International Journal of Difference Equations
ISSN 0973-6069, Volume 10, Number 2, pp. 221–232 (2015)
http://campus.mst.edu/ijde

Existence of some Positive Solutions to Fractional Difference Equations

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Abstract

The main objective of this paper is to study the existence of solutions to some basic fractional difference equations. The tool employed is Krasnosel'skii fixed point theorem which guarantees at least two positive solutions.

AMS Subject Classifications: 39A10, 26A33.
Keywords: Fractional difference equation, existence, positive solution.

1 Introduction

The theory of fractional calculus and associated fractional differential equations in continuous case has received great attention. However, very limited progress has been done

Received March 12, 2015; Accepted July 26, 2015
Communicated by Martin Bohner

in the development of the theory of finite fractional difference equations. But, recently a remarkable research work has been made in the theory of fractional difference equations. Diaz and Osler [10] introduced a discrete fractional difference operator defined as an infinite series.

Recently, a variety of results on discrete fractional calculus have been published by Atici and Eloe [4,5,7] with delta operator. Atici and Sengul [6] provided some initial attempts by using the discrete fractional difference equations to model tumor growth. M. Holm [11] extended his contribution to discrete fractional calculus by presenting a brief theory for composition of fractional sum and difference. Furthermore, Goodrich [1–3] developed some results on discrete fractional calculus in which he used Krasnosel'skii fixed point theorem to prove the existence of initial and boundary value problems. Following this trend, H. Chen, et. al. [13] and S. Kang, et. al. [14] discussed about the positive solutions of BVPs of fractional difference equations depending on parameters. H. Chen, et. al. [8], in their article provided multiple solutions to fractional difference boundary value problems using various fixed point theorems.

In this paper, we consider the boundary value problems of fractional difference equation of the form

$$-\Delta^v y(t) = \lambda h(t+v-1) f(t+v-1, y(t+v-1)), \tag{1.1}$$

$$y(v-2) = y(v+b) = 0, \tag{1.2}$$

where $t \in [0,b]_{\mathbb{N}_0}$, $f : [v-1, v+b]_{\mathbb{N}_{v-1}} \times \mathbb{R} \to \mathbb{R}$ is continuous, $h : [v-1, v+b]_{\mathbb{N}_{v-1}} \longrightarrow [0,\infty)$, $1 < v \leq 2$ and λ is a positive parameter.

The present paper is organized as follows. In Section 2, together with some basic definitions, we will demonstrate some important lemmas and theorem in order to prove our main result. In Section 3, we establish the results for existence of solutions to the boundary value problem (1.1)–(1.2) using Krasnosel'skii fixed point theorem.

2 Preliminaries

In this section, let us first collect some basic definitions and lemmas that are very much important to us in the sequel.

Definition 2.1 (See [3, 7]). We define

$$t^{\underline{v}} = \frac{\Gamma(t+1)}{\Gamma(t+1-v)}, \tag{2.1}$$

for any t and v for which right hand side is defined. We also appeal to the convention that if $t+v-1$ is a pole of the Gamma function and $t+1$ is not a pole, then $t^{\underline{v}} = 0$.

Definition 2.2 (See [3, 7]). The v^{th} fractional sum of a function f, for $v > 0$ is defined as,

$$\Delta^{-v} f(t) = \Delta^{-v} f(t,a) := \frac{1}{\Gamma(v)} \sum_{s=a}^{t-v} (t-s-1)^{\underline{v-1}} f(s), \tag{2.2}$$

for $t \in \{a+v, a+v+1, \dots\} =: \mathbb{N}_{a+v}$. We also define the v^{th} fractional difference for $t^{\underline{v}} = 0$ by $\Delta^{v}_{a+v} f(t) := \Delta^{N} \Delta^{v-N} f(t)$, where $t \in \mathbb{N}_{a+v}$ and $N \in \mathbb{N}$ is chosen so that $0 \le N-1 < v \le N$.

Now we give some important lemmas.

