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Factorization Method Applied to Second-Order (q, h)-Difference Operators

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Abstract

We present several classes of second-order (q, h)-difference operators which can be factorized using first-order operators acting in certain Hilbert spaces.

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1 Introduction

Various versions of the factorization method of second-order q- and (q, h)-difference operators and its applications are studied in [1–3] (see also the references therein).

Recall that, according to [5–7], ladders are (finite or infinite) sequences of real or complex vector spaces V_n, $n \in \mathbb{Z}$, and operators A_n^+, A_n^- acting between them as fol-

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lows:

$$\cdots \quad V_{n-1} \quad \underset{A_n^-}{\overset{A_n^+}{\rightleftarrows}} \quad V_n \quad \underset{A_{n+1}^-}{\overset{A_{n+1}^+}{\rightleftarrows}} \quad V_{n+1} \quad \cdots \tag{1.1}$$

In this paper, we shall assume that the commutator (or the deformed commutator) of the ladder operators is a scalar and we shall investigate the eigenproblem of the loop operators $A_n^+A_n^-$ and $A_{n+1}^-A_{n+1}^+$.

2 (q,h)-Operators in Hilbert Spaces

Throughout this paper, we assume that $0 < q < 1$, $h > 0$. Denote the q-number by

$$[j]_q = \frac{1-q^j}{1-q}$$

and the (q,h)-interval $[a,b]_{q,h}$ by

$$[a,b]_{q,h} := \{q^j a + [j]_q h,\ j \in \mathbb{N} \cup \{0\}\} \cup \{q^j b + [j]_q h,\ j \in \mathbb{N} \cup \{0\}\},$$

where $a,\ b \in \mathbb{R}$. The (q,h)-integral on the (q,h)-interval $[a,b]_{q,h}$ is given by [4, 8]

$$\begin{aligned}\int_a^b \psi(x) d_{q,h}x := &\sum_{j=0}^{\infty} \left((1-q)b - h\right) q^j \psi(q^j b + [j]_q h) \\ &- \sum_{j=0}^{\infty} \left((1-q)a - h\right) q^j \psi(q^j a + [j]_q h).\end{aligned}$$

In this paper, the vector spaces V_n consist of complex valued functions $\psi : [a,b]_{q,h} \to \mathbb{C}$ defined on the (q,h)-interval. We impose an additional structure of Hilbert spaces as follows. Introducing the scalar products by means of the (q,h)-integral

$$\langle\psi|\varphi\rangle_n := \int_a^b \overline{\psi(x)}\varphi(x)\varrho_n(x) d_{q,h}x, \tag{2.1}$$

where ϱ_n are weight functions, we define the Hilbert spaces $\mathcal{H}_n = L^2\left(\mathbb{R}, \varrho_n(x)d_{q,h}x\right)$ consisting of those functions $\psi \in V_n$ which are square integrable, i.e., $0 < \langle\psi|\psi\rangle_n < +\infty$. We also assume that $(1-q)a < h < (1-q)b$, $b > 0,\ b > a$ and $\varrho_n > 0$ to guarantee the positivity of the scalar product. See also [3].

We assume that the operator of multiplication by a function acts as $f_n : \mathcal{H}_n \to \mathcal{H}_{n-1}$. The (q,h)-shift operators $Q : \mathcal{H}_n \to \mathcal{H}_n$ and $Q^{-1} : \mathcal{H}_n \to \mathcal{H}_n$ [6] are given by

$$
\begin{aligned}
&Q\psi(x) := \psi(qx+h), \\
&Q^{-1}\psi(x) := \begin{cases} \psi(q^{-1}x - q^{-1}h) & \text{for} \quad x \neq a \text{ and } x \neq b, \\ 0 & \text{for} \quad x = a \text{ or } x = b. \end{cases}
\end{aligned} \tag{2.2}
$$

We use these operators to define the (q,h)-derivative $\partial_{q,h} : \mathcal{H}_n \to \mathcal{H}_{n-1}$ as follows:

$$
\partial_{q,h}\psi(x) := ((1-q)x - h)^{-1}(1-Q)\psi = \frac{\psi(x) - \psi(qx+h)}{(1-q)x - h},
$$

where 1 is the identity operator.

