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Oscillation of First-Order Neutral Difference Equations with Positive and Negative Coefficients

Hussain Ali Mohamad and **Hala Majid Mohi**
University of Baghdad
College of Science for Women
Department of Mathematics
Baghdad, Iraq
hussainmohamad22@gmail.com
hala.iraq96@yahoo.com

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Abstract

In this paper, necessary and sufficient conditions are obtained to guarantee the oscillation of all solutions of the first-order neutral difference equation with positive and negative coefficients

$$\Delta(y_n - p_n y_{n-m}) + q_n G(y_{n-k}) - r_n G(y_{n-l}) = f_n.$$

Some examples are given to illustrate the obtained results.

1 Introduction

Consider the first-order neutral difference equation with positive and negative coefficients

$$\Delta(y_n - p_n y_{n-m}) + q_n G(y_{n-k}) - r_n G(y_{n-l}) = f_n, \tag{1.1}$$

where Δ is the forward difference operator, p_n, q_n, r_n and f_n are infinite sequences of real numbers. $G \in (\mathbb{R}, \mathbb{R})$ is a function such that $y_n\, G(y_n) > 0$. Throughout this paper the following assumptions are used:

(H$_1$) $\sum_{n=0}^{\infty} r_n < \infty$.

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(H$_2$) $\sum_{n=0}^{\infty} q_n < \infty$.

(H$_3$) There exists a sequence F_n such that $\Delta F_n = f_n$ and $\lim_{n\to\infty} F_n = 0$.

(H$_4$) $\frac{G(y_n)}{y_n} \leq \beta_2$.

(H$_5$) $\frac{G(y_n)}{y_n} \geq \beta_1$.

By a solution of equation (1.1) is a sequence y_n satisfying (1.1) for $n \geq 0$. A nontrivial solution y_n is said to be oscillatory if for every $n_j > 0$ there exists $n \geq n_j$ such that $y_n y_{n+1} \leq 0$. Otherwise the solution is said to be nonoscillatory [2]. In [1], the authors obtained some necessary conditions for the oscillation of all solutions of the linear difference equation with variable delays $\Delta x(n) + p(n)x(\tau(n)) = 0$. In [4], sufficient conditions are obtained for the oscillation of every solution of first-order linear difference equations with positive and negative coefficients of the form

$$x_{n+1} - x_n + p\,x_{n-k} - q\,x_{n-l} = 0,\ n \in \mathbb{N}_0. \tag{A1}$$

It is proved that if $q(k-l) < 1$ and $(p-q)\frac{(k+1)^{k+1}}{k^k} > 1$, then every solution of equation $(A1)$ oscillates. In [5], the authors extended the results in [4] and obtained sufficient conditions for the oscillation of every solution of first-order linear difference equations with several positive and negative coefficients

$$x_{n+1} - x_n + \sum_{i=1}^{m} p_i\, x_{n-k_i} - q_i\, x_{n-l_i} = 0,\ n \in \mathbb{N}_0. \tag{A2}$$

It is proved that if $\sum_{i=1}^{m} q_i(k_i - l_i) \geq 1$ and $\sum_{i=1}^{m}(p_i - q_i)\frac{(k_i+1)^{k_i+1}}{k_i^{k_i}} > 1$, then every solution of equation $(A2)$ oscillates. In [6], the authors established sufficient conditions for all solutions of equation (1.1) to be either oscillatory or to tend to zero. In this paper, sufficient conditions are obtained for all solutions and for bounded solutions of equation (1.1) to oscillate. Examples are given to illustrate the obtained results.

2 Main Result

In this section, we use two sequences (2.2) and (2.9) in order to extract some sufficient conditions for the oscillation of all solutions to the equation (1.1). For simplicity, set

$$z_n = y_n - p_n y_{n-m}. \tag{2.1}$$

Let the sequence w_n be defined as

$$w_n = z_n + \sum_{i=n+l-k}^{n-1} r_i G(y_{i-l}) - F_n, \quad l < k. \tag{2.2}$$

The following result is based on [3, Theorem 7.6.1, p. 184].