Lemma 2.3 (See [3, 7]). *Let t and v be any numbers for which $t^{\underline{v}}$ and $t^{\underline{v-1}}$ are defined. Then $\Delta t^{\underline{v}} = v t^{\underline{v-1}}$.*

Lemma 2.4 (See [3, 7]). *Let $0 \le N-1 < v \le N$. Then*

$$\Delta^{-v} \Delta^{v} y(t) = y(t) + C_1 t^{\underline{v-1}} + C_2 t^{\underline{v-2}} + \cdots + C_N t^{\underline{v-N}}, \tag{2.3}$$

for some $C_i \in \mathbb{R}$ with $1 \le i \le N$.

Lemma 2.5 (See [7]). *Let $1 < v \le 2$ and $f : [v-1, v+b]_{\mathbb{N}_{v-1}} \times \mathbb{R} \to \mathbb{R}$ be given. Then the solution of fractional boundary value problem $-\Delta^{v} y(t) = f(t+v-1, y(t+v-1))$, $y(v-2) = y(v+b+1) = 0$ is given by*

$$y(t) = \sum_{s=0}^{b+1} G(t,s) f(s+v-1, y(s+v-1)), \tag{2.4}$$

where Green's function $G : [v-1, v+b]_{\mathbb{N}_{v-1}} \times [0, b+1]_{\mathbb{N}_0} \to \mathbb{R}$ is defined by

$$G(t,s) = \frac{1}{\Gamma v} \begin{cases} \dfrac{t^{\underline{v-1}} (v+b-s)^{\underline{v-1}}}{(v+b-1)^{\underline{v-1}}} - (t-s-1)^{\underline{v-1}}, & 0 \le s < t-v+1 \le b+1 \\ \dfrac{t^{\underline{v-1}} (v+b-s)^{\underline{v-1}}}{(v+b-1)^{\underline{v-1}}}, & 0 \le t-v+1 < s \le b+1, \end{cases} \tag{2.5}$$

Lemma 2.6 (See [7]). *The Green's function $G(t,s)$ given Lemma 2.5 satisfies*

1. *$G(t,s) \ge 0$ for each $(t,s) \in [v-2, v+b]_{\mathbb{N}_{v-2}} \times [0,\, b+1]_{\mathbb{N}_0}$,*
2. *$\max\limits_{t \in [v-2, v+b]_{\mathbb{N}_{v-2}}} G(t,s) = G(s+v-1, s)$ for each $s \in [0,b]_{\mathbb{N}_0}$ and*
3. *There exists a number $\gamma \in (0,1)$ such that*

$$\min_{t \in \left[\frac{v+b}{4}, \frac{3(v+b)}{4}\right]_{\mathbb{N}_{v-2}}} G(t,s) \ge \max_{t \in [v-2, v+b]_{\mathbb{N}_{v-2}}} G(t,s) = \gamma G(s+v-1, s)$$

for $s \in [0,b]_{\mathbb{N}_0}$.

Now we give the solution of fractional boundary value problem (1.1)–(1.2), if it exists.

Theorem 2.7. *Let* $f : [v-1, v+b]_{\mathbb{N}_{v-1}} \times \mathbb{R} \to \mathbb{R}$ *be given. A function* y *is a solution to the discrete fractional boundary value problem* (1.1)–(1.2) *iff it is a fixed point of the operator*

$$Fy(t) = \lambda \sum_{s=0}^{b} G(t,s)\, h(s+v-1)\, f(s+v-1, y(s+v-1)), \tag{2.6}$$

where $G(t,s)$ *is given in Lemma 2.5.*

Proof. From Lemma 2.4, we find that a general solution to problem (1.1)–(1.2) is

$$y(t) = -\Delta^{-v}\lambda h(t+v-1) f(t+v-1, y(t+v-1)) + C_1 t^{\underline{v-1}} + C_2 t^{\underline{v-2}},$$

from the boundary condition $y(v-2) = 0$,

$$\begin{aligned} y(v-2) =&\ -\Delta^{-v}\lambda h(t+v-1) f(t+v-1, y(t+v-1))\big|_{t=v-2} \\ &+ C_1 (v-2)^{\underline{v-1}} + C_2 (v-2)^{\underline{v-2}} \\ =&\ -\frac{1}{\Gamma v}\sum_{s=0}^{t-v}(t-s-1)^{\underline{v-1}}\lambda h(s+v-1) f(t+v-1, y(t+v-1))\Bigg|_{t=v-2} \\ &+ C_2\Gamma(v-1) \\ =&\ C_2\Gamma(v-1) \\ =&\ 0, \end{aligned}$$

therefore, $C_2 = 0$.