3 Weight Functions

We assume that the scalar products (2.1) are defined by using the weight functions $\varrho_n : [a,b]_{q,h} \to \mathbb{R}$ which are related by the recursion relations

$$
\varrho_{n-1}(x) = \eta_n(x)\varrho_n(x) \tag{3.1}
$$

and

$$
\varrho_{n-1}(x) = B_n(qx+h)\varrho_n(qx+h), \tag{3.2}
$$

where η_n, B_n are real-valued functions on $[a,b]_{q,h}$. The compatibility condition between (3.1) and (3.2) reads as

$$
B_n(qx+h)\varrho_n(qx+h) = \eta_n(x)\varrho_n(x). \tag{3.3}
$$

Additionally, we impose the boundary conditions

$$
B_n(a)\varrho_n(a) = B_n(b)\varrho_n(b) = 0.
$$

Introducing the functions

$$
A_n(x) := \frac{B_n(x) - \eta_n(x)}{(1-q)x - h},
$$

we can rewrite formula (3.3) in the form of the (q,h)-Pearson equation

$$
\partial_{q,h}\left(B_n\varrho_n\right) = A_n\varrho_n. \tag{3.4}
$$

Equation (3.4) corresponds to the Pearson equation in the limit $q \to 1$, $h \to 0$.

4 Adjoint Operators

Let us find adjoint operators to the operators defined in the previous section. By the definition, we have $f_n^* : \mathcal{H}_{n-1} \to \mathcal{H}_n$ and

$$\begin{aligned}
\langle f_n\psi_n|\varphi_{n-1}\rangle_{n-1} &= \int_a^b \overline{f_n(x)\psi_n(x)}\varphi_{n-1}(x)\varrho_{n-1}(x)d_{q,h}x \\
&= \int_a^b \overline{\psi_n(x)f_n(x)}\varphi_{n-1}(x)\eta_n(x)\varrho_n(x)d_{q,h}x = \langle\psi_n|\overline{f_n}\eta_n\varphi_{n-1}\rangle_n,
\end{aligned}$$

where we use equation (3.1). Thus, $f_n^* = \overline{f_n}\eta_n$.

To find the adjoint operator $Q^* : \mathcal{H}_n \to \mathcal{H}_n$ we compute

$$\begin{aligned}
\langle Q^*\psi_n|\varphi_n\rangle_n &= \langle\psi_n|Q\varphi_n\rangle_n = \int_a^b \overline{\psi_n(x)}\varphi_n(qx+h)\varrho_n(x)d_{q,h}x \\
&= \sum_{j=0}^{\infty}((1-q)b-h)\,q^j\overline{\psi_n(q^jb+[j]_qh)}\varphi_n(q^{j+1}b+[j+1]_qh)\varrho_n(q^jb+[j]_qh) \\
&\quad -\sum_{j=0}^{\infty}((1-q)a-h)\,q^j\overline{\psi_n(q^ja+[j]_qh)}\varphi_n(q^{j+1}a+[j+1]_qh)\varrho_n(q^ja+[j]_qh) \\
&= \sum_{m=1}^{\infty}((1-q)b-h)\,q^{m-1}\overline{\psi_n(q^{m-1}b+[m-1]_qh)}\varphi_n(q^mb+[m]_qh) \\
&\qquad \times\varrho_n(q^{m-1}b+[m-1]_qh) \\
&\quad -\sum_{m=1}^{\infty}((1-q)a-h)\,q^{m-1}\overline{\psi_n(q^{m-1}a+[m-1]_qh)}\varphi_n(q^ma+[m]_qh) \\
&\qquad \times\varrho_n(q^{m-1}a+[m-1]_qh).
\end{aligned}$$

In this sum, the expression for $m = 0$, i.e.,

$$\begin{aligned}
&((1-q)b-h)\,\overline{\psi_n(q^{-1}b-q^{-1}h)}\varphi_n(b)\varrho_n(q^{-1}b-q^{-1}h) \\
&-((1-q)a-h)\,a\overline{\psi_n(q^{-1}a-q^{-1}h)}\varphi_n(a)\varrho_n(q^{-1}a-q^{-1}h),
\end{aligned}$$

does not appear because of the second equation in (2.2). From equation (3.3), we obtain for $x \neq a$ and $x \neq b$ that

$$\begin{aligned}
\langle Q^*\psi_n|\varphi_n\rangle_n &= \int_a^b \overline{\psi_n(q^{-1}x-q^{-1}h)}\varphi_n(x)\varrho_n(x)\frac{B_n(x)}{\eta_n(q^{-1}x-q^{-1}h)}q^{-1}d_{q,h}x \\
&= \left\langle q^{-1}\frac{B_n}{Q^{-1}\eta_n}(Q^{-1}\psi_n)\middle|\varphi_n\right\rangle_n.
\end{aligned}$$

5 Factorization of Second-Order (q,h)-Difference Operators

In this section, we shall consider several cases of factorization of certain second-order (q,h)-difference operators.