Theorem 2.1 (See [3]). *Assume that $\{p_n\}$ is a nonnegative sequence of real numbers and let k be a positive integer. Suppose that*

$$\liminf_{n\to\infty} \sum_{i=n-k}^{n-1} p_i > \frac{k^{k+1}}{(k+1)^{k+1}}.$$

Then the difference inequality

$$y_{n+1} - y_n + p_n\, y_{n-k} \geq 0, \quad n \in \mathbb{N}_0$$

cannot have eventually negative solutions.

The next lemma is useful to prove our results.

Lemma 2.2. *Assume $0 \leq p_n \leq 1$, $q_n - r_{n+l-k} \geq 0$ and suppose that (H_1), (H_3)–(H_4) hold. Let y_n be a nonoscillatory bounded solution of equation* (1.1)*. Then $w_n \geq 0$ for $n \geq n_0 \geq 0$.*

Proof. Let y_n be an eventually positive (the proof of the case y_n is eventually negative is similar and will be omitted) bounded solution of (1.1). From (2.1) and (2.2), we obtain

$$\Delta w_n = -(q_n - r_{n+l-k}) G(y_{n-k}) \leq 0. \tag{2.3}$$

Hence w_n is a nonincreasing sequence and $\lim_{n\to\infty} w_n = L$, where $-\infty \leq L < \infty$. Since y_n is bounded, let

$$\limsup_{n\to\infty} y_n = h^* \geq 0.$$

So there exists a subsequence $\{n_j\}$ of $\{n\}$ such that

$$\lim_{j\to\infty} n_j = \infty, \quad \lim_{j\to\infty} y_{n_j} = h^*.$$

Let $y_n \leq \delta_1$. Then $G(y_n) \leq \beta_2 \delta_1 = \delta_2$, where δ_1, δ_2 are positive constants. We claim that $L \geq 0$. Otherwise $L < 0$, and then there exists $n_1 \geq n_0$ and $\alpha < 0$ such that $w_n \leq \alpha < 0$ for $n \geq n_1$. From (2.2), we get

$$\begin{aligned}
y_{n_j} &= w_{n_j} + p_{n_j} y_{n_j-m} + \sum_{i=n_j+l-k}^{n_j-1} r_i G(y_{i-l}) + F_{n_j} \\
&\leq \alpha + p_{n_j} y_{n_j-m} + \delta_2 \sum_{i=n_j+l-k}^{n_j-1} r_i + F_{n_j} \\
&\leq \alpha + p_{n_j} y_{n_j-m} + \varepsilon \leq \alpha + y_{n_j-m} + \varepsilon, \quad \varepsilon > 0.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the last inequality leads to $y_{n_j} \leq \alpha + y_{n_j - m}$. As $j \to \infty$, it follows that $h^* \leq \alpha + h^*$, which is a contradiction. So our claim is established and so $L \geq 0$, that is, $w_n \geq 0$. □

Theorem 2.3. *Assume* $0 \leq p_n < 1$, $q_n - r_{n+l-k} \geq 0$. *Suppose that* (H_1), (H_3)–(H_5) *hold, and*

$$\liminf_{n\to\infty} \Big(p_n + \beta_2 \sum_{i=n+l-k}^{n-1} r_i\Big) < 1, \qquad m, l < k, \tag{2.4}$$

$$\limsup_{n\to\infty} \big(q_n - r_{n+l-k}\big) > \frac{1}{\beta_1}. \tag{2.5}$$

Then every solution of equation (1.1) *oscillates.*

Proof. Let $\{y_n\}$ be a positive solution of (1.1) for $n \geq n_0 \geq 0$. From equations (1.1), (2.1) and (2.2), we obtain

$$\Delta w_n = -\big(q_n - r_{n+l-k}\big)G(y_{n-k}) \leq 0.$$

Hence w_n is a nonincreasing sequence and $\lim_{n\to\infty} w_n = L$, where $-\infty \leq L < \infty$. We consider two cases:

