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Oscillation of Higher Order Fractional Nonlinear Difference Equations

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Abstract

In this article, we consider higher order fractional nonlinear difference equation of the type

$$\begin{aligned}\Delta^{\alpha}x(t)+f_1(t,x(t+\alpha))=&v(t)+f_2(t,x(t+\alpha)),\ t\in\mathbb{N}_0,\ m-1<\alpha\leq m\\ \Delta^{\alpha-k}x(t)\,|_{t=0}=&x_k,\ k=1,2,\ldots,m-1\end{aligned}$$

where Δ^{α} is Riemann–Liouville like discrete fractional difference operator of order α, $m-1<\alpha\leq m$, $m\geq 1$ is an integer. We obtain some oscillation criteria for this equation.

AMS Subject Classifications: 39A10.
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1 Introduction

In the present work, we consider the higher order fractional nonlinear difference equation

$$\begin{aligned}\Delta^{\alpha}x(t)+f_1(t,x(t+\alpha))=&v(t)+f_2(t,x(t+\alpha)),\ t\in\mathbb{N}_0,\ m-1<\alpha\leq m\\ \Delta^{\alpha-k}x(t)\,|_{t=a}=&x_k,\quad k=1,2,\ldots,m-1,\end{aligned}\tag{1.1}$$

where Δ^{α} denotes Riemann–Liouville like discrete fractional difference operator of order α with $m-1<\alpha\leq m$. $f_i:[0,\infty)\times\mathbb{R}\to\mathbb{R}$, $i=1,2$ and v are continuous with respect to t and x, $\mathbb{N}_a=\{a,a+1,a+2,\ldots\}$. We establish some oscillation theorem for higher-order fractional nonlinear difference equations.

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2 Preliminaries and Lemmas

In this section, we presented several definitions and lemmas for discrete fractional calculus.

Definition 2.1 (See [3])**.** Let $v > 0$. The v-th fractional sum of f is defined by

$$\Delta^{-v} f(t) = \frac{1}{\Gamma(v)} \sum_{s=0}^{t-v} (t-s-1)^{(v-1)} f(s).$$

Note that f is defined for $s = a \bmod (1)$ and $\Delta^{-v} f$ is defined for $t = (a+v) \bmod (1)$; in particular, Δ^{-v} maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+v}$ where $\mathbb{N}_t = \{t, t+1, t+2, \ldots\}$ and $t^{(v)} = \dfrac{\Gamma(t+1)}{\Gamma(t-v+1)}$.

Definition 2.2 (See [3])**.** The μ-th fractional difference is defined as

$$\Delta^{\mu} f(t) = \Delta^{m-v} f(t) = \Delta^{m} \Delta^{-v} f(t),$$

where $\mu > 0$ and $m-1 < \mu < m$, where m denotes a positive integer, and $-v = \mu - m$.

Theorem 2.3 (See [2])**.** *Let f be a real-valued function defined on $\mathbb{N}_a$ and $\mu, v > 0$. Then the following equalities hold:*

$$\Delta^{-v}\left[\Delta^{-\mu} f(t)\right] = \Delta^{-(\mu+v)} f(t) = \Delta^{-\mu}\left[\Delta^{-v} f(t)\right],$$

$$\Delta^{-v} \Delta f(t) = \Delta \Delta^{-v} f(t) - \frac{(t-a)^{(v-1)}}{\Gamma(v)} f(a).$$

Lemma 2.4 (See [3])**.** *Let $\mu \neq 1$ and assume $\mu + v + 1$ is not a positive integer. Then*

$$\Delta^{-v} t^{(\mu)} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+v+1)} t^{(\mu+v)}.$$

Lemma 2.5 (Young's inequality)**.**

(i) Let $X, Y \geq 0$, $u > 1$ and $\frac{1}{u} + \frac{1}{v} = 1$. Then

$$XY \leq \frac{1}{u} X^{u} + \frac{1}{v} Y^{v}, \tag{2.1}$$

where the equality holds if and only if $Y = X^{u-1}$.