5.1 Case 1

First we shall consider the ladder (1.1), where the spaces V_n are the Hilbert spaces $\mathcal{H}_n$ and the raising operators (operators of creation) A_n^+ are adjoint to the lowering operators (operators of annihilation) A_n^-.

Let us take the annihilation operators $A_n^- = \mathbf{A}_n : \mathcal{H}_n \to \mathcal{H}_{n-1}$ and the creation operators $A_n^+ = \mathbf{A}_n^* : \mathcal{H}_{n-1} \to \mathcal{H}_n$ in the following form:

$$\mathbf{A}_n = \partial_{q,h} + f_n, \tag{5.1}$$
$$\mathbf{A}_n^* = (\partial_{q,h} + f_n)^* = B_n\left(-\partial_{q,h}Q^{-1} + f_n\right) - A_n\left(1 + \left((1-q)x - h\right) f_n\right), \tag{5.2}$$

where we assume that f_n are real-valued functions.

The first-order (q,h)-difference operators given by (5.1) and (5.2) can be rewritten in the form

$$\mathbf{A}_n = -\frac{1}{(1-q)x-h}Q + f_n + \frac{1}{(1-q)x-h},$$
$$\mathbf{A}_n^* = -\frac{B_n}{(1-q)x-h}Q^{-1} + \left(f_n + \frac{1}{(1-q)x-h}\right)\eta_n.$$

We say that the second-order (q,h)-difference operators $\mathbf{H}_n$ admit a factorization if there exist sequences of the first-order (q,h)-difference operators (5.1), (5.2) and constants a_k such that

$$\mathbf{H}_n = \mathbf{A}_n^*\mathbf{A}_n + a_n = \mathbf{A}_{n+1}\mathbf{A}_{n+1}^* + a_{n+1} \quad \text{for} \quad n \in \mathbb{N} \cup \{0\}. \tag{5.3}$$

Necessary and sufficient conditions for the consistency of factorization formula (5.3) are given as follows (by collecting the coefficients of Q, Q^{-1} and the identity operator):

1.

$$\begin{aligned}&\left(f_n(x) + \frac{1}{(1-q)x-h}\right)\eta_n(x) \\ &= \left(f_{n+1}(qx+h) + \frac{q^{-1}}{(1-q)x-h}\right)\eta_{n+1}(qx+h),\end{aligned} \tag{5.4}$$

2.

$$\left(f_n(q^{-1}x - q^{-1}h) + \frac{q}{(1-q)x-h}\right) B_n(x) = \left(f_{n+1}(x) + \frac{1}{(1-q)x-h}\right) B_{n+1}(x), \tag{5.5}$$

3.

$$\frac{qB_n(x) - q^{-1}B_{n+1}(qx+h)}{((1-q)x-h)^2} + a_n - a_{n+1} = \left(f_{n+1}(x) + \frac{1}{(1-q)x-h}\right)^2 \eta_{n+1}(x) - \left(f_n(x) + \frac{1}{(1-q)x-h}\right)^2 \eta_n(x). \tag{5.6}$$

We note that the expressions in the parenthesis of (5.4) and (5.5) are the same after changing $x \to (x-h)/q$. We shall need this observation in the calculations below.

In general, we get an infinite system of (nonlinear) equations. It is difficult to find its general solution. However, after imposing certain assumptions below, we are able to find a solution in a closed form.

Assume that

$$B_{n+1}(x) = q^\gamma B_n(x). \tag{5.7}$$

Substituting (5.7) into (5.4) and (5.5), we find the following recursion formulas:

$$\eta_{n+1}(qx+h) = q^\gamma \eta_n(x), \tag{5.8}$$

$$f_{n+1}(qx+h) + \frac{q^{-1}}{(1-q)x-h} = \left(f_n(x) + \frac{1}{(1-q)x-h}\right) q^{-\gamma}. \tag{5.9}$$

Next, from (5.7), (5.8) and (5.9), we have for $n \in \mathbb{N} \cup \{0\}$:

$$B_n(x) = q^{\gamma n} B_0(x), \tag{5.10}$$

$$\eta_n(x) = q^{\gamma n} \eta_0(q^{-n}x - [n]_q q^{-n}h), \tag{5.11}$$

$$f_n(x) = q^{-\gamma n} f_0(q^{-n}x - [n]_q q^{-n}h) - \frac{1-q^{(1-\gamma)n}}{(1-q)x-h}.$$

