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A Nonautonomous Discrete Epidemic Model with Isolation

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Abstract

For a nonautonomous discrete SIQR model, we obtain conditions for permanence and extinction of the disease and we establish the global asymptotical stability when our conditions determine extinction. We also discuss the particular autonomous and periodic situations, where our conditions are sharp thresholds between permanence and extinction of the disease.

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1 Introduction

The study of epidemic models with an isolation or quarantined class is a common subject in mathematical epidemiology. In this paper we consider a nonautonomous discrete SIQR model, an epidemiological model where the population is divided in four compartments. These compartments denoted S, I, Q and R correspond respectively to uninfected individuals that are susceptible to the disease (susceptible compartment), the individuals that are infected and not isolated (infective compartment), the isolated individuals (isolated compartment) and the recovered and immune individuals (recovered compartment). This model can be translated into the following system of nonautonomous difference equations:

$$\begin{cases} S_{n+1} - S_n = \Lambda_n - \beta_n S_{n+1} I_n - d_n S_{n+1} \\ I_{n+1} - I_n = \beta_n S_{n+1} I_n - \left[\gamma_n + \sigma_n + d_n + (\alpha_1)_n\right] I_{n+1} \\ Q_{n+1} - Q_n = \sigma_n I_{n+1} - [d_n + (\alpha_2)_n + \varepsilon_n] Q_{n+1} \\ R_{n+1} - R_n = \gamma_n I_{n+1} + \varepsilon_n Q_{n+1} - d_n R_{n+1} \end{cases}, \quad n = 0, 1, \cdots. \tag{1.1}$$

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The parameters used have the following meaning: d_n is the per capita natural mortality rate; γ_n is the recovery rate; σ_n is the removal rate from the infectives; $(\alpha_1)_n$ is the disease related death in the infectives; $(\alpha_2)_n$ is the disease related death in the isolated class; and ε_n is the removal rate from the isolated class. In this paper we consider the incidence into the infective class given by $\beta_n S_{n+1} I_n$. Note that it is assumed in this paper that the isolated individuals are perfectly separated from the others and thus do not infect the susceptibles. It is also assumed that the infection confers lifelong immunity upon recovery like in the case of mumps, measles and rubella.

Our model is a nonautonomous and discrete time counterpart of one of the six continuous-time models considered in [11], where the authors discuss two family of models with isolation. In that paper, for each family, three types of incidence function are considered: mass-action, standard and quarantined adjusted incidences. Our model is derived by applying Mickens nonstandard finite difference method [17] to the continuous SIQR model with mass-action incidence in [11]. A continuous-time nonautonomous version of our model was considered in [24], where a family of models with general incidence functions is discussed.

In autonomous epidemic models the basic reproduction number R_0 (the average number of secondary infections produced when one infected individual is introduced to a host population where everyone is susceptible [12]) is a fundamental tool in determining the extinction/persistence of the disease. Usually, when $R_0 < 1$ the disease is extinct and if $R_0 > 1$ we have persistence.

In the nonautonomous case the situation is not so simple. In spite of this, several results related to the permanence and extinction of the disease for nonautonomous epidemic models have been obtained [13, 15, 16, 18–20, 24–27, 30, 31].

Epidemic models with isolation or/and quarantine were addressed in several papers in the literature. An SIQR model for childhood diseases was considered by Feng and Thieme [7] and it was showed that isolation can lead to self-sustained oscillations. Some dynamical aspects of Feng and Thieme's model were studied by Wu and Feng [28] that found Hopf and homoclinic bifurcations associated to the unfolding of a normal form derived from the model. Models with quarantine and a latent class were considered by Gerberry and Milner [9] and by Yi, Zhao and Zhang [29]. A model with twelve different classes was proposed by Safi and Gumel [23] to study the impact of quarantine and isolation in limiting the spread of an infectious disease in the population. The same authors studied a delayed model including a quarantined class [21] and recently [22] they considered a two patch model and showed that the imperfect nature of quarantine in the two patches could lead to backward bifurcations when the associated reproduction number of the model is less than unity.

A different approach to modelling quarantine was considered by Arino, Jordan and van den Driessche [2] that studied a model where a disease that can be transmitted between different species and different patches and where quarantine is present in the form of travel restriction between patches.

Note that, with the exception of the model in [24], all the models referred above are autonomous models. In spite of this, periodic or quasiperiodic behaviour is seldom observed in the process of disease transmission [1, 3–6, 8, 10, 14]. Note also that all the above models are continuous. We should mention that the formulation of the nonautonomous discrete SIQR model considered in this work was inspired by the use of a nonstandard finite difference method to obtain a discrete version of a SIRVS model in a recent paper by Zhang [32].

2 Nonautonomous Discrete SIQR Model

In this section, we make some assumptions, establish some properties of the solutions of our model and obtain some auxiliary results.

We will assume that

H1) (Λ_n), (d_n), (γ_n), (σ_n), $((\alpha_1)_n)$, $((\alpha_2)_n)$ and (ε_n) are bounded and nonnegative sequences;

H2) there is $\omega \in \mathbb{N}$ such that

$$\limsup_{n\to+\infty} \prod_{k=n}^{n+\omega} \frac{1}{1+d_k} < 1.$$

It follows from H2) that there are constants $K > 0$ and $\theta \in]0,1[$ such that

$$\prod_{k=m}^{n-1} \frac{1}{1+d_k} < K\theta^{n-m}, \tag{2.1}$$

for any $m, n \in \mathbb{N}$.