(ii) Let $X \geq 0$, $Y > 0$, $0 < u < 1$ and $\frac{1}{u} + \frac{1}{v} = 1$. Then

$$XY \geq \frac{1}{u} X^{u} + \frac{1}{v} Y^{v}, \tag{2.2}$$

where the equality holds if and only if $Y = X^{u-1}$.

Lemma 2.6. *The equivalent fractional Taylor's difference formula of* (1.1) *is, for* $t \in \mathbb{N}_\alpha$*,*

$$x(t) = \sum_{k=1}^{m} \frac{(t-a)^{(\alpha-k)}}{\Gamma(\alpha-k+1)} \Delta^{\alpha-k} x(a) + \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-s-1)^{(\alpha-1)} \left[v(s) + f_2(s, x(s+\alpha)) - f_1(s, x(s+\alpha)) \right]. \tag{2.3}$$

Proof. We consider the equation

$$\Delta^\alpha x(t) + f_1(t, x(t+\alpha)) = v(t) + f_2(t, x(t+\alpha)).$$

To find a fractional Taylor's difference formula, we can apply the $\Delta^{-\alpha}$ operator to each order side of this equation. So that, we can make generalization for first, second, later higher order fractional nonlinear difference equations. So, firstly we find a fractional Taylor's difference formula of first order fractional nonlinear difference equation. We write

$$\Delta^\alpha x(t) = v(t) + f_2(t, x(t+\alpha)) - f_1(t, x(t+\alpha)).$$

By applying the $\Delta^{-\alpha}$ operator to each order side of this equation, we obtain

$$\Delta^{-\alpha} \Delta^\alpha x(t) = \Delta^{-\alpha} \left[v(t) + f_2(t, x(t+\alpha)) - f_1(t, x(t+\alpha)) \right].$$

We find the left side this equation. The right side is clear from the definition. From Theorem 2.3

$$\begin{aligned} \Delta^{-\alpha} \Delta^\alpha x(t) =& \Delta^{-\alpha} \Delta \Delta^{-(1-\alpha)} x(t) \\ =& \Delta \Delta^{-\alpha} \Delta^{-(1-\alpha)} x(t) - \frac{(t-a)^{(\alpha-1)}}{\Gamma(\alpha)} \Delta^{\alpha-1} x(a) \\ =& x(t) - \frac{(t-a)^{(\alpha-1)}}{\Gamma(\alpha)} \Delta^{\alpha-1} x(a), \end{aligned}$$

we obtain

$$\begin{aligned} x(t) -& \frac{(t-a)^{(\alpha-1)}}{\Gamma(\alpha)} \Delta^{\alpha-1} \\ &= \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t-s-1)^{(\alpha-1)} \left[v(s) + f_2(t, x(t+\alpha)) - f_1(t, x(t+\alpha)) \right], \end{aligned}$$

$$\begin{aligned} x(t) =& \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t-s-1)^{(\alpha-1)} \left[v(s) + f_2(t, x(t+\alpha)) - f_1(t, x(t+\alpha)) \right] \\ &+ \frac{(t-a)^{(\alpha-1)}}{\Gamma(\alpha)} \Delta^{\alpha-1} x(a). \end{aligned}$$

Now, to find a fractional Taylor's difference formula for second order, we consider

$$\Delta^{\alpha}x(t)+f_1\left(t,x\left(t+\alpha\right)\right)=v(t)+f_2\left(t,x\left(t+\alpha\right)\right).$$

If we apply the $\Delta^{-\alpha}$ operator, then we obtain,

$$\Delta^{-\alpha}\Delta^{\alpha}x\left(t\right)=\Delta^{-\alpha}\left[v(t)+f_2\left(t,x\left(t+\alpha\right)\right)-f_1\left(t,x\left(t+\alpha\right)\right)\right].$$