On the other hand, using boundary condition $y(v+b) = 0$

$$\begin{aligned} y(v+b) =&\ -\Delta^{-v}\lambda h(t+v-1) f(t+v-1, y(t+v-1))\big|_{t=v+b} \\ &+ C_1 (v+b)^{\underline{v-1}} + C_2 (v+b)^{\underline{v-2}} \\ =&\ -\frac{1}{\Gamma v}\sum_{s=0}^{t-v}(t-s-1)^{\underline{v-1}}\lambda h(s+v-1) f(t+v-1, y(t+v-1))\Bigg|_{t=v+b} \\ &+ C_1 (v+b)^{\underline{v-1}} \\ =&\ 0, \end{aligned}$$

$$C_1 (v+b)^{\underline{v-1}} = \frac{1}{\Gamma v}\sum_{s=0}^{t-v}(t-s-1)^{\underline{v-1}}\lambda h(s+v-1)\, f(s+v-1, y(s+v-1))\big|_{t=v+b}$$

$$C_1 = \frac{1}{\Gamma v (v+b)^{\underline{v-1}}}\sum_{s=0}^{b}(v+b-s-1)^{\underline{v-1}}\lambda h(s+v-1) f(s+v-1, y(s+v-1)).$$

Using C_1 and C_2 in $y(t)$, we get

$$y(t) = -\frac{1}{\Gamma v}\sum_{s=0}^{t-v}(t-s-1)^{\underline{v-1}}\lambda h(s+v-1)f(s+v-1, y(s+v-1))$$

$$+\frac{t^{\underline{v-1}}}{\Gamma v(v+b)^{\underline{v-1}}}\sum_{s=0}^{b}(v+b-s-1)^{\underline{v-1}}\lambda h(s+v-1)f(s+v-1, y(s+v-1))$$

$$\begin{aligned} y(t) = &\sum_{s=0}^{t-v}\left\{\frac{t^{\underline{v-1}}(v+b-s-1)^{\underline{v-1}}}{\Gamma v(v+b)^{\underline{v-1}}} - \frac{(t-s-1)^{\underline{v-1}}}{\Gamma v}\right\}\lambda h(s+v-1) \\ &\cdot f(s+v-1, y(s+v-1)) \\ &+\sum_{s=t-v+1}^{b}\frac{t^{\underline{v-1}}(v+b-s-1)^{\underline{v-1}}}{\Gamma v(v+b)^{\underline{v-1}}}\lambda h(s+v-1)f(s+v-1, y(s+v-1)) \end{aligned}$$

$$y(t) = \sum_{s=0}^{b} G(t,s)\lambda h(s+v-1)f(s+v-1, y(s+v-1)). \tag{2.7}$$

Consequently, we observe that $y(t)$ implies that whenever y is a solution of (1.1)–(1.2), y is a fixed point of (2.6), as desired. □

Theorem 2.8 (See [12])**.** *Let E be a Banach space, and let $\mathcal{K} \subset E$ be a cone in E. Assume that Ω_1 and Ω_2 are open sets contained in E s. t. $0 \in \Omega_1$ and $\overline{\Omega}_1 \subseteq \Omega_2$, and let $S : \mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \mathcal{K}$ be a completely continuous operator such that either*

1. *$\|Sy\| \le \|y\|$ for $y \in \mathcal{K} \cap \partial\Omega_1$ and $\|Sy\| \ge \|y\|$ for $y \in \mathcal{K} \cap \partial\Omega_2$; Or*
2. *$\|Sy\| \ge \|y\|$ for $y \in \mathcal{K} \cap \partial\Omega_1$ and $\|Sy\| \le \|y\|$ for $y \in \mathcal{K} \cap \partial\Omega_2$*

Then S has at least one fixed point in $\mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3 Main Result

To prove our main result, let us state all required theorems for the existence of positive solutions to problem (1.1)–(1.2).