Given a bounded sequence (h_n) we will use the notation

$$h^u = \sup_{n\geqslant 0} h_n \quad \text{and} \quad h^\ell = \inf_{n\geqslant 0} h_n.$$

Consider the auxiliary equation

$$x_{n+1} = \frac{\Lambda_n}{1+d_n} + \frac{1}{1+d_n} x_n \tag{2.2}$$

where Λ_n and d_n are the birth and death rate in (1.1). The solutions of this equation will be important to determine threshold conditions for the permanence and extinction of the disease. In the next lemma some properties of the solutions of equation (2.2) are established.

Lemma 2.1. *We have the following:*

i) All solutions (x_n) *of equation* (2.2) *with initial condition* $x_0 \geqslant 0$ *are nonnegative for all* $n \in \mathbb{N}$*;*

ii) All solutions (x_n) *of equation* (2.2) *with initial condition* $x_0 > 0$ *are positive for all* $n \in \mathbb{N}$*;*

iii) Given a solution (x_n) *of equation* (2.2) *with initial condition* $x_0 \in [\Lambda^\ell/d^u, \Lambda^u/d^\ell]$ *we have*

$$\frac{\Lambda^\ell}{d^u} \leqslant x_n \leqslant \frac{\Lambda^u}{d^\ell}$$

for all $n \in \mathbb{N}$*;*

iv) Each fixed solution (x_n) *of* (2.2) *with initial condition* $x_0 \geqslant 0$ *is bounded and globally uniformly attractive on* $[0, +\infty)$*;*

v) There is a constant $D > 0$ *such that, if* (x_n) *is a solution of* (2.2) *and* $(\tilde{x}_n)$ *is a solution of the system*

$$x_{n+1} = \frac{\Lambda_n + x_n + f_n}{1 + d_n}, \quad n = 0, 1, \dots \tag{2.3}$$

with $\tilde{x}_0 = x_0$ *then*

$$\sup_{n \geqslant 0} |\tilde{x}_n - x_n| \leqslant D \sup_{n \geqslant 0} |f_n|.$$

Proof. Properties i) and ii) are immediate. If $x_n \leqslant \Lambda^u/d^\ell$ we have

$$x_{n+1} = \frac{\Lambda_n}{1 + d_n} + \frac{x_n}{1 + d_n} \leqslant \frac{\Lambda^u}{1 + d^\ell} + \frac{\Lambda^u}{(1 + d^\ell)d^\ell} = \frac{\Lambda^u}{d^\ell}$$

and, if $x_n \geqslant \Lambda^\ell/d^u$ we have

$$x_{n+1} = \frac{\Lambda_n}{1 + d_n} + \frac{x_n}{1 + d_n} \geqslant \frac{\Lambda^\ell}{1 + d^u} + \frac{\Lambda^\ell}{(1 + d^u)d^u} = \frac{\Lambda^\ell}{d^u}$$

and property iii) follows. The solution of the nonautonomous first-order difference equation (2.2) it is given by

$$x_{n+1} = \left(\prod_{m=0}^{n-1} \frac{1}{1 + d_m} \right) x_0 + \sum_{m=0}^{n-1} \Lambda_m \left(\prod_{k=m}^{n-1} \frac{1}{1 + d_k} \right). \tag{2.4}$$

By (2.4) and H2), we have

$$x_{n+1} \leqslant x_0 K \theta^{n-m} + \sum_{m=0}^{n-1} \Lambda_m K \theta^{n-m} \leqslant x_0 K + \Lambda^u K \frac{\theta}{1 - \theta},$$

and thus (x_n) is bounded.

Letting (y_n) be some other solution of (2.2), we have by (2.4) and

$$|x_{n+1} - y_{n+1}| = \left(\prod_{m=0}^{n-1} \frac{\Lambda_m}{1+d_m}\right) |x_0 - y_0| \leqslant \Lambda^u K \theta^n |x_0 - y_0| \to 0,$$

as $n \to +\infty$ and we obtain property iv).

Give solutions (x_n) of (2.2) and $(\tilde{x}_n)$ of (2.3) such that $x_0 = \tilde{x}_0$ we define $u_n = |x_n - \tilde{x}_n|$ and note that $u_0 = 0$. By (2.4) we have

$$\begin{aligned} u_{n+1} &\leqslant \frac{f_n}{1+d_n} + \frac{1}{1+d_n} u_n \\ &\leqslant \frac{f_n}{1+d_n} + \frac{1}{1+d_n}\left(\frac{f_{n-1}}{1+d_{n-1}} + \frac{1}{1+d_{n-1}} u_{n-1}\right) \\ &\leqslant \cdots \\ &\leqslant \left(\prod_{m=0}^{n-1} \frac{1}{1+d_m}\right) u_0 + \sum_{m=0}^{n-1} f_m \left(\prod_{k=m}^{n-1} \frac{1}{1+d_k}\right) \qquad (2.5) \\ &= \sum_{m=0}^{n-1} f_m \left(\prod_{k=m}^{n-1} \frac{1}{1+d_k}\right) \\ &\leqslant \frac{K\theta}{1-\theta} \sup_{m \geqslant 0} f_m, \end{aligned}$$

and we obtain v) setting $D = K\theta/(1-\theta)$. □

Next we state some simple facts about problem (1.1).