Note that, in this case, the system of equations (5.6) can be rewritten in the following form (after changing $x \to q^n x + [n]_q h$):

$$q^{2n(\gamma-1)} \frac{qB_0(q^n x + [n]_q h) - q^{\gamma-1} B_0(q^{n+1}x + [n+1]_q h)}{((1-q)x-h)^2} + q^{\gamma n}(a_n - a_{n+1}) = q^{-\gamma}\left(f_0(q^{-1}x - q^{-1}h) + \frac{q}{(1-q)x-h}\right)^2 \eta_0(q^{-1}x - q^{-1}h) - \left(f_0(x) + \frac{1}{(1-q)x-h}\right)^2 \eta_0(x). \tag{5.12}$$

Since the right-hand side of equation (5.12) does not depend on the parameter n, we get the following condition on the function B_0 (by equating the left-hand side with n and $n = 0$):

$$\frac{qB_0(x) - q^{\gamma-1}B_0(qx+h)}{((1-q)x-h)^2} + a_0 - a_1 = q^{2n(\gamma-1)}\frac{qB_0(q^n x + [n]_q h) - q^{\gamma-1}B_0(q^{n+1}x + [n+1]_q h)}{((1-q)x-h)^2} + q^{\gamma n}\left(a_n - a_{n+1}\right). \tag{5.13}$$

We look for the solutions of (5.13) in the form of a series

$$B_0(x) = \sum b_k \left(x - \frac{h}{(1-q)}\right)^k. \tag{5.14}$$

We do not specify the indices in the sum. Note that

$$B_0(q^n x + [n]_q h) = \sum b_k (x - h/(1-q))^k q^{nk}.$$

If one substitutes the term $b_m(x - h/(1-q))^m$ into (5.13), we get

$$b_m(q^2 - q^{m+\gamma})(1 - q^{n(m-2+2\gamma)})(x - h/(1-q))^m + const.$$

Also if $m = 2$, then we get some constant expression in (5.13). So we conclude that in order to cancel the terms with x, we need to take either $m = 2$ or $m = 2 - \gamma$ or $m = 2 - 2\gamma$.

Substituting (5.14) into (5.13) and comparing the coefficients at the same power of x, we obtain the solution

$$B_0(x) = b_2\left(x - \frac{h}{1-q}\right)^2 + b_1\left(x - \frac{h}{1-q}\right)^{2-\gamma} + b_0\left(x - \frac{h}{1-q}\right)^{2-2\gamma} \tag{5.15}$$

and additionally the condition on the sequence $\{a_n\}$:

$$(q-1)^2(a_1 - a_0) - b_2 q(q^\gamma - 1)(q^{2n\gamma} - 1) + (q-1)^2 q^{n\gamma}(a_n - a_{n+1}) = 0.$$

At the end, we return again to condition (5.12) for $n = 0$ and calculate f_0 from it as follows:

$$f_0(x) = \sqrt{\frac{\frac{q^{\gamma+1}b_2}{(1-q)^2} + \frac{q^\gamma(a_0-a_1)}{1-q^\gamma} + b_3\left(x - \frac{h}{1-q}\right)^{-\gamma} + \frac{q^{1-\gamma}b_0}{(1-q)^2}\left(x - \frac{h}{1-q}\right)^{-2\gamma}}{\eta_0(x)} - \frac{1}{(1-q)x-h}}, \tag{5.16}$$

where η_0 is a freely chosen function. The detailed derivation of this formula is given in Appendix A.

Note that if $\gamma = 0$, then

$$B_0(x) = b_2\left(x - \frac{h}{1-q}\right)^2 + \tilde{b}_2\left(x - \frac{h}{1-q}\right)^2 \log\left(x - \frac{h}{1-q}\right),$$

$$f_0(x) = \sqrt{\frac{b_3 + \left(\frac{a_1-a_0}{\log q} + \frac{q\tilde{b}_2}{(1-q)^2}\right)\log\left(x - \frac{h}{1-q}\right)}{\eta_0(x)}} - \frac{1}{(1-q)x-h} \tag{5.17}$$

and $a_{n+1} - a_n = a_1 - a_0$, which leads to $a_n = n(a_1 - a_0) + a_0$.