Case 1. y_n is unbounded. So there exists a subsequence $\{n_j\}$ of $\{n\}$ such that $\lim_{j\to\infty} n_j = \infty$, $\lim_{j\to\infty} y_{n_j} = \infty$ and $y_{n_j} = \max\{y_n : \ n_0 \leq n \leq n_j\}$. By using ($H_3$), ($H_4$), we obtain from (2.2)

$$\begin{aligned} w_{n_j} &= y_{n_j} - p_{n_j} y_{n_j - m} - \sum_{i=n_j+l-k}^{n_j - 1} r_i G(y_{i-l}) - F_{n_j} \\ &\geq \left(1 - p_{n_j} - \beta_2 \sum_{i=n_j+l-k}^{n_j-1} r_i\right) y_{n_j} - F_{n_j}. \end{aligned}$$

Hence $\lim_{j\to\infty} w_{n_j} = \infty$ leads to a contradiction.

Case 2. y_n is bounded. Then by Lemma 2.2, it follows that $w_n \geq 0$ for $n \geq n_0$. From (2.3), we have

$$-w_n \leq w_{n+1} - w_n = -\big(q_n - r_{n+l-k}\big)G(y_{n-k}),$$

so

$$w_n \geq \beta_1\big(q_n - r_{n+l-k}\big)y_{n-k}. \tag{2.6}$$

From (2.2), we get

$$y_n = w_n + p_n y_{n-m} + \sum_{i=n+l-k}^{n-1} r_i G(y_{i-l}) + F_n,$$

$$y_n \geq w_n + F_n \geq w_n - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, for sufficiently large n, it follows that

$$y_{n-k} \geq w_{n-k}. \tag{2.7}$$

Substituting (2.7) in (2.6), we obtain

$$w_n \geq \beta_1 \big(q_n - r_{n+l-k}\big) w_{n-k},$$

so

$$\frac{w_n}{w_{n-k}} \geq \beta_1 \big(q_n - r_{n+l-k}\big).$$

$\big(q_n - r_{n+l-k}\big)\beta_1 \leq 1$ leads to a contradiction. The proof is complete. □

Example 2.4. Consider the first-order neutral difference equation

$$\Delta\Big(y_n - \Big(\frac{1}{2}\Big)^{n+1} y_{n-2}\Big) + \frac{405}{32} y_{n-4} - 3\Big(\frac{1}{2}\Big)^{n+3} y_{n-2} = \frac{2}{9}\Big(\frac{1}{2}\Big)^n \Big(-\frac{3}{2}\Big)^n, \tag{2.8}$$

where

$$m = 2,\ k = 4,\ l = 2,\ p_n = \Big(\frac{1}{2}\Big)^{n+1},\ q_n = \frac{405}{64},\ r_n = 3\Big(\frac{1}{2}\Big)^{n+4},$$

$$G(y_n) = 2y_n,\ \beta_1 = \beta_2 = 2,\ f_n = \frac{2}{9}\Big(\frac{1}{2}\Big)^n \Big(-\frac{3}{2}\Big)^n.$$

One can find that all conditions of Theorem 2.3 hold as follows:

$$p_n = \Big(\frac{1}{2}\Big)^{n+1} < 1,\ n \geq 0,\ q_n - r_{n+l-k} = \frac{405}{64} - 3\Big(\frac{1}{2}\Big)^{n+2} > 0,\ n \geq 0,$$

$$\limsup_{n\to\infty} \Big(p_n + \beta_1 \sum_{i=n+l-k}^{n-1} r_i\Big) = \lim_{n\to\infty} \Big(\Big(\frac{1}{2}\Big)^{n+1} + 6 \sum_{i=n-2}^{n-1} \Big(\frac{1}{2}\Big)^{i+4}\Big) = 0,$$

$$\liminf_{n\to\infty} \big(q_n - r_{n+l-k}\big) = \lim_{n\to\infty} \Big(\frac{405}{64} - 3\Big(\frac{1}{2}\Big)^{n+2}\Big) = \frac{405}{64} > \frac{1}{2}.$$

So according to Theorem 2.3, every solution of (2.8) oscillates. For instance,

$$y_n = \Big(-\frac{3}{2}\Big)^n$$

is such a solution.