Lemma 2.2. *We have the following:*

i) All solutions $((S_n, I_n, Q_n, R_n))$ *of* (1.1) *with* $S_0 \geqslant 0$, $I_0 \geqslant 0$, $Q_0 \geqslant 0$ *and* $R_0 \geqslant 0$ *satisfy* $S_n \geqslant 0$, $I_n \geqslant 0$, $Q_n \geqslant 0$ *and* $R_n \geqslant 0$ *for all* $n \geqslant 0$*;*

ii) All solutions $((S_n, I_n, Q_n, R_n))$ *of* (1.1) *with* $S_0 > 0$, $I_0 > 0$, $Q_0 > 0$ *and* $R_0 > 0$ *satisfy* $S_n > 0$, $I_n > 0$, $Q_n > 0$ *and* $R_n > 0$ *for all* $n \geqslant 0$*;*

iii) If $((S_n, I_n, Q_n, R_n))$ *is a solution of* (1.1) *with nonnegative initial conditions then there is a constant* $L > 0$ *such that*

$$\limsup_{n \to +\infty} (S_n + I_n + Q_n + R_n) \leqslant L.$$

Proof. By the first two equations in (1.1), we conclude that

$$S_{n+1} = \frac{\Lambda_n + S_n}{1 + d_n + \beta_n I_n}$$

and

$$I_{n+1} = \frac{(\beta_n S_{n+1} + 1)I_n}{1 + \gamma_n + \sigma_n + d_n + (\alpha_1)_n}.$$

Thus, if $S_0, I_0 > 0$, we conclude that $S_1 > 0$, and therefore $I_1 > 0$. Proceeding by induction on n we can easily conclude that $S_n, I_n > 0$ for all $n \in \mathbb{N}_0$. By the last two equations in (1.1), we have

$$Q_{n+1} = \frac{Q_n + \sigma_n I_{n+1}}{1 + d_n + \varepsilon_n + (\alpha_2)_n} \tag{2.6}$$

and

$$R_{n+1} = \frac{R_n + \gamma_{n+1} I_{n+1} + \varepsilon_n Q_{n+1}}{1 + d_n}.$$

Thus, if $Q_0, R_0, I_0, S_0 > 0$, we have by the previous argument $S_n, I_n > 0$ for all $n \in \mathbb{N}_0$, and we conclude that $Q_1 > 0$, and therefore $R_1 > 0$. Proceeding by induction on n we can easily conclude that $Q_n, R_n > 0$ for all $n \in \mathbb{N}_0$. Similar arguments allow us to draw similar conclusions for nonnegative initial conditions instead of positive initial conditions. We have proved i) and ii).

Writing $N_n = S_n + I_n + Q_n + R_n$ and adding the equations in (1.1), we obtain

$$N_{n+1} = \frac{\Lambda_n}{1 + d_n} + \frac{1}{1 + d_n}(N_n - (\alpha_1)_n I_{n+1} - (\alpha_2)_n Q_{n+1}) \leqslant \frac{\Lambda_n}{1 + d_n} + \frac{1}{1 + d_n} N_n.$$

By comparison we have $N_n \leqslant x_n$, where (x_n) is the solution of (2.2) with $x_0 = N_0$ and, by iv) in Lemma 2.1, we have iii). □

3 Extinction and Persistence of the Infectives

In this section we will establish our main results on the persistence and extinction of the disease. For each solution (x_n^*) of (2.2) with $x_0 > 0$ and each $\lambda > 0$, define the numbers

$$\mathcal{R}_p(\lambda) = \liminf_{n \to +\infty} \prod_{k=n}^{n+\lambda} \frac{1 + \beta_k x_{k+1}^*}{1 + \gamma_k + \sigma_k + d_k + (\alpha_1)_k} \tag{3.1}$$

and

$$\mathcal{R}_e(\lambda) = \limsup_{n \to +\infty} \prod_{k=n}^{n+\lambda} \frac{1 + \beta_k x_{k+1}^*}{1 + \gamma_k + \sigma_k + d_k + (\alpha_1)_k}. \tag{3.2}$$

We will see that the numbers $\mathcal{R}_p(\lambda)$ and $\mathcal{R}_e(\lambda)$ do not depend on the particular positive solution of (2.2). This numbers play the role of the basic reproduction number in the nonautonomous setting.

We have the following result.

Lemma 3.1. *The numbers $\mathcal{R}_p(\lambda)$ and $\mathcal{R}_e(\lambda)$ are independent of the particular solution (x_n^*) with $x_0^* > 0$ of* (2.2).

Proof. Since we don't know whether the numbers in (3.1) and (3.2) depend on the particular solution of (2.2) or not, write $\mathcal{R}_p(x,\lambda)$ and $\mathcal{R}_e(x,\lambda)$ for the numbers in (3.1) and (3.2) corresponding to the solution $x = (x_n)$ of (2.2).