For this, let

$$\eta := \frac{1}{\sum\limits_{s=0}^{b} G(s+v-1, s)h(s+v-1)},$$

$$\sigma := \frac{1}{\gamma\sum_{s=\left[\frac{b+v}{4}-v+1\right]}^{\left[\frac{3(b+v)}{4}-v+1\right]} G\left(\left[\frac{b-v}{2}\right]+v, s\right)},$$

where η and σ are well defined by Lemma 2.6 and γ is the number given by Lemma 2.6.3.

In the sequel, we present some conditions on f that will imply the existence of positive solutions.

H1: $\exists$ a number $r > 0$ such that $f(t, y) \leq \dfrac{\eta r}{\lambda}$ whenever $0 \leq y \leq r$.

H2: $\exists$ a number $r > 0$ such that $f(t, y) \geq \dfrac{\sigma r}{\lambda}$ whenever $\gamma r \leq y \leq r$.

H3: $\displaystyle\lim_{y\to 0^+} \min_{t\in[v-2,v+b]_{\mathbb{N}_{v-2}}} \frac{f(t,y)}{y} = +\infty.$

H4: $\displaystyle\lim_{y\to \infty^+} \min_{t\in[v-2,v+b]_{\mathbb{N}_{v-2}}} \frac{f(t,y)}{y} = +\infty.$

For our purpose, let E be a Banach space defined by

$$E = \Big\{ y : [v-2, v+b]_{\mathbb{N}_{v-2}} \to \mathbb{R},\ \ y(v-2) = y(v+b) = 0 \Big\}, \tag{3.1}$$

with norm $\|y\| = \max |y(t)|$, $t \in [v-2, v+b]_{\mathbb{N}_0}$.

Also, define the cones

$$\mathcal{K}_0 = \left\{ y \in E : 0 \leq y(t),\ \min_{t\in\left[\frac{v+b}{4}, \frac{3(v+b)}{4}\right]} y(t) \geq \gamma \|y(t)\| \right\}. \tag{3.2}$$

In order to prove our first existence result, let us prove the following important lemma.

Lemma 3.1. $F(\mathcal{K}_0) \subseteq \mathcal{K}_0$ *i.e.,* F *leaves the cone* $\mathcal{K}_0$ *invariant.*

Proof. Observe that

$$\begin{aligned}
\min_{t\in\left[\frac{v+b}{4}, \frac{3(v+b)}{4}\right]} (Fy)(t) &= \min_{t\in\left[\frac{v+b}{4}, \frac{3(v+b)}{4}\right]} \sum_{s=0}^{b} G(t,s)\lambda h(s+v-1) \\
&\quad \cdot f(s+v-1, y(s+v-1)) \\
&\geq \gamma \sum_{s=0}^{b} G(t,s)\lambda h(s+v-1) f(s+v-1, y(s+v-1)) \\
&\geq \gamma \max_{t\in[v-2,v+b]_{N_{v-2}}} \sum_{s=0}^{b} G(t,s)\lambda h(s+v-1) \\
&\quad \cdot f(s+v-1, y(s+v-1)) \\
&= \gamma \|Fy\|,
\end{aligned}$$

which implies $Fy \in \mathcal{K}_0$. $\square$

Theorem 3.2. *Assume that $\exists$ distinct numbers $r_1 > 0$ and $r_2 > 0$ with $r_1 < r_2$ such that f satisfies the condition $H1$ at r_1 and $H2$ at r_2. Then the fractional boundary value problem* (1.1)–(1.2) *has at least one positive solution say y_0 satisfying $r_1 \leq \|y_0\| \leq r_2$.*

