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Asymptotic Stability of a Discrete Version of the Heavy Ball with Friction Dynamical System

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Abstract

In this paper boundedness and asymptotic behavior of a discrete version of nonlinear heavy ball with friction dynamical system is studied. Our results extend the previous results of the first author [7] to the nonhomogeneous case and for more general assumptions on the parameters.

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1 Introduction

Let H be a real Hilbert space with inner product $< \cdot, \cdot >$ and norm $|\cdot|$. We denote weak convergence in H by $\rightharpoonup$ and strong convergence by $\to$. Let A be a nonempty subset of $H \times H$ to which we shall refer as a (nonlinear) possibly multivalued operator in H. A is called monotone (resp. strongly monotone) iff $< y_2 - y_1, x_2 - x_1 > \geq 0$ (resp. $< y_2 - y_1, x_2 - x_1 > \geq \alpha |x_2 - x_1|^2$ for some $\alpha > 0$), for all $[x_i, y_i] \in A$, $i = 1, 2$. A is called maximal monotone if A is monotone and $R(I + A) = H$, where I is the identity operator of H. Given any function $\varphi : H \to]-\infty, +\infty]$ (not necessarily convex) with domain $D(\varphi)$, its subdifferential is the multivalued operator $\partial\varphi$, defined as

$$\partial\varphi(x) := \{w \in H \mid \varphi(x) - \varphi(y) \leq < w, x - y >, \quad \forall y \in H\}.$$

The function φ is called proper iff $\varphi \not\equiv +\infty$. It is a well-known result that if φ is a proper, convex, and lower semicontinuous function, then $\partial\varphi$ is a maximal monotone

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operator. We refer the reader to the book by Morosanu [9] in order to understand monotone operators and subdifferential of convex functions in Hilbert spaces.

Let A be a maximal monotone operator on a real Hilbert space H and γ a positive real constant. The following second order dissipative system of maximal monotone type

$$\begin{cases} u''(t) + \gamma u'(t) + Au(t) \ni 0, \\ u(0) = u_0, \quad u'(0) = u_1, \end{cases} \tag{1.1}$$

is called heavy ball with friction dynamical system, because when $A = \nabla\varphi$ (the gradient of φ), this system is a model for damping oscillation of a heavy ball on the graph of φ. The asymptotic behavior of (1.1) and its discrete version at infinity is a subject of many recent investigations. Attouch and Alvarez [3], Alvarez [2] and Attouch, Goudou and Redont [4] studied the dynamical system (1.1), when $A = \nabla\varphi$, where φ is a continuously differentiable and convex function on H. When $A = \nabla\varphi$, equation (1.1) provides a dynamical approach to optimization problems, because the solution of (1.1) converges weakly to a minimum point of φ. To the best of our knowledge, the problem of convergence of solutions to (1.1) for general maximal monotone operator A is still open. For numerical and practical purposes, as well as to get an algorithm for approximation of a zero of a maximal monotone operator or a minimum point of a convex function (when $A = \partial\varphi$), it is useful to consider the discrete version of (1.1). In [2] Alvarez using the following approximations for u' and u'':

$$u'(t) = \frac{u(t+h) - u(t)}{h} + O(h), \tag{1.2}$$

$$u''(t) = \frac{u(t+h) - 2u(t) + u(t-h)}{h} + O(h^2), \tag{1.3}$$

obtained the following discretization of (1.1).

$$u_n - u_{n+1} + \alpha_n(u_n - u_{n-1}) \in \lambda_{n+1} A u_{n+1}, \tag{1.4}$$

where $0 \le \alpha_n \le 1$ and λ_n is a positive sequence. When $A = \nabla\varphi$, Alvarez proved convergence of the sequence u_n to a minimum point of φ. Jules and Maingé [6] considered the iterative method (1.4) for a co-coercive operator A and obtained the weak convergence of the sequence u_n to an element of $A^{-1}(0)$. They also showed that (1.4) has a better rate of convergence than the standard proximal point algorithm (when $\alpha_n \equiv 0$). Alvarez and Attouch [1] obtained the weak convergence of u_n given by (1.4) for general maximal monotone operators with appropriate assumptions on λ_n and α_n.

In [7] the first author replaced the approximation (1.2) by the following approximation for $u'(t)$

$$u'(t) = \frac{u(t+h) - u(t-h)}{2h} + O(h^2), \tag{1.5}$$

which is better than (1.2) and obtained the following discretization of (1.1).