For the left side,

$$\begin{aligned}\Delta^{-\alpha}\Delta^{\alpha}x\left(t\right)=&\Delta^{-\alpha}\Delta^{2}\Delta^{-(2-\alpha)}x\left(t\right)\\=&\Delta^{-\alpha}\Delta\Delta\Delta^{-(2-\alpha)}x\left(t\right)\\=&\Delta\Delta^{-\alpha}\Delta\Delta^{-(2-\alpha)}x\left(t\right)-\frac{(t-a)^{(\alpha-1)}}{\Gamma\left(\alpha\right)}\Delta^{\alpha-1}x(a)\\=&\Delta^{1-\alpha}\Delta\Delta^{-(2-\alpha)}x\left(t\right)-\frac{(t-a)^{(\alpha-1)}}{\Gamma\left(\alpha\right)}\Delta^{\alpha-1}x(a)\\=&\Delta\Delta^{1-\alpha}\Delta^{-(2-\alpha)}x\left(t\right)-\frac{(t-a)^{(\alpha-1-1)}}{\Gamma\left(\alpha-1\right)}\Delta^{\alpha-1-1}x(a)\\&-\frac{(t-a)^{(\alpha-1)}}{\Gamma\left(\alpha\right)}\Delta^{\alpha-1}x(a)\\=&x\left(t\right)-\frac{(t-a)^{(\alpha-2)}}{\Gamma\left(\alpha-1\right)}\Delta^{\alpha-2}x(a)-\frac{(t-a)^{(\alpha-1)}}{\Gamma\left(\alpha\right)}\Delta^{\alpha-1}x(a)\\=&x\left(t\right)-\sum_{k=1}^{2}\frac{(t-a)^{(\alpha-k)}}{\Gamma\left(\alpha-k+1\right)}\Delta^{\alpha-k}x(a).\end{aligned}$$

So, we find the fractional Taylor difference formula of the second order fractional nonlinear difference equation

$$\begin{aligned}x(t)=&\frac{1}{\Gamma\left(\alpha\right)}\sum_{s=0}^{t-\alpha}\left(t-s-1\right)^{(\alpha-1)}\left[v\left(s\right)+f_2\left(t,x\left(t+\alpha\right)\right)-f_1\left(t,x\left(t+\alpha\right)\right)\right]\\&+\sum_{k=1}^{2}\frac{(t-a)^{(\alpha-k)}}{\Gamma\left(\alpha-k+1\right)}\Delta^{\alpha-k}x(a).\end{aligned}$$

Similarly, we do this process for the higher-order fractional nonlinear difference. We can write

$$\begin{aligned}
\Delta^{-\alpha}\Delta^{\alpha}x\,(t) =&\Delta^{-\alpha}\Delta^{m}\Delta^{-(m-\alpha)}x\,(t)\\
=&\Delta^{-\alpha}\Delta\Delta^{m-1}\Delta^{-(m-\alpha)}x\,(t)\\
=&\Delta\Delta^{-\alpha}\Delta^{m-1}\Delta^{-(m-\alpha)}x\,(t)-\frac{(t-a)^{(\alpha-1)}}{\Gamma\,(\alpha)}\Delta^{\alpha-1}x(a)\\
=&\Delta^{1-\alpha}\Delta\Delta^{m-2}\Delta^{-(m-\alpha)}x\,(t)-\frac{(t-a)^{(\alpha-1)}}{\Gamma\,(\alpha)}\Delta^{\alpha-1}x(a)\\
=&\Delta\Delta^{1-\alpha}\Delta^{m-2}\Delta^{-(m-\alpha)}x\,(t)-\frac{(t-a)^{(\alpha-2)}}{\Gamma\,(\alpha-1)}\Delta^{\alpha-2}x(a)\\
&-\frac{(t-a)^{(\alpha-1)}}{\Gamma\,(\alpha)}\Delta^{\alpha-1}x(a)\\
=&\Delta^{2-\alpha}\Delta\Delta^{m-3}\Delta^{-(m-\alpha)}x\,(t)-\frac{(t-a)^{(\alpha-2)}}{\Gamma\,(\alpha-1)}\Delta^{\alpha-2}x(a)\\
&-\frac{(t-a)^{(\alpha-1)}}{\Gamma\,(\alpha)}\Delta^{\alpha-1}x(a)\\
=&\Delta\Delta^{2-\alpha}\Delta^{m-3}\Delta^{-(m-\alpha)}x\,(t)-\frac{(t-a)^{(\alpha-3)}}{\Gamma\,(\alpha-2)}\Delta^{\alpha-3}x(a)\\
&-\frac{(t-a)^{(\alpha-2)}}{\Gamma\,(\alpha-1)}\Delta^{\alpha-2}x(a)-\frac{(t-a)^{(\alpha-1)}}{\Gamma\,(\alpha)}\Delta^{\alpha-1}x(a).
\end{aligned}$$