Proof. As F is completely continuous operator and $F : \mathcal{K}_0 \to \mathcal{K}_0$, let

$$\Omega_1 = \{y \in \mathcal{K}_0 : \|y\| \geq r_1\}.$$

Then for any $y \in \mathcal{K}_0 \cap \partial\Omega_1$, we have from H1

$$\begin{aligned}
\|Fy\| &= \max_{t\in[v-2,v+b]_{N_{v-2}}} \lambda \sum_{s=0}^{b} G(t,s)\, h(s+v-1)\, f(s+v-1, y(s+v-1)), \\
&\leq \lambda \sum_{s=0}^{b} G(s+v-1,s)\, h(s+v-1)\, f(s+v-1, y(s+v-1)) \\
&\leq \lambda \frac{\eta r_1}{\lambda} \sum_{s=0}^{b} G(s+v-1,s)h(s+v-1) \\
&= r_1 \qquad (3.3)\\
&= \|y\|.
\end{aligned}$$

Hence, $\|Fy\| = \|y\|$ for $y \in \mathcal{K}_0 \cap \partial\Omega_1$.

Now, let $\Omega_2 = \{y \in \mathcal{K}_0 : \|y\| \leq r_2\}$. Then for any $y \in \mathcal{K}_0 \cap \partial\Omega_2$ we have from H2

$$\begin{aligned}
Fy(t) &= \lambda \sum_{s=0}^{b} G(t,s)\, h(s+v-1)\, f(s+v-1, y(s+v-1)), \\
&\geq \lambda \sum_{s=\left[\frac{b+v}{4}-v+1\right]}^{\left[\frac{3(b+v)}{4}-v+1\right]} G(t,s)\, h(s+v-1)\, f(s+v-1, y(s+v-1)) \\
&\geq \lambda \sum G(t,s)h(s+v-1)\frac{\sigma r_2}{\lambda} \\
&= r_2 \qquad (3.4)\\
&= \|y\|.
\end{aligned}$$

Hence, $\|Fy\| \geq \|y\|$ for $y \in \mathcal{K}_0 \cap \partial\Omega_2$.

So, it follows from Theorem 2.8 that there exists $y_0 \in \mathcal{K}_0$ such that $Fy_0 = y_0$ i. e., fractional boundary value problem (1.1)–(1.2) has a positive solution, say y_0 satisfying $r_1 \leq \|y_0\| \leq r_2$. □

In the next theorem, we give the existence of at least two positive solutions.

Theorem 3.3. *Assume that f satisfies condition $H1$ and $H3$. Then the fractional boundary value problem* (1.1)–(1.2) *has at least two positive solutions, say y_1 and y_2 such that $0 \leq \|y_1\| < m < \|y_2\|$.*

Proof. From the assumptions, $\exists\, \varepsilon > 0$ and $r > 0$ with $r < m$ s. t. for $0 \leq y \leq r$, $f(t,y) \geq \dfrac{(\sigma+\varepsilon)}{\lambda} y, \qquad t \in [v-2, v+b]_{\mathbb{N}_{v-2}}$.

Let $r_1 \in (0,r)$ and $\left[\dfrac{b-v}{2}\right] + v \in \left[\dfrac{b+v}{4}, \dfrac{3(b+v)}{4}\right]$.

Hence, for $y \in \partial\Omega_r$, we have

$$\begin{aligned}
(Fy)\left(\left[\frac{b-v}{2}\right]+v\right) &= \sum_{s=0}^{b} G\left(\left[\frac{b-v}{2}\right]+v, s\right)\lambda h(s+v-1) \\
&\cdot f(s+v-1, y(s+v-1)) \\
&\geq \lambda \sum_{s=0}^{b} G\left(\left[\frac{b-v}{2}\right]+v, s\right) h(s+v-1)\frac{(\sigma+\varepsilon)}{\lambda} y \\
&\geq \lambda \frac{(\sigma+\varepsilon)}{\lambda}\|y\| \sum_{s=\left[\frac{v+b}{4}-v+1\right]}^{s=\left[\frac{3(v+b)}{4}-v+1\right]} G\left(\left[\frac{b-v}{2}\right]+v, s\right) \\
&\cdot h(s+v-1) \\
&> \sigma\|y\| \cdot \frac{1}{\sigma} \\
&= \|y\| = r.
\end{aligned} \tag{3.5}$$

Thus, $\|Fy\| > \|y\|$, for $y \in \mathcal{K}_0 \cap \partial\Omega_r$.