Let $x^* = (x_n^*)$ and $y^* = (y_n^*)$ be distinct solutions of (2.2). By iv) in Lemma 2.1,

$$x_k^* - \varepsilon \leqslant y_k^* \leqslant x_k^* + \varepsilon$$

for k sufficiently large, say $k \geqslant N$. Therefore

$$\begin{aligned}
&\prod_{k=n}^{n+\lambda} \frac{1+\beta_k y_{k+1}^*}{1+\gamma_k+\sigma_k+d_k+(\alpha_1)_k} \\
&\leqslant \prod_{k=n}^{n+\lambda} \frac{1+\beta_k x_{k+1}^* + \beta_k\varepsilon}{1+\gamma_k+\sigma_k+d_k+(\alpha_1)_k} \\
&\leqslant \prod_{k=n}^{n+\lambda} \left(\frac{1+\beta_k x_{k+1}^*}{1+\gamma_k+\sigma_k+d_k+(\alpha_1)_k} + \varepsilon E\right) \\
&\leqslant \prod_{k=n}^{n+\lambda} \frac{1+\beta_k x_{k+1}^*}{1+\gamma_k+\sigma_k+d_k+(\alpha_1)_k} + \sum_{j=1}^{\lambda+1} \binom{\lambda+1}{j} \varepsilon^j D^{\lambda+1-j} E^j,
\end{aligned} \tag{3.3}$$

for $n \geqslant N$, where

$$D = \frac{1+\beta^u (x^*)^u}{1+\gamma^\ell+\sigma^\ell+d^\ell+(\alpha_1)^\ell} \quad \text{and} \quad E = \frac{\beta^u}{1+\gamma^\ell+\sigma^\ell+d^\ell+(\alpha_1)^\ell}.$$

By the same reasoning,

$$\begin{aligned}
&\prod_{k=n}^{n+\lambda} \frac{1+\beta_k y_{k+1}^*}{1+\gamma_k+\sigma_k+d_k+(\alpha_1)_k} \\
&\geqslant \prod_{k=n}^{n+\lambda} \frac{1+\beta_k x_{k+1}^* - \beta_k\varepsilon}{1+\gamma_k+\sigma_k+d_k+(\alpha_1)_k} \\
&\geqslant \prod_{k=n}^{n+\lambda} \frac{1+\beta_k x_{k+1}^*}{1+\gamma_k+\sigma_k+d_k+(\alpha_1)_k} - \sum_{j=1}^{\lambda+1} \binom{\lambda+1}{j} \varepsilon^j F^{\lambda+1-j} G^j,
\end{aligned} \tag{3.4}$$

for $n \geqslant N$, where

$$F = \frac{1+\beta^\ell (x^*)^\ell}{1+\gamma^u+\sigma^u+d^u+(\alpha_1)^u} \quad \text{and} \quad G = \frac{\beta^\ell}{1+\gamma^u+\sigma^u+d^u+(\alpha_1)^u}.$$

By (3.3) and (3.4) we have

$$\begin{aligned}&\mathcal{R}_p(x^*,\lambda)-\sum_{j=1}^{\lambda+1}\binom{\lambda+1}{j}\varepsilon^j F^{\lambda+1-j}G^j\\ &\leqslant \mathcal{R}_p(y^*,\lambda)\\ &\leqslant \mathcal{R}_p(x^*,\lambda)+\sum_{j=1}^{\lambda+1}\binom{\lambda+1}{j}\varepsilon^j D^{\lambda+1-j}E^j\end{aligned}$$

and similarly

$$\begin{aligned}&\mathcal{R}_e(x^*,\lambda)-\sum_{j=1}^{\lambda+1}\binom{\lambda+1}{j}\varepsilon^j F^{\lambda+1-j}G^j\\ &\leqslant \mathcal{R}_e(y^*,\lambda)\\ &\leqslant \mathcal{R}_e(x^*,\lambda)+\sum_{j=1}^{\lambda+1}\binom{\lambda+1}{j}\varepsilon^j D^{\lambda+1-j}E^j.\end{aligned}$$

By the arbitrariness of $\varepsilon > 0$ we obtain the result. □

We state our theorem on the extinction of the disease.

Theorem 3.2. *If there is a constant $\lambda > 0$ such that $\mathcal{R}_e(\lambda) < 1$ then the infectives (I_n) go to extinction in* (1.1) *and any disease-free solution $((S_n^*, 0, 0, 0))$ of* (1.1)*, where (S_n^*) is a solution of* (2.2)*, is globally asymptotically attractive.*

Proof. Since $\mathcal{R}_e < 1$ then there are $\varepsilon_0, \varepsilon > 0$ and $N \in \mathbb{N}$ such that

$$\prod_{k=n}^{n+\lambda}\frac{1+\beta_k(x_{k+1}^*+\varepsilon_0)}{1+\gamma_k+\sigma_k+d_k+(\alpha_1)_k} < 1-\varepsilon, \tag{3.5}$$

for $n \geqslant N$. By the first two equations in (1.1), we conclude that

$$S_{n+1}=\frac{\Lambda_n+S_n}{1+d_n+\beta_n I_n}\leqslant\frac{\Lambda_n}{1+d_n}+\frac{1}{1+d_n}S_n$$

and thus $S_n \leqslant x_n$, where (x_n) is a solution of (2.2) with $S_0 = x_0$. By iv) in Lemma 2.1 we have $|x_n - x_n^*| \leqslant \varepsilon_0$ for sufficiently large n, say $n \geqslant N_1 \geqslant N$. Thus

