find that in addition to B^*, there are several other constants that are important for understanding the nature of (1.1). We define them as

$$B^- = \frac{2\gamma\alpha D + \alpha - \alpha\sqrt{1 + 4\gamma D}}{2\gamma^2},$$

and

$$B^+ = \frac{2\gamma\alpha D + \alpha + \alpha\sqrt{1 + 4\gamma D}}{2\gamma^2}. \tag{3.25}$$

The next lemma will show the relationship between these and other previously defined constants. The proof is omitted.

Lemma 3.10.

$$B^- < \frac{\alpha D}{\gamma} < B^+ < B^*$$

Lemma 3.11. *The following are true.*

1. *If* $\frac{\alpha D}{\gamma} < B \leq B^+$, *then* $\max\left\{\frac{\alpha}{\gamma B - \alpha D}, \frac{\gamma B}{D(\gamma B - \alpha D) + B}\right\} = \frac{\alpha}{\gamma B - \alpha D}$.

2. *If* $B > B^+$ *then* $\min\left\{\frac{\alpha}{\gamma B - \alpha D}, \frac{\gamma B}{D(\gamma B - \alpha D) + B}\right\} = \frac{\alpha}{\gamma B - \alpha D}$.

Proof. Let b be the quadratic function,

$$b(z) = \gamma^2 z^2 - (2\gamma\alpha D + \alpha)z + \alpha^2 D^2, \tag{3.26}$$

whose roots are B^- and B^+.

We prove the second statement, while the proof of the first follows similarly. Assume that $B > B^+$ then, by the nature of (3.26), it follows that

$$\begin{aligned}
0 &\leq \gamma^2 B^2 - (2\gamma\alpha D + \alpha)B + \alpha^2 D^2 \\
\alpha D\gamma B - \alpha^2 D^2 + \alpha B &\leq \gamma^2 B^2 - \alpha D\gamma B \\
\alpha D(\gamma B - \alpha D) + \alpha B &\leq \gamma B(\gamma B - \alpha D) \\
\frac{\alpha}{\gamma B - \alpha D} &\leq \frac{\gamma B}{D(\gamma B - \alpha D) + B},
\end{aligned}$$

and

$$\begin{aligned}
\frac{\alpha}{\gamma B - \alpha D} &\leq \frac{\gamma B}{D(\gamma B - \alpha D) + B} \\
\alpha D(\gamma B - \alpha D) + \alpha B &\leq \gamma B(\gamma B - \alpha D) \\
\alpha D\gamma B - \alpha^2 D^2 + \alpha B &\leq \gamma^2 B^2 - \alpha D\gamma B \\
\gamma^2 B^2 - (2\gamma\alpha D + \alpha)B + \alpha^2 D^2 &\geq 0.
\end{aligned} \tag{3.27}$$

Thus, concludes the proof. $\square$

Our goal is to show that depending on the value of B, there will be a specific closed and bounded invariant and attracting interval. If $\frac{\alpha D}{\gamma} < B < B^+$, then we will show that $[(\gamma B - \alpha D)/B, \alpha/(\gamma B - \alpha D)]$ is such an interval, while if $B > B^+$, then the interval will be $[\alpha/(\gamma B - \alpha D), (\gamma B - \alpha D)/B]$.

We will define the interval

$$K = \begin{cases} \left[\frac{\gamma B - \alpha D}{B}, \frac{\alpha}{\gamma B - \alpha D}\right] & \text{, if } \frac{\alpha D}{\gamma} < B < B^+ \\ \\ \left[\frac{\alpha}{\gamma B - \alpha D}, \frac{\gamma B - \alpha D}{B}\right] & \text{, if } B > B^+ \end{cases}.$$

Amazingly, when we have $B = B^+$, we are able to determine the unique equilibrium exactly. In this case, the interval K degenerates to a single point.