If in addition to (5.7) $f_n(x) = ((q-1)x+h)^{-1}$ for any n, then from (5.6), we get the first-order difference equation for the function B_0:

$$a_n - a_{n+1} + \frac{q^{2+\gamma n}B_0(x) - q^{\gamma+\gamma n}B_0(qx+h)}{q((q-1)x+h)^2} = 0. \tag{5.18}$$

Searching for a solution in the form (5.14), we get

$$B_0(x) = b_1\left(x - \frac{h}{1-q}\right)^{2-\gamma} + b_2\left(x - \frac{h}{1-q}\right)^2$$

and

$$a_n - a_{n+1} = \frac{b_2 q^{1+\gamma n}(q^\gamma - 1)}{(q-1)^2},$$

or, after the telescopic summation,

$$a_n = -\frac{b_2 q(q^{\gamma n} - 1)}{(q-1)^2} + a_0.$$

If, in addition, $\gamma = 0$, then we can find a solution of (5.18) in a different form:

$$B_0(x) = b_2\left(x - \frac{h}{1-q}\right)^2 \log\left(x - \frac{h}{1-q}\right)$$

with $a_n - a_{n+1} = b_2 q \log q/(q-1)^2$, or $a_n = a_0 - b_2 nq \log q/(q-1)^2$.

5.2 Case 2

Changing the factorization condition to

$$\mathbf{H}_n = \mathbf{A}_n^*\mathbf{A}_n + a_n = q^d\mathbf{A}_{n+1}\mathbf{A}_{n+1}^* + a_{n+1} \quad \text{for} \quad n \in \mathbb{N} \cup \{0\}, \tag{5.19}$$

we get a similar system to (5.4)–(5.6), where in equations (5.4) and (5.5), we need the factor q^d in second lines (in the right-hand sides) and equation (5.6) reads

$$\frac{qB_n(x) - q^{d-1}B_{n+1}(qx+h)}{((1-q)x-h)^2} + a_n - a_{n+1}$$
$$= q^d\left(f_{n+1}(x) + \frac{1}{(1-q)x-h}\right)^2 \eta_{n+1}(x) - \left(f_n(x) + \frac{1}{(1-q)x-h}\right)^2 \eta_n(x).$$

Similarly as in the previous case, it is easy to show that the solution of the above condition (5.19), assuming (5.7), is given by the formulas (5.10), (5.11) and

$$
\begin{aligned}
&f_n(x) = q^{-(\gamma+d)n} f_0(q^{-n}x - [n]_q q^{-n}h) - \frac{1-q^{(1-\gamma-d)n}}{(1-q)x-h},\\
&B_0(x) = b_2\left(x-\frac{h}{1-q}\right)^2 + b_1\left(x-\frac{h}{1-q}\right)^{2-\gamma-d} + b_0\left(x-\frac{h}{1-q}\right)^{2-2(\gamma+d)},\\
&(q-1)^2(a_1-a_0) - b_2 q(q^{\gamma+d}-1)(q^{2n(\gamma+d)}-1) + (q-1)^2 q^{n(\gamma+2d)}(a_n - a_{n+1}) = 0.
\end{aligned}
$$

Moreover, if $\gamma \neq d$, then f_0 is the same as in (5.16) with the only change $\gamma \to \gamma + d$ and η_0 is arbitrary.

If $\gamma = -d$, then we have

$$B_0(x) = b_2\left(x-\frac{h}{1-q}\right)^2 + \tilde{b}_2\left(x-\frac{h}{1-q}\right)^2 \log\left(x-\frac{h}{1-q}\right)$$

and

$$a_{n+1} - a_n = q^{-dn}(a_1 - a_0),$$

or, after the telescopic summation,

$$a_n = a_0 - q^d \frac{1-q^{-dn}}{1-q^d}(a_1 - a_0).$$

The function f_0 is the same as in (5.17).

If, in addition to (5.7), $f_n(x) = ((q-1)x+h)^{-1}$ for any n, then

$$B_0(x) = b_1\left(x-\frac{h}{1-q}\right)^{2-d-\gamma} + b_2\left(x-\frac{h}{1-q}\right)^2$$

and

$$a_n = -\frac{b_2 q(q^{n\gamma}-1)(q^{d+\gamma}-1)}{(q-1)^2(q^\gamma-1)} + a_0.$$

If, in addition, $\gamma = -d$, then

$$B_0(x) = b_2\left(x-\frac{h}{1-q}\right)^2 \log\left(x-\frac{h}{1-q}\right)$$

with

$$a_n = a_0 + \frac{b_2 q^{1+d(1-n)}(1-q^{dn})\log q}{(q-1)^2(q^d-1)}.$$

5.3 Case 3

Let us consider a periodic case. We assume that $\mathcal{H}_n = \mathcal{H}_{n+2}$. Then the weight functions satisfy $\varrho_0(x) = \varrho_{2k}(x)$, $\varrho_1(x) = \varrho_{2k-1}(x)$, $k \in \mathbb{Z}$. Recursive relations (3.1) give

$$\left.\begin{array}{l} \varrho_{-1}(x) = \eta_0(x)\varrho_0(x) \\ \varrho_0(x) = \eta_1(x)\varrho_{-1}(x) \end{array}\right\} \Longrightarrow \eta_0(x)\eta_1(x) = 1. \tag{5.20}$$