By repeating this process, we find

$$\begin{aligned}
\Delta^{-\alpha}\Delta^{\alpha}x\,(t) =&\Delta^{m-\alpha-1}\Delta\Delta^{m-m}\Delta^{-(m-\alpha)}x\,(t)\\
&-\frac{(t-a)^{(\alpha-m+1)}}{\Gamma\,(\alpha-m+2)}\Delta^{\alpha-m+1}x(a)-\ldots-\frac{(t-a)^{(\alpha-1)}}{\Gamma\,(\alpha)}\Delta^{\alpha-1}x(a)\\
=&\Delta\Delta^{m-\alpha-1}\Delta^{-(m-\alpha)}x\,(t)-\frac{(t-a)^{(\alpha-m)}}{\Gamma\,(\alpha-m+1)}\Delta^{\alpha-m}x(a)\\
&-\frac{(t-a)^{(\alpha-m+1)}}{\Gamma\,(\alpha-m+2)}\Delta^{\alpha-m+1}x(a)-\ldots-\frac{(t-a)^{(\alpha-1)}}{\Gamma\,(\alpha)}\Delta^{\alpha-1}x(a)\\
=&x\,(t)-\sum_{k=1}^{m}\frac{(t-a)^{(\alpha-k)}}{\Gamma\,(\alpha-k+1)}\Delta^{\alpha-k}x(a).
\end{aligned}$$

We can write

$$\begin{aligned}
x(t) =&\frac{1}{\Gamma\,(\alpha)}\sum_{s=0}^{t-\alpha}(t-s-1)^{(\alpha-1)}\left[v\,(s)+f_2\,(t,x\,(t+\alpha))-f_1\,(t,x\,(t+\alpha))\right]\\
&+\sum_{k=1}^{m}\frac{(t-a)^{(\alpha-k)}}{\Gamma\,(\alpha-k+1)}\Delta^{\alpha-k}x(a).
\end{aligned}$$

This completes the proof. □

3 Main Results

We will use the conditions

$$x f_i(t,x) > 0 \ (i = 1,2), \quad x \neq 0, \ t \geq a \tag{3.1}$$

and

$$|f_1(t,x)| \geq p_1(t)|x|^{\beta} \text{ and } |f_2(t,x)| \leq p_2(t)|x|^{\gamma}, \quad x \neq 0, \ t \geq a, \tag{3.2}$$

where $p_1, p_2 \in C\left([a,\infty), \mathbb{R}^+\right)$ and $\beta, \gamma > 0$ are real numbers. We obtain some oscillation theorems for (1.1) without the condition (3.1) but with the condition (3.2) and the following conditions:

$$|f_1(t,x)| \leq p_1(t)|x|^{\beta} \text{ and } |f_2(t,x)| \geq p_2(t)|x|^{\gamma}, \quad x \neq 0, \ t \geq a, \tag{3.3}$$

where $p_1, p_2 \in C\left([a,\infty), \mathbb{R}^+\right)$ and $\beta, \gamma > 0$ are constants.