In the next result, the sequence W_n will be used, where

$$W_n = y_n - p_n y_{n-m} + \sum_{i=n-l+k}^{n-1} q_i\, G(y_{i-k}) - F_n. \tag{2.9}$$

Lemma 2.5. *Assume $p_n \geq 1$, $r_n - q_{n-l+k} \geq 0$. Suppose that (H_2)–(H_4) hold. Let y_n be a nonoscillatory bounded solution of equation* (1.1). *Then $W_n \leq 0$ for $n \geq n_0 \geq 0$.*

Proof. Let y_n be an eventually positive and bounded solution of (1.1) (the case when y_n is eventually negative is similar and will be omitted). From (1.1), (2.1) and (2.9), we obtain

$$\Delta W_n = (r_n - q_{n-l+k})G(y_{n-l}) \geq 0. \tag{2.10}$$

Hence W_n is a nondecreasing sequence and $\lim_{n\to\infty} W_n = L$, where $-\infty < L \leq \infty$. Let

$$\liminf_{n\to\infty} y_n = h_* \geq 0.$$

So there exists a subsequence $\{n_j\}$ of $\{n\}$ such that

$$\lim_{j\to\infty} n_j = \infty, \quad \lim_{j\to\infty} y_{n_j} = h_*.$$

Let $y_n \leq \delta_1$. Then $G(y_n) \leq \beta_2$, $\delta_1 = \delta_2$, where δ_1, δ_2 are positive constants. We claim that $W_n \leq 0$ eventually. Otherwise if $W_n > 0$, then there exists $n_1 \geq n_0$ and $\alpha > 0$ such that $W_n \geq \alpha > 0$ for $n \geq n_1$. From (2.9), we get

$$\begin{aligned} y_{n_j} &= W_{n_j} + p_{n_j} y_{n_j-m} - \sum_{i=n_j-l+k}^{n_j-1} q_i G(y_{i-k}) + F_{n_j} \\ &\geq \alpha + p_{n_j} y_{n_j-m} - \delta_2 \sum_{i=n_j-l+k}^{n_j-1} q_i + F_{n_j}. \end{aligned}$$

By using (H_2)–(H_3), we get

$$y_{n_j} \geq \alpha + p_{n_j} y_{n_j-m} - \varepsilon \geq \alpha + y_{n_j-m} - \varepsilon, \quad \varepsilon > 0.$$

Since $\varepsilon > 0$ is arbitrary, the last inequality leads to $y_{n_j} \geq \alpha + y_{n_j-m}$ for sufficiently large n. As $j \to \infty$, it follows that $h_* \geq \alpha + h_*$, which is a contradiction. So our claim is established and we have $W_n \geq 0$. □

Theorem 2.6. *Assume $p_n \geq 1$, $r_n - q_{n-l+k} \geq 0$. Suppose that (H_2)–(H_4) hold, and*

$$\liminf_{n\to\infty} \sum_{i=n-l+k}^{n-1} \frac{r_i - q_{i-l+k}}{p_{i-l+m}} > \frac{\big(l-m\big)^{l-m+1}}{\beta_1\big(l-m+1\big)^{l-m+1}}, \quad m,k < l. \tag{2.11}$$