Lemma 3.12. *If* $B = B^+$ *then the unique equilibrium of* (1.1) *is*

$$\bar{x} = \frac{\gamma B - \alpha D}{B} = \frac{\alpha}{\gamma B - \alpha D}.$$

Proof. We leave it to the reader to verify that if $B = B^+$, then $\dfrac{\gamma B - \alpha D}{B} = \dfrac{\alpha}{\gamma B - \alpha D}$. Notice that this implies that

$$\left(\frac{\gamma B - \alpha D}{B}\right)^2 = \left(\frac{\alpha}{\gamma B - \alpha D}\right)^2 = \left(\frac{\alpha}{\gamma B - \alpha D}\right)\left(\frac{\gamma B - \alpha D}{B}\right) = \frac{\alpha}{B}.$$

Let h be the cubic equation defined in (3.1). We show that

$$\begin{aligned}
h\left(\frac{\gamma B - \alpha D}{B}\right) &= D\frac{\alpha}{B}\frac{\gamma B - \alpha D}{B} + (B+1)\frac{\alpha}{B} - \gamma\frac{\gamma B - \alpha D}{B} - \alpha \\
&= \frac{\alpha D(\gamma B - \alpha D)}{B^2} + \alpha + \frac{\alpha}{B} - \frac{\gamma(\gamma B - \alpha D)}{B} - \alpha \\
&= \frac{\alpha D(\gamma B - \alpha D) - \gamma B(\gamma B - \alpha D)}{B^2} + \alpha + \frac{\alpha}{B} - \alpha \\
&= \frac{(\gamma B - \alpha D)(\alpha D - \gamma B)}{B^2} + \frac{\alpha}{B} \\
&= -\frac{(\gamma B - \alpha D)^2}{B^2} - \frac{\alpha}{B} \\
&= \frac{\alpha}{B} - \frac{\alpha}{B} = 0.
\end{aligned}$$

Hence, $\bar{x} = \dfrac{\gamma B - \alpha D}{B} = \dfrac{\alpha}{\gamma B - \alpha D}$. $\square$

Our next lemma shows that consecutive terms of our solution oscillate around this interval K.

Lemma 3.13. *Let* $\{x_n\}_{n=-1}^{\infty}$ *be a solution of* (1.1). *The following statements are true.*

1. *If* $\dfrac{\alpha D}{\gamma} < B \le B^+$, *then:*

   (a) *if* $x_n \le \dfrac{\alpha}{\gamma B - \alpha D}$, *then* $x_{n+1} \ge \dfrac{\gamma B - \alpha D}{B}$;

   (b) *if* $x_n \ge \dfrac{\gamma B - \alpha D}{B}$, *then* $x_{n+1} \le \dfrac{\alpha}{\gamma B - \alpha D}$.

2. *If* $B > B^+$*, then:*

   (a) *if* $x_n \leq \dfrac{\gamma B - \alpha D}{B}$*, then* $x_{n+1} \geq \dfrac{\alpha}{\gamma B - \alpha D}$*;*

   (b) *if* $x_n \geq \dfrac{\alpha}{\gamma B - \alpha D}$*, then* $x_{n+1} \leq \dfrac{\gamma B - \alpha D}{B}$*.*

Proof. We give the proof for the case when $\dfrac{\alpha D}{\gamma} < B \leq B^+$. The proof for the other case follows similarly, and will be omitted.

Suppose that $x_n \leq \dfrac{\alpha}{\gamma B - \alpha D}$. Then, by using Inequality (2.8), we can see that

$$\begin{aligned}
x_{n+1} &= \frac{\alpha + \gamma x_{n-1}}{B x_n + D x_n x_{n-1} + x_{n-1}} \\
&\geq \frac{\alpha + \gamma x_{n-1}}{B\left(\frac{\alpha}{\gamma B - \alpha D}\right) + D\left(\frac{\alpha}{\gamma B - \alpha D}\right) x_{n-1} + x_{n-1}} \\
&= \frac{\alpha + \gamma x_{n-1}}{\frac{\alpha B}{\gamma B - \alpha D} + \left(\frac{\gamma B}{\gamma B - \alpha D}\right) x_{n-1}} \\
&\geq \min\left\{\frac{\alpha(\gamma B - \alpha D)}{\alpha B}, \frac{\gamma(\gamma B - \alpha D)}{\gamma B}\right\} = \frac{\gamma B - \alpha D}{B}.
\end{aligned}$$