Theorem 3.1. *Suppose that condition* (3.1) *and* (3.2) *hold with $\beta > \gamma$. If*

$$\liminf_{t\to\infty} t^{(1-\alpha)} \sum_{s=a}^{t-\alpha} (t-s-1)^{(\alpha-1)} [v(s) + H(s)] = -\infty \tag{3.4}$$

and

$$\limsup_{t\to\infty} t^{(1-\alpha)} \sum_{s=a}^{t-\alpha} (t-s-1)^{(\alpha-1)} [v(s) - H(s)] = \infty \tag{3.5}$$

for every sufficiently large T, where $H(s) = (\beta/\gamma - 1)[\gamma p_2(s)/\beta]^{\beta/(\beta-\gamma)} p_1^{\gamma/(\gamma-\beta)}(s)$, then every solution of equation (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of equation (1.1). Suppose that x is an eventually positive solution of (1.1). Then there exists $T_1 > a$ such that $x(t) > 0$ for $t \geq T_1$. Let $s \geq T_1$ and take $X = |x|^{\gamma}(s)$, $Y = \gamma p_2(s)/(\beta p_1(s))$, $u = \beta/\gamma$ and $v = \beta/(\beta-\gamma)$. Then from (1.1), we see,

$$\begin{aligned} p_2(s)|x|^{\gamma}(s) - p_1(s)|x|^{\beta}(s) &= \frac{\beta p_1(s)}{\gamma}\left[|x|^{\gamma}\frac{\gamma p_2(s)}{\beta p_1(s)} - \frac{1}{\beta/\gamma}\left(|x|^{\gamma}(s)\right)^{\beta/\gamma}\right] \\ &= \frac{\beta p_1(s)}{\gamma}\left[XY - \frac{1}{u}X^u\right] \\ &\leq \frac{\beta p_1(s)}{\gamma}\frac{1}{v}Y^v = H(s) \quad \text{for } s \geq T_1, \end{aligned} \tag{3.6}$$

where H is defined as in Theorem 3.1. From (2.3), (3.1), (3.2) and (3.6), we obtain

$$\begin{aligned}\Gamma(\alpha)x(t) =& \Gamma(\alpha)\sum_{k=1}^{m}\frac{(t-a)^{(\alpha-k)}}{\Gamma(\alpha-k+1)}\Delta^{\alpha-k}x(a)\\ &+\sum_{s=a}^{T_1}(t-s-1)^{(\alpha-1)}\left[v(s)+f_2(s,x(s+\alpha))-f_1(s,x(s+\alpha))\right]\\ &+\sum_{s=T_1}^{t-\alpha}(t-s-1)^{(\alpha-1)}\left[v(s)+f_2(s,x(s+\alpha))-f_1(s,x(s+\alpha))\right]\\ \le& \Phi(t)+\psi(t,T_1)\\ &+\sum_{s=T_1}^{t-\alpha}(t-s-1)^{(\alpha-1)}\left[v(s)+p_2(s)x^{\gamma}(s)-p_1(s)x^{\beta}(s)\right]\\ \le& \Phi(t)+\psi(t,T_1)+\sum_{s=T_1}^{t-\alpha}(t-s-1)^{(\alpha-1)}\left[v(s)+H(s)\right] \quad \text{for } t\ge T_1,\end{aligned} \tag{3.7}$$

where

$$\Phi(t)=\Gamma(\alpha)\sum_{k=1}^{m}\frac{(t-a)^{(\alpha-k)}}{\Gamma(\alpha-k+1)}\Delta^{\alpha-k}x(a)=\Gamma(\alpha)\sum_{k=1}^{m}\frac{(t-a)^{(\alpha-k)}}{\Gamma(\alpha-k+1)}x_k \tag{3.8}$$

and

$$\psi(t,T_1)=\sum_{s=a}^{T_1}(t-s-1)^{(\alpha-1)}\left[v(s)+f_2(s,x(s+\alpha))-f_1(s,x(s+\alpha))\right]. \tag{3.9}$$

Multiplying (3.7) by $t^{(1-\alpha)}$, for $t\ge T_1$,

$$\begin{aligned}0<t^{(1-\alpha)}\Gamma(\alpha)x(t)\le& t^{(1-\alpha)}\Phi(t)+t^{(1-\alpha)}\psi(t,T_1)\\ &+t^{(1-\alpha)}\sum_{s=T_1}^{t-\alpha}(t-s-1)^{(\alpha-1)}\left[v(s)+H(s)\right].\end{aligned} \tag{3.10}$$