$$S_n \leqslant x_n \leqslant x_n^* + \varepsilon_0,$$

for $n \geqslant N_1$. Therefore, by (3.5),

$$\begin{aligned} I_{n+1} &= \frac{(1+\beta_n S_{n+1})I_n}{1+\gamma_n+\sigma_n+d_n+(\alpha_1)_n} \\ &\leqslant \frac{(1+\beta_n(x^*_{n+1}+\varepsilon_0))I_n}{1+\gamma_n+\sigma_n+d_n+(\alpha_1)_n} \\ &< (1-\varepsilon)\, I_{n-\lambda-1} \\ &< (1-\varepsilon)^{\lfloor n/(\lambda+1)\rfloor}\, I_{n-\lfloor n/(\lambda+1)\rfloor(\lambda+1)} \\ &\leqslant dC^n, \end{aligned}$$

for $n \geqslant N_1$, where $C = (1-\varepsilon)^{1/(\lambda+1)}$ and $d = \max_{j=0,\dots,\lambda} I_j$. We conclude that $I_n \to 0$ as $n \to +\infty$ and we have extinction of the infectives.

Let $((S_n, I_n, Q_n, R_n))$ be any solution of (1.1) and consider $((S^*_n, 0, 0, 0))$, where (S^*_n) is a solution of (2.2), that is a disease-free solution of (1.1).

Since $I_n \to 0$ as $n \to +\infty$, given $\delta > 0$ there is $T \in \mathbb{N}$ such that $I_n < \delta$ for $n \geqslant T$. Letting $U_n = S_n - S^*_n$, we have, by the first equation in (1.1),

$$U_{n+1} - U_n = -d_n U_{n+1} - \beta_n S_{n+1} I_n,$$

for $n \geqslant T$. Thus, by iii) in Lemma 2.2,

$$-\beta^u L\delta < (1+d_n)U_{n+1} - U_n < 0,$$

for n sufficiently large. We get, for $\delta > 0$ sufficiently small

$$\begin{aligned} U_{n+1} &> -\frac{\beta^u L\delta}{1+d_n} + \frac{1}{1+d_n}U_n \\ &> -\frac{\beta^u L\delta}{1+d_n} + \frac{1}{1+d_n}\left(-\frac{\beta^u K\delta}{1+d_{n-1}} + \frac{1}{1+d_{n-1}}U_{n-1}\right) \\ &> \cdots \\ &> \left(\prod_{m=0}^{n-1}\frac{1}{1+d_m}\right)U_0 - \sum_{m=0}^{n-1}(\beta^u L\delta)^{m+1}\left(\prod_{k=m}^{n-1}\frac{1}{1+d_k}\right) \\ &> \left(\prod_{m=0}^{n-1}\frac{1}{1+d_m}\right)U_0 - \delta\beta^u L\sum_{m=0}^{n-1}K\theta^{n-m} \\ &> \left(\prod_{m=0}^{n-1}\frac{1}{1+d_m}\right)U_0 - \frac{\beta^u LK\theta}{1-\theta}\,\delta \\ &> -\frac{\beta^u LK\theta}{1-\theta}\,\delta. \end{aligned}$$

Similarly,

$$U_{n+1} < \frac{1}{1+d_n}U_n < \left(\prod_{m=0}^{n-1}\frac{1}{1+d_m}\right)U_0.$$

Since
$$\prod_{m=0}^{n-1} \frac{1}{1+d_m} \to 0 \quad \text{as} \quad n \to +\infty,$$
given $\delta > 0$, we have $|U_{n+1}| < M\delta$, where $M = \beta^u LK\theta/(1-\theta)$, for sufficiently large n. We conclude that $|U_n| \to 0$ as $n \to +\infty$ and thus
$$S_n \to S_n^* \quad \text{as} \quad n \to +\infty. \tag{3.6}$$
Again, since $I_n \to 0$ as $n \to +\infty$, given $\delta > 0$ we have
$$\begin{aligned} Q_{n+1} &< \frac{\sigma^u}{1+d_n+\varepsilon_n+(\alpha_2)_n}\delta + \frac{1}{1+d_n+\varepsilon_n+(\alpha_2)_n} Q_n \\ &< \frac{1}{1+d_n}\sigma^u\delta + \frac{1}{1+d_n} Q_n \\ &< \left(\prod_{m=0}^{n-1} \frac{1}{1+d_m}\right) Q_0 + \frac{\sigma^u\theta}{1-\theta} K\delta, \end{aligned} \tag{3.7}$$
for sufficiently large n, and, since $\delta > 0$ is arbitrary, we conclude that $Q_n \to 0$ as $n \to +\infty$. Since, $Q_n, I_n \to 0$ as $n \to +\infty$, given $\delta > 0$ we have
$$\begin{aligned} R_{n+1} &= \frac{R_n + \gamma_{n+1} I_{n+1} + \varepsilon_n Q_{n+1}}{1+d_n} \\ &< \frac{1}{1+d_n}(\gamma^u + \varepsilon^u)\delta + \frac{1}{1+d_n} R_n \\ &< \left(\prod_{m=0}^{n-1} \frac{1}{1+d_m}\right) R_0 + \frac{(\gamma^u+\varepsilon^u)\theta}{1-\theta} K\delta, \end{aligned} \tag{3.8}$$
for sufficiently large n and, since $\delta > 0$ is arbitrary, $R_n \to 0$ as $n \to +\infty$.