Suppose that $x_n \geq \dfrac{\gamma B - \alpha D}{B}$. Then, by using Lemma 3.11:

$$\begin{aligned}
x_{n+1} &= \frac{\alpha + \gamma x_{n-1}}{B x_n + D x_n x_{n-1} + x_{n-1}} \\
&\leq \frac{\alpha + \gamma x_{n-1}}{B\left(\frac{\gamma B - \alpha D}{B}\right) + D\left(\frac{\gamma B - \alpha D}{B}\right) x_{n-1} + x_{n-1}} \\
&= \frac{\alpha + \gamma x_{n-1}}{(\gamma B - \alpha D) + \left(\frac{D(\gamma B - \alpha D)}{B} + 1\right) x_{n-1}} \\
&= \frac{\alpha + \gamma x_{n-1}}{(\gamma B - \alpha D) + \left(\frac{D(\gamma B - \alpha D) + B}{B}\right) x_{n-1}} \\
&\leq \max\left\{\frac{\alpha}{\gamma B - \alpha D}, \frac{\gamma B}{D(\gamma B - \alpha D) + B}\right\} = \frac{\alpha}{\gamma B - \alpha D}.
\end{aligned}$$

Thus, concludes the proof. $\square$

The next lemma establishes that the interval K is invariant. The proof follows from Lemma 3.13, and will be omitted.

Lemma 3.14. *If* $\dfrac{\alpha D}{\gamma} < B$ *then* K *is an invariant interval. In other words, if* $x_n \in K$ *then* $x_{n+1} \in K$*.*

Lemma 3.15. *If* $\dfrac{\alpha D}{\gamma} < B$ *then* K *is an attracting interval. In other words, there exists* $N \in \mathbb{Z}^+$ *such that* $x_n \in K$ *for all* $n \geq N$*.*

Proof. We will give the proof for $\dfrac{\alpha D}{\gamma} < B \leq B^+$. The proof for the other case follows similarly and will be omitted.

Let $I = \liminf\limits_{n\to\infty} x_n$ and $S = \limsup\limits_{n\to\infty} x_n$. Then, if either $I \in K$ or $S \in K$, then we are done, since by Lemma 3.14, if $I \in K$, then $S \in K$ and vice versa. For the sake of contradiction, assume that $I \notin K$ and $S \notin K$. It follows from Lemma 3.13 that $I < \dfrac{\gamma B - \alpha D}{B}$ and $S > \dfrac{\alpha}{\gamma B - \alpha D}$. Thus, there is an open neighborhood O containing S such that $O \cap K = \emptyset$. By Theorem 2.6, let S_{n+1} be a full-limiting sequence such that $\lim\limits_{n\to\infty} S_{n+1} = S$. Thus, there exists a positive integer N, such that $S_n \in O$ for $n \geq N$. According to Lemma 3.13, if $S_n > \dfrac{\alpha}{\gamma B - \alpha D} \geq \dfrac{\gamma B - \alpha D}{B}$, then $S_{n+1} < \dfrac{\alpha}{\gamma B - \alpha D}$, which is a contradiction. Thus, it must be the case that both I and S are in the interval K. □

4 Proof of Theorem 1.1

We have defined the constant B^+ in (3.25), and by Lemma 3.10, we know that $B^+ < B^*$. We will prove Theorem 1.1 by partitioning the values of B as follows. For $B \leq B^+$, we will use Theorem 2.1 to show that, with the exception of a single value of B, the unique equilibrium is a global attractor. For this single exception, we use Theorem 2.2. Next, for $B^+ < B \leq B^*$, we will use Theorem 2.3 to show that every solution converges to the unique equilibrium. Finally, for $B > B^*$, we use Theorem 2.3 and Theorem 3.9 to show that every solution converges to a not necessarily prime period-two solution, and there exists a unique prime period-two solution.

Theorem 4.1. *If* $B \leq B^+$ *and* $B \neq \dfrac{\alpha D}{\gamma} - 1$ *then the unique positive equilibrium of* (1.1) *is globally asymptotically stable.*

Proof. Clearly $B^+ < B^*$, thus Theorem 3.5 implies that $\bar{x}$ is locally asymptotically stable in this region of parameters. It suffices to show that under these conditions $\bar{x}$ is a global attractor.

This proof will use the Theorem 2.1.