On the other hand, suppose $H3$ holds, then there exists $\tau > 0$ and $R_1 > 0$ such that $f(t,y) \geq \dfrac{(\sigma+\tau)}{\lambda} y, \forall\, y \geq R_1, t \in [v-2, v+b]_{\mathbb{N}_{v-2}}$.

Now let R such that, $R > \max\left(m, {R_1}/_{\gamma}\right)$ then, we have

$$\begin{aligned}
(Fy)\left(\left[\frac{b-v}{2}\right]+v\right) &= \sum_{s=0}^{b} G\left(\left[\frac{b-v}{2}\right]+v, s\right)\lambda h(s+v-1) \\
&\cdot f(s+v-1, y(s+v-1)) \\
&\geq \lambda \sum_{s=0}^{b} G\left(\left[\frac{b-v}{2}\right]+v, s\right) h\left(s+v-1\right)\frac{(\sigma+\tau)}{\lambda} y \\
&\geq \lambda \frac{(\sigma+\tau)}{\lambda}\|y\| \sum_{s=\left[\frac{b+v}{4}-v+1\right]}^{\left[\frac{3(b+v)}{4}-v+1\right]} G\left(t,s\right) h\left(s+v-1\right) \\
&\geq \sigma\|y\| \cdot \frac{1}{\sigma} \\
&= R.
\end{aligned} \tag{3.6}$$

Hence, $\|Fy\| \geq \|y\|$ for $y \in \mathcal{K}_0 \cap \partial\Omega_R$.

Now, for any $y \in \partial\Omega_m$, $H1$ implies that, $f(t,y) \leq \dfrac{\eta m}{\lambda}$, $t \in [v-2, v+b]_{\mathbb{N}_{v-2}}$.

Let

$$\begin{aligned} Fy(t) &= \lambda \sum_{s=0}^{b} G(t,s)\, h(s+v-1)\, f(s+v-1, y(s+v-1)) \\ &\leq \lambda \sum_{s=0}^{b} G(s+v-1,s)\, h(s+v-1) \cdot \frac{\eta m}{\lambda} \\ &= \eta m \cdot \frac{1}{\eta} \\ &= m = \|y\|. \end{aligned} \tag{3.7}$$

Hence, $\|Fy\| \leq \|y\|$ for $y \in \mathcal{K}_0 \cap \partial\Omega_m$.

Therefore from Theorem 2.8, there are two fixed points y_1 and y_2 of operator F s. t. $0 \leq \|y_1\| < m < \|y_2\|$. □

Theorem 3.4. *Suppose conditions $H2$ and $H4$ hold, $f > 0$ for $t \in [v-2, v+b]_{\mathbb{N}_{v-2}}$. Then the fractional boundary value problem* (1.1)–(1.2) *has at least two positive solutions, say y_1 and y_2 such that* $0 \leq \|y_1\| < m < \|y_2\|$.

Proof. Suppose that $H2$ holds. Then there exists $\varepsilon > 0$ $(\varepsilon < \eta)$ and $0 < r < m$ such that $f(t,y) \leq \dfrac{(\eta-\varepsilon)}{\lambda} y$, $0 \leq y \leq r$, $\quad t \in [v-2, v+b]_{\mathbb{N}_{v-2}}$.

Let $r_1 \in (0, r)$, then for $y \in \partial\Omega_{r_1}$, we have

$$\begin{aligned} Fy(t) &= \lambda \sum_{s=0}^{b} G(t,s)\, h(s+v-1)\, f(s+v-1, y(s+v-1)) \\ &\leq \lambda \sum_{s=0}^{b} G(s+v-1,s)\, h(s+v-1) \cdot \frac{(\eta-\varepsilon)}{\lambda} r_1 \\ &< \eta r_1 \sum_{s=0}^{b} G(s+v-1,s)\, h(s+v-1) \\ &< \eta r_1 \cdot \frac{1}{\eta} \\ &= r_1 = \|y\|. \end{aligned} \tag{3.8}$$

Hence, we have $\|Fy\| < \|y\|$ for $y \in \partial\Omega_{r_1}$.