Take $T_2>T_1$. Now we look at the cases $0<\alpha\le 1$ and $\alpha>1$, respectively.
Case (i). Let $0<\alpha\le 1$. Then we get $m=1$, $\Phi(t)=x_1(t-a)^{(\alpha-1)}$ for $t\ge T_2$,

$$\left|t^{(1-\alpha)}\Phi(t)\right|=|x_1|\,t^{(1-\alpha)}(t-a)^{(\alpha-1)}\le|x_1|\left(\frac{T_2}{T_2-a}\right)^{(1-\alpha)}=c_1(T_2) \tag{3.11}$$

and

$$\left|t^{(1-\alpha)}\psi(t,T_1)\right|$$

$$
\begin{aligned}
&= \left| t^{(1-\alpha)} \sum_{s=a}^{T_1} (t-s-1)^{(\alpha-1)} \left[v(s) + f_2(s, x(s+\alpha)) - f_1(s, x(s+\alpha)) \right] \right| \\
&\leq \sum_{s=a}^{T_1} t^{(1-\alpha)} (t-s-1)^{(\alpha-1)} \left| v(s) + f_2(s, x(s+\alpha)) - f_1(s, x(s+\alpha)) \right| \\
&\leq \sum_{s=a}^{T_1} \left(\frac{T_2}{T_2-s-1} \right)^{(1-\alpha)} \left| v(s) + f_2(s, x(s+\alpha)) - f_1(s, x(s+\alpha)) \right| \\
&= c_2(T_1, T_2) \quad \text{for } t \geq T_2. \qquad (3.12)
\end{aligned}
$$

We can see from (3.10), (3.11), (3.12) that

$$
t^{(1-\alpha)} \sum_{s=T_1}^{t-\alpha} (t-s-1)^{(\alpha-1)} \left[v(s) + H(s) \right] > -\left[c_1(T_2) + c_2(T_1, T_2) \right] \quad \text{for } t \geq T_2.
$$

Hence,

$$
\liminf_{t \to \infty} t^{(1-\alpha)} \sum_{s=T_1}^{t-\alpha} (t-s-1)^{(\alpha-1)} \left[v(s) + H(s) \right] \geq -\left[c_1(T_2) + c_2(T_1, T_2) \right] > -\infty,
$$

which contradicts (3.4).
Case (ii). Let $\alpha > 1$. Then we have, for $m \geq 2$,

$$
\begin{aligned}
\left| t^{(1-\alpha)} \Phi(t) \right| &= \left| t^{(1-\alpha)} \Gamma(\alpha) \sum_{k=1}^{m} \frac{(t-a)^{(\alpha-k)}}{\Gamma(\alpha-k+1)} x_k \right| \\
&\leq \Gamma(\alpha) \sum_{k=1}^{m} \frac{|x_k| \, t^{(1-\alpha)} (t-a)^{(\alpha-k)}}{\Gamma(\alpha-k+1)} \\
&\leq \Gamma(\alpha) \sum_{k=1}^{m} \frac{|x_k| \, (T_2-a)^{(1-k)}}{\Gamma(\alpha-k+1)} \\
&= c_3(T_2) \quad \text{for } t \geq T_2 \qquad (3.13)
\end{aligned}
$$

and

$$
\begin{aligned}
&\left| t^{(1-\alpha)} \psi(t, T_1) \right| \\
&= \left| t^{(1-\alpha)} \sum_{s=a}^{T_1} (t-s-1)^{(\alpha-1)} \left[v(s) + f_2(s, x(s+\alpha)) - f_1(s, x(s+\alpha)) \right] \right| \\
&\leq \sum_{s=a}^{T_1} t^{(1-\alpha)} (t-s-1)^{(\alpha-1)} \left| v(s) + f_2(s, x(s+\alpha)) - f_1(s, x(s+\alpha)) \right| \\
&\leq \sum_{s=a}^{T_1} \left| v(s) + f_2(s, x(s+\alpha)) - f_1(s, x(s+\alpha)) \right|
\end{aligned}
$$

$$= c_4(T_1) \quad \text{for } t \geq T_2. \tag{3.14}$$

From (3.10), (3.13) and (3.14), we can see

$$t^{(1-\alpha)} \sum_{s=T_1}^{t-\alpha} (t-s-1)^{(\alpha-1)} [v(s) + H(s)] > -[c_3(T_2) + c_4(T_1)] \quad \text{for } t \geq T_2.$$