Finally, by (3.6), (3.7) and (3.8) we conclude that $(S_n, I_n, Q_n, R_n) \to (S_n^*, 0, 0, 0)$ as $n \to +\infty$ and we obtain our result. □

Theorem 3.3. *If there is a constant $\lambda > 0$ such that $R_p(\lambda) > 1$ then the infectives (I_n) are strongly persistent.*

Proof. Assuming that $\mathcal{R}_p > 1$, there is $\varepsilon_0 > 0$ such that
$$\limsup_{n\to+\infty} \prod_{k=n}^{n+\lambda} \frac{1+\beta_k(x_{k+1}^* - \varepsilon_0)}{1+\gamma_k+\sigma_k+d_k+(\alpha_1)_k} > 1. \tag{3.9}$$
Given $\varepsilon_1 > 0$, consider the auxiliary equation
$$x_{n+1} = \frac{\Lambda_n + x_n - \beta_n L\varepsilon_1}{1+d_n}. \tag{3.10}$$

Given $n_0 \in \mathbb{N}$ and $x_0, y_0 \in \mathbb{R}^+$, let (x_n) be the solution of (2.2) with initial condition $x_{n_0} = x_0$ and let $(\bar{x}_n)$ be the solution of (3.10) with $\bar{x}_{n_0} = x_0$. By v) in Lemma 2.1 we have

$$|x_n - \bar{x}_n| \leqslant DL\varepsilon_1 \sup_{n\in\mathbb{N}} \beta_n,$$

for all $n \geqslant n_0$. Thus there is $\varepsilon_1 > 0$ sufficiently small such that

$$|x_n - \bar{x}_n| < \frac{\varepsilon_0}{2} \tag{3.11}$$

for all $n \geqslant n_0$. On the other hand, by iv) in Lemma 2.1, there is $n_1 \in \mathbb{N}$ with $n_1 \geqslant n_0$ such that

$$|x_n - x_n^*| < \frac{\varepsilon_0}{2} \tag{3.12}$$

for all $n \geqslant n_1$.

Let now $((S_n, I_n, Q_n, R_n))$ be any solution of (1.1) with $I_n > 0$ for all $n \in \mathbb{N}$. We claim that

$$\limsup_{n\to+\infty} I_n > \varepsilon_1. \tag{3.13}$$

To prove that (3.13) holds, we will proceed by contradiction. Assuming that it is not true, there is $n_2 \in \mathbb{N}$ such that $I_n < \varepsilon_1$ for all $n \geqslant n_2$. By the first equation in (1.1), we conclude that

$$S_{n+1} = \frac{\Lambda_n + S_n - \beta_n I_n S_{n+1}}{1 + d_n} \geqslant \frac{\Lambda_n + S_n - \beta_n L\varepsilon_1}{1 + d_n}. \tag{3.14}$$

Let $(\bar{x}_n)$ be the solution of (3.10) with initial condition $\bar{x}_{n_2} = S_{n_2}$. Comparing (3.10) with (3.14) we conclude that

$$S_n \geqslant \bar{x}_n, \quad \text{for all} \quad n \geqslant n_2.$$

By (3.11) and (3.12), we get

$$S_n \geqslant \bar{x}_n > x_n - \frac{\varepsilon_0}{2} > x_n^* - \varepsilon_0,$$

for all n sufficiently large, say $n \geqslant n_3 \geqslant n_2$.

From the second equation in (1.1), we conclude that

$$\begin{aligned} I_{n+1} &= \frac{(1 + \beta_n S_{n+1}) I_n}{1 + \gamma_n + \sigma_n + d_n + (\alpha_1)_n} \\ &\geqslant \frac{1 + \beta_n (x_{n+1}^* - \varepsilon_0)}{1 + \gamma_n + \sigma_n + d_n + (\alpha_1)_n} I_n, \end{aligned} \tag{3.15}$$

for all $n \geqslant n_3$. Therefore, by (3.9) and (3.15), we conclude that $I_n \to +\infty$. A contradiction to iii) in Lemma 2.2. Thus the infectives (I_n) are weakly persistent, namely we have (3.13).

We will now use the weak persistence to prove the strong persistence. With no loss of generality we may assume that there is $\varepsilon_0 > 0$ such that

$$\prod_{k=n}^{n+\lambda} \frac{1+\beta_k(x^*_{k+1}-\varepsilon_0)}{1+\gamma_k+\sigma_k+d_k+(\alpha_1)_k} > 1. \tag{3.16}$$

for all $n \in \mathbb{N}$. For each $z_0 = (S_0, I_0, Q_0, R_0)$, denote by $((S_{n,z_0}, I_{n,z_0}, Q_{n,z_0}, R_{n,z_0}))$ the solution of (1.1) with $(S_{n,z_0}(0), I_{n,z_0}(0), Q_{n,z_0}(0), R_{n,z_0}(0)) = (S_0, I_0, Q_0, R_0)$.