Consider the function $f(u,v) = \dfrac{\alpha + \gamma v}{Bu + Duv + v}$. Then, by (3.5), it is clear that $f(u,v)$ is decreasing in u when v is fixed. Further, if $B \leq \dfrac{\alpha D}{\gamma}$, then $f(u,v)$ is decreasing in v when u is fixed. Further, if $\dfrac{\alpha D}{\gamma} < B \leq B^+$, then by Lemma 3.15, there exists

$N \in \mathbb{Z}^+$, such that $x_n \in K = \left[\dfrac{\gamma B - \alpha D}{B}, \dfrac{\alpha}{\gamma B - \alpha D}\right]$ for all $n \geq N$, and which also makes $f(u,v)$ nonincreasing in v.

We will show that the following system has no positive solutions other than $m = M$.

$$\begin{cases} M = \dfrac{\alpha + \gamma m}{(B+1)m + Dm^2}, \\ \\ m = \dfrac{\alpha + \gamma M}{(B+1)M + DM^2} \end{cases} \tag{4.1}$$

Thus

$$\begin{aligned} (B+1)mM + Dm^2M &= \alpha + \gamma m, \\ (B+1)mM + DmM^2 &= \alpha + \gamma M. \end{aligned}$$

Subtracting we obtain

$$\begin{gathered} DmM(m - M) = \gamma(m - M) \\ DmM = \gamma. \end{gathered}$$

Since we can assume $m \neq M$. Thus,

$$m = \frac{\gamma}{DM}. \tag{4.2}$$

By substituting (4.2) into (4.1) we obtain

$$B = \frac{\alpha D}{\gamma} - 1.$$

Thus, if $B \neq \dfrac{\alpha D}{\gamma} - 1$ then Theorem 2.1 implies that the unique positive equilibrium is a global attractor. From Theorem 3.5, we see that in this region the equilibrium is globally asymptotically stable. □

We now look at the point when $B = \dfrac{\alpha D}{\gamma} - 1$ separately.

Theorem 4.2. *Suppose that* $B = \dfrac{\alpha D}{\gamma} - 1$. *Then, every solution of* (1.1) *converges to the unique positive equilibrium,* $\bar{x}$.

Proof. We will use Theorem 2.2. Observe that

$$\frac{d}{dx} x f(x,x) = 0,$$

thus, $xf(x,x)$ is nondecreasing in x. Hence, it follows that $\bar{x}$ is a global attractor. □

Theorem 4.3. *Let $\bar{x}$ be the unique positive equilibrium of* (1.1)*. The following statements are true.*

1. *If $B^+ < B \leq B^*$ then every solution of* (1.1) *converges to $\bar{x}$.*

2. *If $B > B^*$ then every solution of* (1.1) *converges to a period-two solution. Further, the unique equilibrium is unstable, and every prime period-two solution is unique.*

Proof. In this region, $f(u, v)$ is decreasing in u and increasing in v. By Theorem 2.3, the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n+1}\}_{n=-1}^{\infty}$ are eventually monotone. Since they are eventually in J, a bounded interval, they must converge. By Theorem 3.9, if $B \leq B^*$, there are no period-two solutions present, thus every solution must converge to the equilibrium. When $B > B^*$, we know that period-two solutions exist, and every solution converges to a (not necessarily prime) period-two solution. The uniqueness of the prime period-two solution follows from Theorem 3.9.

Lastly, in the case when $B > B^*$ the equilibrium is unstable. To prove this, we apply Theorem 3.5. Further, as a consequence of Theorem 3.9 we know that there exists a unique prime period-two solution when $B > B^*$. This concludes the proof of Theorem 1.1. □

5 Conclusion

We saw that period-two solutions emerge for $B > B^*$, and that the equilibrium is an unstable saddle point in this region. We pose the following open problem.

Open Problem 5.1. Find the basin of attraction for the unique prime period-two solution of (1.1) when $B > B^*$.

An Application of Theorem 1.1 to a System

We will conclude this paper by showing how Theorem 1.1 can be applied to a system of rational difference equations. Let us introduce the system of rational difference equations known as System (12, 17) in the numbering system defined in [4]. Namely,

$$\left\{ \begin{aligned} x_{n+1} &= \frac{\alpha}{\beta x_n + y_n} \\ y_{n+1} &= \frac{x_n}{A + x_n} \end{aligned} \right. \quad \text{for } n = 0, 1, \ldots, \tag{5.1}$$

where the coefficients are positive real numbers and the initial conditions are nonnegative real numbers such that the denominators are never zero.