We note that in the case when $\eta_0(x) = \eta_1(x) = 1$, the weight function is not changed, i.e., $\varrho(x) := \varrho_0(x) = \varrho_1(x)$. This sub-case was considered in [1] in the limit $h \to 0$. In addition, if $B_0(x) = B_1(x) = 1$, then the weight function is, in particular, a constant, i.e., $\varrho(x) = \mathrm{const}$.

In the general situation, from recursive relations (3.2), we obtain $B_0(x) = B_{2k}(x)$, $B_1(x) = B_{2k-1}(x)$. Similarly, we have $\eta_0(x) = \eta_{2k}(x)$, $\eta_1(x) = \eta_{2k-1}(x)$.

Let $\zeta_k(x) = f_k(x) + \dfrac{1}{(1-q)x - h}$, $k \in \mathbb{Z}$. From the factorization condition (5.19) and equations similar to (5.4) and (5.5) with the factor q^d in the right-hand sides, we obtain

1. $$\begin{cases} \zeta_{2k}(x)\eta_0(x) = q^d\zeta_{2k+1}(qx+h)\eta_1(qx+h), \\ \zeta_{2k+1}(x)\eta_1(x) = q^d\zeta_{2k+2}(qx+h)\eta_0(qx+h), \end{cases} \tag{5.21}$$

2. $$\begin{cases} \zeta_{2k}(x)B_0(qx+h) = q^d\zeta_{2k+1}(qx+h)B_1(qx+h), \\ \zeta_{2k+1}(x)B_1(qx+h) = q^d\zeta_{2k+2}(qx+h)B_0(qx+h), \end{cases} \tag{5.22}$$

3. $$\begin{cases} \dfrac{qB_0(x) - q^{d-1}B_1(qx+h)}{((1-q)x-h)^2} + a_{2k} - a_{2k+1} = q^d\zeta^2_{2k+1}(x)\eta_1(x) \\ \qquad\qquad\qquad\qquad\qquad\qquad -\zeta^2_{2k}(x)\eta_0(x), \\ \dfrac{qB_1(x) - q^{d-1}B_0(qx+h)}{((1-q)x-h)^2} + a_{2k+1} - a_{2k+2} = q^d\zeta^2_{2k+2}(x)\eta_0(x) \\ \qquad\qquad\qquad\qquad\qquad\qquad -\zeta^2_{2k+1}(x)\eta_1(x). \end{cases} \tag{5.23}$$

If $\zeta_i = 0$, then

$$\begin{aligned} B_0(x) &= q^{d-2}B_1(qx+h) + q^{-1}(1-q)^2(a_{2k+1} - a_{2k})\left(x - \frac{h}{1-q}\right)^2, \\ B_1(x) &- q^{2d-4}B_1(q^2x + [2]_q h) \\ &= (1-q)^2\left(q^{d-1}(a_{2k+1} - a_{2k}) + q^{-1}(a_{2k+2} - a_{2k+1})\right)\left(x - \frac{h}{1-q}\right)^2. \end{aligned}$$

Then the solution for $d \neq 0$ is

$$\begin{aligned} B_1(x) &= b_2\left(x-\frac{h}{1-q}\right)^{2-d} \\ &+ \frac{(1-q)^2}{1-q^{2d}}\left(q^{d-1}(a_{2k+1}-a_{2k})+q^{-1}(a_{2k+2}-a_{2k+1})\right)\left(x-\frac{h}{1-q}\right)^2 \end{aligned}$$

and the solution for $d = 0$ is

$$\begin{aligned} B_1(x) = b_2\left(x-\frac{h}{1-q}\right)^2 - \frac{(1-q)^2}{2\log q}\left(q^{d-1}(a_{2k+1}-a_{2k})+q^{-1}(a_{2k+2}-a_{2k+1})\right) \\ \times\left(x-\frac{h}{1-q}\right)^2\log\left(x-\frac{h}{1-q}\right). \end{aligned}$$

From now on, we impose the following condition for the second-order (q,h)-difference operators (5.19): $\mathbf{H}_k = \mathbf{H}_{k+2} + \alpha$. More precisely, we require the periodic conditions $\zeta_0(x) = \zeta_{2k}(x)$, $\zeta_1(x) = \zeta_{2k+1}(x)$ and $a_{k+2} = a_k - \alpha$, where α is some constant. Then (5.21)–(5.23) can be rewritten as