Hence,

$$\liminf_{t\to\infty} t^{(1-\alpha)} \sum_{s=T_1}^{t-\alpha} (t-s-1)^{(\alpha-1)} [v(s) + H(s)] \geq -[c_3(T_2) + c_4(T_1)] > -\infty,$$

which contradicts (3.4). Next, we suppose that x is an eventually negative solution of (1.1). Then a similar argument leads to a contradiction with (3.5). The proof is complete. □

Theorem 3.2. *Let* $\alpha \geq 1$ *and suppose that conditions* (3.1) *and* (3.3) *hold with* $\beta < \gamma$. *If*

$$\limsup_{t\to\infty} t^{(1-\alpha)} \sum_{s=a}^{t-\alpha} (t-s-1)^{(\alpha-1)} [v(s) + H(s)] = \infty \tag{3.15}$$

and

$$\liminf_{t\to\infty} t^{(1-\alpha)} \sum_{s=0}^{t-\alpha} (t-s-1)^{(\alpha-1)} [v(s) - H(s)] = -\infty \tag{3.16}$$

for every sufficiently large T*, where* H *is defined as in Theorem 3.1, then every bounded solution of* (1.1) *is oscillatory.*

Proof. Let x be a bounded nonoscillatory solution of (1.1). Then there exist constants M_1 and M_2 such that

$$M_1 \leq x(t) \leq M_2 \quad \text{for } t \geq a. \tag{3.17}$$

We assume that, x is a bounded eventually positive solution of (1.1). Then there exist $T_1 > a$ such that $x(t) > 0$ for $t \geq T_1$. By the inequalities (2.2) and (3.6), we find

$$p_2(s) |x|^{\gamma}(s) - p_1(s) |x|^{\beta} \geq H(s) \quad \text{for } s \geq T_1, \tag{3.18}$$

where H is defined as in Theorem 3.1. Define Φ and ψ as in (3.8) and (3.9), respectively. Similarly to the proof of (3.10), from (2.3), (3.1), (3.3) and (3.18), we get for $t \geq T_1$,

$$\begin{aligned} t^{(1-\alpha)} \Gamma(\alpha) x(t) \geq & t^{(1-\alpha)} \Phi(t) + t^{(1-\alpha)} \psi(t, T_1) \\ & + t^{(1-\alpha)} \sum_{s=T_1}^{t-\alpha} (t-s-1)^{(\alpha-1)} [v(s) + H(s)]. \end{aligned} \tag{3.19}$$

Take $T_2 > T_1$. Next, we consider the cases $\alpha = 1$ and $\alpha > 1$.
Case (i). Let $\alpha = 1$. Then (3.10) and (3.11) are still true. From (3.10), (3.11), (3.17) and (3.19), for $t \geq T_2$,

$$M_2\Gamma(\alpha) \geq -c_1(T_2) - c_2(T_1, T_2) + t^{(1-\alpha)} \sum_{s=T_1}^{t-\alpha} (t-s-1)^{(\alpha-1)} [v(s) + H(s)].$$

Thus, we can see

$$\limsup_{t\to\infty} t^{(1-\alpha)} \sum_{s=T_1}^{t-\alpha} (t-s-1)^{(\alpha-1)} [v(s) + H(s)] \leq c_1(T_2) + c_2(T_1, T_2) + M_2\Gamma(\alpha) < \infty,$$

which contradicts (3.15).
Case (ii). Let $\alpha > 1$. Then (3.12) and (3.13) are still valid. From (3.12), (3.13), (3.17) and (3.19), for $t \geq T_2$,

$$M_2\Gamma(\alpha) t^{(1-\alpha)} \geq -c_3(T_2) - c_4(T_1) + t^{(1-\alpha)} \sum_{s=T_1}^{t-\alpha} (t-s-1)^{(\alpha-1)} [v(s) + H(s)].$$