This system was studied in [3], where it was determined that the system was bounded in both components. Further, a nonautonomous variation of the system was studied in [22] where it was determined that both components are bounded when the coefficients are themselves sequences which are bounded from above and from below by positive constants. We can iterate to obtain the following second-order equation for x_{n+1}, which we can view as a special case of (1.1):

$$x_{n+1} = \frac{\alpha A + \alpha x_{n-1}}{A\beta x_n + \beta x_n x_{n-1} + x_{n-1}} \text{ for } n = 0, 1, \ldots \tag{5.2}$$

The global behavior of the system in (5.1) as well as the equation in (5.2) has been determined by Drymonis and Ladas, see [11], who showed that there is a unique equilibrium which is globally asymptotically stable. We claim that this result will also follow from our Theorem 1.1. Namely, we give the following corollary.

Corollary 5.2. *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be any solution of the system* (5.1). *Then the unique positive equilibrium $(\bar{x}, \bar{y})$ is globally asymptotically stable.*

Proof. After converting the system into the equation (5.2), we can define the new parameters as follows:

$$x_{n+1} = \frac{\alpha A + \alpha x_{n-1}}{A\beta x_n + \beta x_n x_{n-1} + x_{n-1}} := \frac{\alpha' + \gamma' x_{n-1}}{B' x_n + D' x_n x_{n-1} + x_{n-1}}$$

and thus

$$\gamma' B' - \alpha' D' = \alpha A \beta - \alpha A \beta = 0.$$

Thus,

$$\gamma'^2 B' = \gamma' \alpha' D' < \gamma'^2 + \gamma' \alpha' D' + 2\alpha' + 2\alpha' \sqrt{1 + \gamma' D'}.$$

We can now apply case (1) of Theorem 1.1. $\square$

Acknowledgements

This work was supported, in part, by a Creativity and Research Grant from Monmouth University.

References

[1] J. Bektešević, M. R. S. Kulenović, and E. Pilav. Global dynamics of cubic second order difference equation in the first quadrant. *Adv. Difference Equ.*, pages 2015:176, 38, 2015.

[2] A. Brett and M. R. S. Kulenović. Basins of attraction for two-species competitive model with quadratic terms and the singular Allee effect. *Discrete Dyn. Nat. Soc.*, pages Art. ID 847360, 16, 2015.

[3] E. Camouzis, E. Drymonis, G. Ladas, and W. Tikjha. Patterns of boundedness of the rational system $x_{n+1} = \dfrac{\alpha_1}{A_1 + B_1 x_n + C_1 y_n}$ and $y_{n+1} = \dfrac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + C_2 y_n}$. *J. Difference Equ. Appl.*, 18(1):89–110, 2012.

[4] E. Camouzis, M. R. S. Kulenović, G. Ladas, and O. Merino. Rational systems in the plane. *J. Difference Equ. Appl.*, 15(3):303–323, 2009.

[5] E. Camouzis and G. Ladas. When does periodicity destroy boundedness in rational equations? *J. Difference Equ. Appl.*, 12(9):961–979, 2006.

[6] E. Camouzis and G. Ladas. *Dynamics of third-order rational difference equations with open problems and conjectures*, volume 5 of *Advances in Discrete Mathematics and Applications*. Chapman & Hall/CRC, Boca Raton, FL, 2008.

[7] M. Dehghan, C. M. Kent, R. Mazrooei-Sebdani, N. L. Ortiz, and H. Sedaghat. Dynamics of rational difference equations containing quadratic terms. *J. Difference Equ. Appl.*, 14(2):191–208, 2008.

[8] M. Dehghan, C. M. Kent, R. Mazrooei-Sebdani, N. L. Ortiz, and H. Sedaghat. Monotone and oscillatory solutions of a rational difference equation containing quadratic terms. *J. Difference Equ. Appl.*, 14(10-11):1045–1058, 2008.

[9] M. Dehghan, R. Mazrooei-Sebdani, and H. Sedaghat. Global behaviour of the Riccati difference equation of order two. *J. Difference Equ. Appl.*, 17(4):467–477, 2011.