Proceeding by contradiction, if the system is not strong persistent, then there is a sequence of initial values $z_{0,k} = (S_{0,k}, I_{0,k}, Q_{0,k}, R_{0,k})$, $k \in \mathbb{N}$, such that

$$\liminf_{n\to+\infty} I_{n,z_{0,k}} < \frac{\varepsilon_0}{k^2}. \tag{3.17}$$

From (3.13) and (3.16), for each $k \in \mathbb{N}$ there are sequences $(s_{m,k})$ and $(t_{m,k})$ such that

$$0 < s_{1,k} < t_{1,k} < s_{2,k} < t_{2,k} < \cdots < s_{m,k} < t_{m,k} < \cdots, \tag{3.18}$$

$$s_{m,k} \to +\infty \ \text{ as } \ m \to +\infty, \tag{3.19}$$

$$I_{s_{m,k},z_{0,k}} > \frac{\varepsilon_0}{k}, \ \ I_{t_{m,k},z_{0,k}} < \frac{\varepsilon_0}{k^2}, \tag{3.20}$$

and

$$\frac{\varepsilon_0}{k^2} \leqslant I_{n,z_{0,k}} \leqslant \frac{\varepsilon_0}{k}, \ \text{ for all } \ n \in [s_{m,k}, t_{m,k}-1] \cap \mathbb{N}. \tag{3.21}$$

For any $n \in [s_{m,k}, t_{m,k}-1] \cap \mathbb{N}$, we have

$$\begin{aligned} I_{n+1,z_{0,k}} &= \frac{1+\beta_n S_{n+1,z_{0,k}}}{1+\gamma_n+\sigma_n+d_n+(\alpha_1)_n} I_{n,z_{0,k}} \\ &\geqslant \frac{1}{1+\gamma_n+\sigma_n+d_n+(\alpha_1)_n} I_{n,z_{0,k}} \\ &\geqslant \frac{1}{1+\eta} I_{n,z_{0,k}}, \end{aligned}$$

where $\eta = \sup_{n\in\mathbb{N}}(\gamma_n+\sigma_n+d_n+(\alpha_1)_n) > 0$. Therefore, by (3.20), we obtain

$$\frac{\varepsilon_0}{k^2} > I_{t_{m,k},z_{0,k}} \geqslant \left(\frac{1}{1+\eta}\right)^{t_{m,k}-s_{m,k}} I_{s_{m,k},z_{0,k}} > \left(\frac{1}{1+\eta}\right)^{t_{m,k}-s_{m,k}} \frac{\varepsilon_0}{k},$$

and therefore we get

$$t_{m,k} - s_{m,k} > \frac{\ln k}{\ln(1+\eta)} \to +\infty \ \text{ as } \ k \to +\infty.$$

We conclude that we can choose $k_1 \in \mathbb{N}$ such that

$$t_{m,k} - s_{m,k} > n_1 + \lambda + 1,$$

for all $k \geqslant k_1$. For every $k \geqslant k_1$ and $n \in [s_{m,k}+1, t_{m,k}] \cap \mathbb{N}$, we have

$$S_{n+1} \geqslant \frac{\Lambda_n + S_n - \beta_n L \varepsilon_0}{1+d_n}.$$

Let $(\bar{x}_n)$ be a solution of (3.10) with initial condition $\bar{x}_{s_{m,k}+1} = S_{s_{m,k}+1}$. Comparing we get

$$S_n \geqslant \bar{x}_n \text{ for all } n \in [s_{m,k}+1, t_{m,k}] \cap \mathbb{N}. \tag{3.22}$$

Similarly to (3.11) and (3.12), we can conclude, by iv) and v) in Lemma 2.1, that

$$|x_n - \bar{x}_n| < \frac{\varepsilon_0}{2} \text{ and } |x_n - x_n^*| < \frac{\varepsilon_0}{2}, \tag{3.23}$$

for all $n \in [s_{m,k}+1, t_{m,k}] \cap \mathbb{N}$. By (3.22) and (3.23) we conclude that

$$S_n \geqslant \bar{x}_n > x_n - \frac{\varepsilon_0}{2} > x_n^* - \varepsilon_0. \tag{3.24}$$

for all $n \in [s_{m,k}+n_1+1, t_{m,k}] \cap \mathbb{N}$. By (3.24) we have

$$\begin{aligned} I_{n+1,z_{0,k}} &= \frac{1+\beta_n S_{n+1,z_{0,k}}}{1+\gamma_n+\sigma_n+d_n+(\alpha_1)_n} I_{n,z_{0,k}} \\ &\geqslant \frac{1+\beta_n(x_n^*-\varepsilon_0)}{1+\gamma_n+\sigma_n+d_n+(\alpha_1)_n} I_{n,z_{0,k}} \end{aligned} \tag{3.25}$$

for all $n \in [s_{m,k}+n_1+1, t_{m,k}] \cap \mathbb{N}$ and $k \geqslant n_4$. By (3.17) and (3.25) we get

$$\frac{\varepsilon_0}{k^2} > I_{t_{m,k},z_{0,k}} \geqslant I_{t_{m,k}-\lambda,z_{0,k}} \prod_{n=t_{m,k}-\lambda}^{t_{m,k}} \frac{1+\beta_n(x_n^*-\varepsilon_0)}{1+\gamma_n+\sigma_n+d_n+(\alpha_1)_n} I_{n,z_{0,k}} > \frac{\varepsilon_0}{k^2}$$

a contradiction. Thus we conclude that the infectives are strong persistent and the result follow. □

Next we consider some particular situations: the autonomous and periodic cases. Let

$$\mathcal{R}_A = \frac{\beta\Lambda}{d(\gamma+\sigma+d+\alpha_1)}.$$

Note that $\mathcal{R}_A$ matches the autonomous basic reproductive number for the continuous-time system (see for example [11]).