1.
$$\begin{cases} \zeta_0(x)\eta_0(x) = q^d\zeta_1(qx+h)\eta_1(qx+h), \\ \zeta_1(x)\eta_1(x) = q^d\zeta_0(qx+h)\eta_0(qx+h), \end{cases} \tag{5.24}$$

2.
$$\begin{cases} \zeta_0(x)B_0(qx+h) = q^d\zeta_1(qx+h)B_1(qx+h), \\ \zeta_1(x)B_1(qx+h) = q^d\zeta_0(qx+h)B_0(qx+h), \end{cases} \tag{5.25}$$

3.
$$\begin{cases} \dfrac{qB_0(x)-q^{d-1}B_1(qx+h)}{((1-q)x-h)^2} + a_0 - a_1 = q^d\zeta_1^2(x)\eta_1(x) - \zeta_0^2(x)\eta_0(x), \\ \dfrac{qB_1(x)-q^{d-1}B_0(qx+h)}{((1-q)x-h)^2} + a_1 - a_0 + \alpha = q^d\zeta_0^2(x)\eta_0(x) - \zeta_1^2(x)\eta_1(x). \end{cases} \tag{5.26}$$

In the next step from (5.20) and (5.24) using (5.34), we have

$$\zeta_0(x) = \beta\left(x-\frac{h}{1-q}\right)^{-d}\frac{1}{\eta_0(x)}, \tag{5.27}$$

$$\zeta_1(x) = \beta\left(x-\frac{h}{1-q}\right)^{-d}\eta_0(x), \tag{5.28}$$

where β is a constant. Substituting (5.27) and (5.28) into equations (5.25), they reduce to

$$B_0(qx+h) = \eta_0(x)\eta_0(qx+h)B_1(qx+h). \tag{5.29}$$

Finally, applying (5.27)–(5.29) to equations (5.26), we get

$$\begin{cases} \dfrac{q\eta_0(q^{-1}x - q^{-1}h)\eta_0(x)B_1(x) - q^{d-1}B_1(qx+h)}{((1-q)x-h)^2} \\ = \beta^2\left(x - \dfrac{h}{1-q}\right)^{-2d}\left(q^d\eta_0(x) - \dfrac{1}{\eta_0(x)}\right) + a_1 - a_0, \\ \dfrac{qB_1(x) - q^{d-1}\eta_0(x)\eta_0(qx+h)B_1(qx+h)}{((1-q)x-h)^2} \\ = \beta^2\left(x - \dfrac{h}{1-q}\right)^{-2d}\left(q^d\dfrac{1}{\eta_0(x)} - \eta_0(x)\right) + a_0 - a_1 - \alpha. \end{cases} \tag{5.30}$$

5.3.1 Special case $\eta_0(x) = 1$

If we put $\eta_0(x) = 1$, then equations (5.30) simplify to

$$\frac{qB_1(x) - q^{d-1}B_1(qx+h)}{((1-q)x-h)^2} = \beta^2\left(q^d - 1\right)\left(x - \frac{h}{1-q}\right)^{-2d} + a_1 - a_0, \tag{5.31}$$

$$\alpha = 2(a_0 - a_1).$$

Equation (5.31) is of the type (5.35), and its solutions are expressed using (5.36) in the form

$$\begin{aligned} B_1(x) =& b_2\left(x - \frac{h}{1-q}\right)^{2-d} + (1-q)^2\beta^2\frac{q^d-1}{(1-q^{-d})q}\left(x - \frac{h}{1-q}\right)^{2-2d} \\ &+ \frac{(1-q)^2}{(1-q^d)q}(a_1 - a_0)\left(x - \frac{h}{1-q}\right)^2 \end{aligned}$$

for $d \neq 0$. The above sub-case in the limit $h \to 0$ is related to the case considered in [1].