When $\lim_{t\to\infty} t^{(1-\alpha)} = 0$, we obtain

$$\limsup_{t\to\infty} t^{(1-\alpha)} \sum_{s=T_1}^{t-\alpha} (t-s-1)^{(\alpha-1)} [v(s) + H(s)] \leq c_3(T_2) + c_4(T_1) < \infty,$$

which contradicts (3.15). Finally, we suppose that x is a bounded eventually negative solution of (1.1). Then a similar argument leads to a contradiction with (3.16). The proof is complete. □

4 Results with the Fractional Caputo Like Difference Operator

In this section, we first give a definition for the Caputo like difference.

Definition 4.1 (See [1]). Let $\mu > 0$ and $m - 1 < \mu < m$, where m denotes a positive integer, $m = \lceil \mu \rceil$, $\lceil . \rceil$ ceiling of number. Set $v = m - \mu$. The μ-th fractional Caputo like difference is defined as

$$\Delta_*^{\alpha} f(t) = \Delta^{-v}(\Delta^m f(t)) = \frac{1}{\Gamma(v)} \sum_{s=a}^{t-v} (t-s-1)^{(v-1)} (\Delta^m f)(s), \quad \forall t \in \mathbb{N}_{a+v}.$$

Here Δ^m is the m-th order forward difference operator

$$\left(\Delta^m f\right)(s) = \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} f(s+k).$$

In this section, we study for the Caputo fractional operator, the oscillation of the initial value problem

$$\begin{aligned} \Delta_*^{\alpha} x(t) + f_1(t, x(t+\alpha)) =& v(t) + f_2(t, x(t+\alpha)), \quad t > a \geq 0 \\ \Delta^k x(t)\,|_{t=a} =& x_k, \quad k = 0, 1, \ldots, m-1, \end{aligned} \tag{4.1}$$

where $\Delta_*^{\alpha} x(t)$ is the Caputo type like discrete fractional difference operator of order α with $m-1 < \alpha \leq m$. $f_i : [0,\infty) \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2$ and v are continuous with respect to t and x, $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$. The corresponding Caputo type fractional Taylor's difference formula

$$x(t) = \sum_{k=0}^{m-1} \frac{(t-a)^{(k)}}{k!} \Delta^k x(a) + \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-s-1)^{(\alpha-1)} \Delta_*^{\alpha} x(s), \quad t \in \mathbb{N}_{a+\alpha}.$$

Theorem 4.2. *Suppose that* (3.1) *and* (3.2) *hold with* $\beta > \gamma$. *If*

$$\liminf_{t\to\infty} t^{(1-m)} \sum_{s=a}^{t-\alpha} (t-s-1)^{(\alpha-1)} \left[v(s) + H(s)\right] = -\infty$$

and

$$\limsup_{t\to\infty} t^{(1-m)} \sum_{s=a}^{t-\alpha} (t-s-1)^{(\alpha-1)} \left[v(s) - H(s)\right] = \infty$$

for every sufficiently large T, *where* H *is defined as in Theorem 3.1, then every solution of* (4.1) *is oscillatory.*

Theorem 4.3. *Let* $\alpha \geq 1$ *and suppose that* (3.1) *and* (3.3) *hold with* $\beta < \gamma$. *If*

$$\liminf_{t\to\infty} t^{(1-m)} \sum_{s=a}^{t-\alpha} (t-s-1)^{(\alpha-1)} \left[v(s) - H(s)\right] = -\infty$$

and

$$\limsup_{t\to\infty} t^{(1-m)} \sum_{s=a}^{t-\alpha} (t-s-1)^{(\alpha-1)} \left[v(s) + H(s)\right] = \infty$$

for every sufficiently large T, *where* H *is defined as in Theorem 3.1, then every bounded solution of* (4.1) *is oscillatory.*

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