Then every bounded solution of equation (1.1) *oscillates.*

Proof. For the sake of contradiction, assume that $\{y_n\}$ is a positive and bounded solution of (1.1) for $n \geq n_0 \geq 0$. From equations (1.1), (2.1) and (2.9), it follows that (2.10) holds, that is

$$\Delta W_n = \big(r_n - q_{n-l+k}\big)G(y_{n-l}) \geq 0.$$

Hence W_n is nondecreasing and $\lim_{n\to\infty} W_n = L$, where $-\infty < L < \infty$. Since $\{y_n\}$ is bounded, by Lemma 2.5, it follows that $W_n \le 0$. From (2.10), we get

$$W_{n+1} - W_n - \big(r_n - q_{n-l+k}\big)G(y_{n-l}) = 0,$$

so

$$W_{n+1} - W_n - \beta_1\big(r_n - q_{n-l+k}\big)y_{n-l} \ge 0. \tag{2.12}$$

From (2.9), we get

$$\begin{aligned} W_{n_j} &= y_{n_j} - p_{n_j}y_{n_j-m} + \sum_{i=n_j-l+k}^{n_j-1} q_i G(y_{i-k}) - F_{n_j} \\ &\ge -p_{n_j}y_{n_j-m} - F_{n_j} \ge -p_{n_j}y_{n_j-m} - \varepsilon, \quad \varepsilon > 0. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, for sufficiently large n it follows that $-p_{n_j}y_{n_j-m} \le W_{n_j}$, so

$$y_{n_j-m} \ge -\frac{1}{p_{n_j}}W_{n_j},$$

and hence

$$y_{n_j-l} \ge -\frac{1}{p_{n_j-l+m}}W_{n_j-l+m}. \tag{2.13}$$

Substituting (2.13) in (2.12), we get

$$W_{n_j+1} - W_{n_j} + \frac{r_{n_j} - q_{n_j-l+k}}{p_{n_j-l+m}}W_{n_j-(l-m)} \ge 0.$$

By Theorem 2.1 and in virtue of condition (2.11), it follows that the last inequality cannot have eventually negative solutions, which is a contradiction. The proof is complete. □

Example 2.7. Consider the first-order neutral difference equation

$$\begin{aligned} \Delta\left(y_n - \left(1 + \left(\frac{1}{e}\right)^n\right)y_{n-1}\right) + e^{-3}\left(\frac{1}{e}\right)^n y_{n-2} - e^{-4}\big(1 + 2e + e^2\big)y_{n-3} \\ = -\left(\frac{1}{e}\right)^{n-1}\left(-\frac{1}{e}\right)^n, \end{aligned} \tag{2.14}$$

where

$$m = 1,\ k = 2,\ l = 3,\ p_n = 1 + \left(\frac{1}{e}\right)^n,\ q_n = e^{-3}\left(\frac{1}{e}\right)^n,\ r_n = e^{-4}\big(1 + 2e + e^2\big),$$

$$G(y_n) = y_n,\ \beta_1 = \beta_2 = 1,\ f_n = -e\left(\frac{1}{e}\right)^n\left(-\frac{1}{e}\right)^n.$$

One can find that all conditions of Theorem 2.6 hold as follows:

$$p_n = 1+\left(\frac{1}{e}\right)^n \geq 1,\ n \geq 0,\ r_n - q_{n-l+k} = e^{-4}\left(1+2e+e^2\right) - e^{-3}\left(\frac{1}{e}\right)^{n-1} > 0,\ n \geq 0,$$

$$\sum_{n=0}^{\infty} q_n = \sum_{n=0}^{\infty} e^{-3}\left(\frac{1}{e}\right)^n = \frac{e^{-2}}{e-1} < \infty,$$

$$\begin{aligned} \liminf_{n\to\infty} \sum_{i=n-l+m}^{n-1} \frac{(r_i - q_{i-l+k})}{p_{i-l+m}} &= \lim_{n\to\infty} \sum_{n-2}^{n-1} \frac{e^{-4}(1+2e+e^2) - e^{-2}(\frac{1}{e})^i}{1+e^2(\frac{1}{e})^i} \\ &= 2e^{-4}(1+2e+e^2) > \left(\frac{2}{3}\right)^3. \end{aligned}$$

So according to Theorem 2.6, every bounded solution of (2.14) oscillates. For instance,

$$y_n = \left(-\frac{1}{e}\right)^n$$

is such a solution.

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