Moreover, for $d = 0$, it is easy to see that we have

$$B_1(x) = b_2\left(x - \frac{h}{1-q}\right)^2 + \frac{(1-q)^2}{q\log q}(a_0 - a_1)\left(x - \frac{h}{1-q}\right)^2\log\left(x - \frac{h}{1-q}\right).$$

5.3.2 Special case $d = 0$

If we put $d = 0$, then equations (5.30) simplify to

$$\begin{aligned} &q\eta_0(q^{-1}x - q^{-1}h)\eta_0(x)B_1(x) - q^{-1}B_1(qx+h) \\ &= (1-q)^2\left(x - \frac{h}{1-q}\right)^2(a_1 - a_0) + \beta^2\left(x - \frac{h}{1-q}\right)^2\left(\eta_0(x) - \frac{1}{\eta_0(x)}\right), \\ &qB_1(x) - q^{-1}\eta_0(x)\eta_0(qx+h)B_1(qx+h) \\ &= (1-q)^2\left(x - \frac{h}{1-q}\right)^2(a_0 - a_1 - \alpha) + \beta^2\left(x - \frac{h}{1-q}\right)^2\left(\frac{1}{\eta_0(x) - \eta_0(x)}\right). \end{aligned}$$

Moreover, if $\alpha = 0$ and $\eta_0(qx+h) = -1/\eta_0(x)$, then we get

$$B_1(x) = q^{-1}(1-q)^2\left(-\beta^2\eta_0(x) + \frac{a_0 - a_1}{2}\right)\left(x - \frac{h}{1-q}\right)^2.$$

Appendix. First-Order Nonhomogeneous (q,h)-Difference Equations

Let us consider an equation in the following form (first-order homogeneous (q,h)-difference equation):

$$y(x) - q^d y(qx+h) = 0. \tag{5.32}$$

We will look for a solution in the form

$$y(x) = \left(x - \frac{h}{1-q}\right)^r f\left(x - \frac{h}{1-q}\right), \tag{5.33}$$

where $r \in \mathbb{R}$ and f is a nonzero analytic function. Substituting (5.33) into (5.32), we find

$$r = -d, \quad f(z) = f(qz),$$

where $z = x - h/(q-1)$, and hence the solution is constant. Finally, the solution of (5.32) has the form

$$y(x) = \left(x - \frac{h}{1-q}\right)^{-d} f(0). \tag{5.34}$$

If we consider a first-order nonhomogeneous (q,h)-difference equation

$$y(x) - q^d y(qx+h) = g(x), \tag{5.35}$$

then its solution has the form

$$y(x) = \left(x - \frac{h}{1-q}\right)^{-d} f(0) + \sum_{k=0}^{\infty} q^{kd} g(q^k x + [k]_q h). \tag{5.36}$$

In the case when the function g is the form $g(x) = g_0\left(x - \frac{h}{1-q}\right)^{-d}$ (or if it contains such a term), then the solution of (5.35) has the form

$$y(x) = \left(x - \frac{h}{1-q}\right)^{-d} f(0) - \frac{g_0}{\log q}\left(x - \frac{h}{1-q}\right)^{-d}\log\left(x - \frac{h}{1-q}\right).$$

Now, we come back to equation (5.12) for $n = 0$. After substitution (5.15) (and changing $x \to qx + h$), we obtain

$$\left(f_0(x) + \frac{1}{(1-q)x-h}\right)^2 \eta_0(x) - q^\gamma \left(f_0(qx+h) + \frac{q^{-1}}{(1-q)x-h}\right)^2 \eta_0(qx+h)$$
$$= q^\gamma (a_0 - a_1) + \frac{q^{\gamma+1} b_2 (1-q^\gamma)}{(1-q)^2} + \frac{q^{1-\gamma} b_0 (1-q^{-\gamma})}{(1-q)^2} \left(x - \frac{h}{1-q}\right)^{-2\gamma}.$$

We can solve the equation above for $\left(f_0(x) + \frac{1}{(1-q)x-h}\right)^2 \eta_0(x)$ using (5.36) and obtain

$$\begin{aligned}\left(f_0(x) + \frac{1}{(1-q)x-h}\right)^2 \eta_0(x) &= \left(x - \frac{h}{1-q}\right)^{-\gamma} b_3 + \sum_{k=0}^{\infty} q^{k\gamma} \Bigg(q^\gamma (a_0 - a_1) \\ &+ \frac{q^{\gamma+1} b_2 (1-q^\gamma)}{(1-q)^2} + q^{-2\gamma k} \frac{q^{1-\gamma} b_0 (1-q^{-\gamma})}{(1-q)^2} \left(x - \frac{h}{1-q}\right)^{-2\gamma}\Bigg) \\ &= \left(x - \frac{h}{1-q}\right)^{-\gamma} b_3 + \frac{q^\gamma (a_0 - a_1)}{1-q^\gamma} + \frac{q^{\gamma+1} b_2}{(1-q)^2} + \frac{q^{1-\gamma} b_0}{(1-q)^2} \left(x - \frac{h}{1-q}\right)^{-2\gamma}.\end{aligned} \tag{5.37}$$

Finally we can use equation (5.37) to express f_0 as (5.16).

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