Corollary 3.4 (Autonomous system)**.** *Assume that* $\Lambda_n = \Lambda$, $\beta_n = \beta$, $d_n = d$, $\gamma_n = \gamma$, $\sigma_n = \sigma$, $\varepsilon_n = \varepsilon$, $(\alpha_1)_n = \alpha_1$ *and* $(\alpha_2)_n = \alpha_2$. *Then, if* $\mathcal{R}_A < 1$, *the infectives in system* (1.1) *go to extinction and, if* $\mathcal{R}_A > 1$, *the infectives in system* (1.1) *are strong persistent.*

Proof. Note that, in the autonomous situation, $x_n^* = \Lambda/d$ is a particular positive solution of (2.2). Thus

$$\mathcal{R}_e(\lambda) = \limsup_{n\to+\infty} \prod_{k=n}^{n+\lambda} \frac{1+\beta\Lambda/d}{1+\gamma+\sigma+d+(\alpha_1)} = \left(\frac{1+\beta\Lambda/d}{1+\gamma+\sigma+d+(\alpha_1)}\right)^{\lambda}.$$

Thus

$$\mathcal{R}_e(\lambda) < 1 \quad \Leftrightarrow \quad 1+\beta\Lambda/d < 1+\gamma+\sigma+d+(\alpha_1) \quad \Leftrightarrow \quad \mathcal{R}_A < 1$$

and similarly

$$\mathcal{R}_e(\lambda) > 1 \quad \Leftrightarrow \quad 1+\beta\Lambda/d > 1+\gamma+\sigma+d+(\alpha_1) \quad \Leftrightarrow \quad \mathcal{R}_A > 1.$$

The result follows. □

Define now

$$\mathcal{R}_P = \prod_{k=n}^{n+\omega} \frac{1+\beta_k x_{k+1}^*}{1+\gamma_k+\sigma_k+d_k+(\alpha_1)_k}, \tag{3.26}$$

where (x_n^*) is the positive ω-periodic solution of the auxiliary equation (2.2) (that exists as we will see).

Corollary 3.5 (Periodic system)**.** *Assume that there is $\omega \in \mathbb{N}$ such that $\Lambda_{n+\omega} = \Lambda_n$, $\beta_{n+\omega} = \beta_n$, $d_{n+\omega} = d_n$, $\gamma_{n+\omega} = \gamma_n$, $\sigma_{n+\omega} = \sigma_n$, $\varepsilon_{n+\omega} = \varepsilon_n$, $(\alpha_1)_{n+\omega} = (\alpha_1)_n$ and $(\alpha_2)_{n+\omega} = (\alpha_2)_n$, for all $n \in \mathbb{N}$. Then, if $\mathcal{R}_P < 1$, the infectives in system* (1.1) *go to extinction and, if $\mathcal{R}_P > 1$, the infectives in system* (1.1) *are strong persistent.*

Additionally system (1.1) *always has a disease-free periodic solution, $((S_n^*, 0, 0, 0))$, where (S_n^*) is the disease-free periodic solution of* (2.2)*, and this solution is globally asymptotically stable when $\mathcal{R}_P < 1$.*

Proof. It follows from Theorem 3.2 that, when $\mathcal{R}_P < 1$, the infectives in system (1.1) go to extinction and it follows from Theorem 3.3 that, if $\mathcal{R}_P > 1$, the infectives in system (1.1) are strong persistent.

Consider the continuous function $F : [\Lambda^\ell/d^u, \Lambda^u/d^\ell] \to [\Lambda^\ell/d^u, \Lambda^u/d^\ell]$ given by

$$F(x) = \left(\prod_{m=0}^{\omega-1} \frac{1}{1+d_m}\right) x + \sum_{m=0}^{\omega-1} \Lambda_m \left(\prod_{k=m}^{\omega-1} \frac{1}{1+d_k}\right).$$

By iii) in Lemma 2.1, F is well defined a function from the convex and compact set $[\Lambda^\ell/d^u, \Lambda^u/d^\ell]$ onto itself. Thus, by Brower's fixed point theorem, we conclude that F has a fixed point u_0.

It is immediate that the sequence (y_n) given by $y_0 = u_0$ and

$$y_{n+1} = \frac{\Lambda_n}{1+d_n} + \frac{1}{1+d_n} y_n, \quad n = 1, 2, \cdots$$

is a ω-periodic solution of (2.2). Writing $S_n^* = y_n$, we easily conclude that $((S_n^*, 0, 0, 0))$ is a disease-free ω-periodic solution of (1.1). Its global asymptotic stability when $\mathcal{R}_P < 1$ follows from Theorem 3.2. □

We note that the number in (3.26) is a sharp threshold between persistence and extinction of the disease in the periodic case.

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