[10] M. DiPippo, E. Janowski, and M. R. S. Kulenović. Global asymptotic stability for quadratic fractional difference equation. *Adv. Difference Equ.*, pages 2015:179, 13, 2015.

[11] E. Drymonis and G. Ladas. On the global character of the rational system $\dfrac{\alpha_1}{A_1 + B_1 x_n + y_n}$ and $\dfrac{\alpha_2 + \beta_2 x_n}{A_2 + B_2 x_n + C_2 y_n}$. *Sarajevo Journal of Mathematics*, 8(21):293–309, 2012.

[12] M. Garić-Demirović, M. R. S. Kulenović, and M. Nurkanović. Basins of attraction of certain homogeneous second order quadratic fractional difference equation. *J. Concr. Appl. Math.*, 13(1-2):35–50, 2015.

[13] E. A. Grove and G. Ladas. *Periodicities in nonlinear difference equations*, volume 4 of *Advances in Discrete Mathematics and Applications*. Chapman & Hall/CRC, Boca Raton, FL, 2005.

[14] E. J. Janowski and M. R. S. Kulenović. Existence of a period-two solution in linearizable difference equations. *Discrete Dyn. Nat. Soc.*, pages Art. ID 421545, 9, 2013.

[15] S. Jašarević and M. R. S. Kulenović. Basins of attraction of equilibrium and boundary points of second-order difference equations. *J. Difference Equ. Appl.*, 20(5-6):947–959, 2014.

[16] S. Kalabušić, M. R. S. Kulenović, and M. Mehuljić. Global period-doubling bifurcation of quadratic fractional second order difference equation. *Discrete Dyn. Nat. Soc.*, pages Art. ID 920410, 13, 2014.

[17] S. Kalabušić, M. R. S. Kulenovicć, and M. Mehuljić. Global dynamics and bifurcations of two quadratic fractional second order difference equations. *J. Computational Analysis and Applications*, 21(1):132–143, 2016.

[18] G. Karakostas. Convergence of a difference equation via the full limiting sequences method. *Differential Equations and Dynamical Systems*, 1:289–294, 1993.

[19] G. Karakostas. Asymptotic 2-periodic difference equations with diagonally self-invertible responses. *J. Difference Equ. Appl.*, 6:329–335, 2000.

[20] C. M. Kent and H. Sedaghat. Global attractivity in a quadratic-linear rational difference equation with delay. *J. Difference Equ. Appl.*, 15(10):913–925, 2009.

[21] C. M. Kent and H. Sedaghat. Global attractivity in a rational delay difference equation with quadratic terms. *J. Difference Equ. Appl.*, 17(4):457–466, 2011.

[22] Y. Kostrov and Z. Kudlak. On the boundedness character of a system of difference equations with non-constant coefficients. *Comm. Appl. Nonlinear Anal.*, 2015.

[23] M. R. S. Kulenović and G. Ladas. *Dynamics of second order rational difference equations*. Chapman & Hall/CRC, Boca Raton, FL, 2002. With open problems and conjectures.

[24] M. R. S. Kulenović, E. Pilav, and E. Silić. Local dynamics and global attractivity of a certain second-order quadratic fractional difference equation. *Advances in Difference Equations*, 2014(68), 2014.

[25] M. R. S. Kulenović and M. Pilling. Global dynamics of a certain two-dimensional competitive system of rational difference equations with quadratic terms. *J. Comput. Anal. Appl.*, 19(1):156–166, 2015.

[26] Mustafa R. S. Kulenović and Orlando Merino. *Discrete dynamical systems and difference equations with* $Mathematica$. Chapman & Hall/CRC, Boca Raton, FL, 2002.

[27] D. Mitrinović. *Elementary Inequalities*. P. Noordhoff LTD, 1964.

[28] H. Sedaghat. On third-order rational difference equations with quadratic terms. *J. Difference Equ. Appl.*, 14(8):889–897, 2008.

[29] H. Sedaghat. Global behaviours of rational difference equations of orders two and three with quadratic terms. *J. Difference Equ. Appl.*, 15(3):215–224, 2009.

[30] H. Sedaghat. Periodic and chaotic behavior in a class of second order difference equations. In *Advances in discrete dynamical systems*, volume 53 of *Adv. Stud. Pure Math.*, pages 311–318. Math. Soc. Japan, Tokyo, 2009.