On the other hand, suppose that $H4$ holds, then there exists $0 < \tau < \eta$ and $R_0 > 0$ s. t. $f(t,y) \leq \tau\eta$, $y \geq R_0$, $t \in [v-2, v+b]_{\mathbb{N}_{v-2}}$.

Denote $M = \max\limits_{(t,y) \in [v-2,v+b]_{\mathbb{N}_{v-2}} \times [0,R_0]} f(t,y)$. Then

$$0 \leq f(t,y) \leq \frac{(\tau y + M)}{\lambda}, \quad 0 \leq y < \infty.$$

Let $R_2 > \max\left\{\dfrac{M}{(\eta-\tau)}, 2m\right\}$. For $y \in \partial\Omega_{R_2}$, we have

$$\begin{aligned}
\|Fy\| &= \max_{t\in[v-2,v+b]_{\mathbb{N}_{v-2}}} \lambda \sum_{s=0}^{b} G(t,s)\, h(s+v-1)\, f(s+v-1, y(s+v-1)) \\
&\le \lambda \sum_{s=0}^{b} G(s+v-1,s)\, h(s+v-1)\, f(s+v-1, y(s+v-1)) \\
&\le \lambda \frac{(\tau\|y\|+M)}{\lambda} \sum_{s=0}^{b} G(s+v-1,s) h(s+v-1) \\
&= \lambda \frac{(\tau R_2 + M)}{\lambda} \cdot \frac{1}{\eta} \\
&< R_2 = \|y\|.
\end{aligned} \tag{3.9}$$

Hence, we have $\|Fy\| < \|y\|$ for $y \in \partial\Omega_{R_2}$.

Finally, for any $y \in \partial\Omega_m$, since $\gamma m \le y(t) \le m$ for $t \in \left[\dfrac{b+v}{4}, \dfrac{3(b+v)}{4}\right]$, we have

$$\begin{aligned}
(Fy)\left(\left[\frac{b-v}{2}\right]+v\right) &= \sum_{s=0}^{b} G\left(\left[\frac{b-v}{2}\right]+v, s\right) \lambda h(s+v-1) \\
&\quad \cdot f(s+v-1, y(s+v-1)) \\
&> \lambda\sigma\gamma m \sum_{s=\left[\frac{b+v}{4}-v+1\right]}^{\left[\frac{3(b+v)}{4}-v+1\right]} G\left(\left[\frac{b-v}{2}\right]+v, s\right) h(s+v-1) \\
&= m = \|y\|.
\end{aligned} \tag{3.10}$$

Hence, $\|Fy\| > \|y\|$, for $y \in \mathcal{K}_0 \cap \partial\Omega_m$.

Therefore, by Theorem 2.8, the proof is complete. □

Example 3.5. Consider the fractional boundary value problem

$$\Delta^{\frac{5}{4}} y(t) = -\lambda \frac{1}{100} e^{\left(t+\frac{1}{4}\right)} \left(t+\frac{1}{4}\right) \left\{ y^{\frac{1}{2}}\left(t+\frac{1}{4}\right) + y^2\left(t+\frac{1}{4}\right) \right\} \tag{3.11}$$

$$y\left(-\frac{3}{4}\right) = 0, \quad y\left(\frac{25}{4}\right) = 0, \tag{3.12}$$

where $v = \frac{5}{4}$, $b = 5$, $f(t,y) = \frac{1}{100} t\left(y^{\frac{1}{2}} + y^2\right)$, $h(t) = e^t$. With a simple computation we can verify that $\eta > 0.0021$. $f : [0,\infty) \times [0,\infty) \to [0,\infty)$ and $h : [0,\infty) \to [0,\infty)$ and $f(t,y)$ satisfies the conditions $H1$ and $H3$, will have at least one positive solution.

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