$$\begin{cases} (1-\alpha_n)u_n + \alpha_n u_{n-1} - u_{n+1} \in \lambda_{n+1} A u_{n+1}, \\ u_0, u_1 \in H, \end{cases} \tag{1.6}$$

where α_n (resp. λ_n) is a nonnegative (resp. positive) sequence and A is a maximal monotone operator. The difference inclusion (1.6) can be rewritten in explicit form based on the resolvent operator as follows

$$u_{n+1} = J_{\lambda_{n+1}}((1-\alpha_n)u_n + \alpha_n u_{n-1}). \tag{1.7}$$

It seems that the difference inclusion (1.6) is more stable than (1.4), because if $\{u_n\}$ is the solution of (1.6) (equivalently (1.7)) with initial data $u_0,\ u_1$, and u'_n is the solution of (1.6) with initial data $u'_0,\ u'_1$, then by (1.7) since J_λ is nonexpansive (see [9, Theorem 1.3 on page 21]) we have:
$|u_{n+1} - u'_{n+1}| \leq (1-\alpha_n)|u_n - u'_n| + \alpha_n |u_{n-1} - u'_{n-1}|$
$\leq Max\{|u_n - u'_n|,\ |u_{n-1} - u'_{n-1}|\} \leq ... \leq Max\{|u_1 - u'_1|,\ |u_0 - u'_0|\}$.
In this paper, we consider the following nonhomogeneous case of (1.6)

$$\begin{cases} (1-\alpha_n)u_n + \alpha_n u_{n-1} - u_{n+1} + e_n \in \lambda_{n+1} A u_{n+1} \\ u_0, u_1 \in H, \end{cases} \tag{1.8}$$

where e_n is the error sequence in H. We extend and improve the results of [7] with more general assumptions on the parameters α_n, λ_n and the error e_n.

In Section 2, we prove the boundedness of the sequence $\{u_n\}$ for coercive maximal monotone operators. We also show the relation between the boundedness of $\{u_n\}$ and the assumption $A^{-1}(0) \neq \varnothing$. Section 3 is devoted to the weak convergence of the bounded sequence $\{u_n\}$ given by (1.8) and its weighted average. In Section 4, we consider the weak convergence and the rate of convergence in the subdifferential case. Under suitable assumptions on the parameters and the operator A, the strong convergence of the algorithm is established in Section 5. We denote the weighted average of the sequence u_n by $w_n := (\sum_{k=1}^{n} \lambda_k)^{-1}(\sum_{k=1}^{n} \lambda_k u_k)$, and the element $\dfrac{(1-\alpha_n)u_n + \alpha_n u_{n-1} - u_{n+1} + e_n}{\lambda_{n+1}}$ by Au_{n+1} for simplicity.

Definition 1.1. Given a bounded sequence $\{u_n\}$ in H, the asymptotic center c of $\{u_n\}$ is defined as follows (see [5]): for every $q \in H$, let $\varphi(q) = \lim_{n\to+\infty} \sup |u_n - q|^2$. Then φ is a continuous and strictly convex function on H, satisfying $\varphi(q) \to +\infty$ as $|q| \to +\infty$. Thus φ achieves its minimum on H at a unique point c, called the asymptotic center of the sequence $\{u_n\}$.

Throughout the paper, we assume $0 \leq \alpha_n \leq 1$, for all $n > 0$.

2 Boundedness

In this section, we study the boundedness of the sequence $\{u_n\}$ generated by (1.8). We present the relation between the boundedness of $\{u_n\}$ and the assumption $A^{-1}(0) \neq \varnothing$.

Theorem 2.1. *Let A be a coercive maximal monotone operator. If the sequence $\{\frac{|e_n|}{\lambda_{n+1}}\}$ is bounded, then the sequence $\{u_n\}$ is bounded.*

Proof. Let $M > 0$ be such that for each $n \geq 1$, $\frac{|e_n|}{\lambda_{n+1}} < M$. By coerciveness of A, there exist $K > 0$ and $y_0 \in H$ such that for all $[x, y] \in A$, with $|x - y_0| > K$, $\frac{< y, x - y_0 >}{|x - y_0|} > M$. If there exists n such that $|u_{n+1} - y_0| > K$, using (1.8), we get

$$\lambda_{n+1} M|u_{n+1} - y_0| \leq < \lambda_{n+1} A u_{n+1}, u_{n+1} - y_0 >$$
$$= < (1 - \alpha_n)u_n + \alpha_n u_{n-1} + e_n - u_{n+1}, u_{n+1} - y_0 >$$
$$= < (1 - \alpha_n)(u_n - y_0) + \alpha_n(u_{n-1} - y_0) + e_n, u_{n+1} - y_0 > -|u_{n+1} - y_0|^2$$
$$\leq |(1 - \alpha_n)(u_n - y_0) + \alpha_n(u_{n-1} - y_0) + e_n||u_{n+1} - y_0| - |u_{n+1} - y_0|^2$$
$$\leq \big((1 - \alpha_n)|u_n - y_0| + \alpha_n|u_{n-1} - y_0| + |e_n|\big)|u_{n+1} - y_0| - |u_{n+1} - y_0|^2.$$

Thus

$$|u_{n+1} - y_0| \leq (1 - \alpha_n)|u_n - y_0| + \alpha_n|u_{n-1} - y_0| + \lambda_{n+1}(\frac{|e_n|}{\lambda_{n+1}} - M)$$
$$\leq (1 - \alpha_n)|u_n - y_0| + \alpha_n|u_{n-1} - y_0| \leq \max\{|u_n - y_0|, |u_{n-1} - y_0|\}$$
$$\leq ... \leq \max\{|u_1 - y_0|, |u_0 - y_0|\}.$$

Hence, for all $n \geq 1$, $|u_{n+1} - y_0| \leq \max\{|u_1 - y_0|, |u_0 - y_0|, K\}$. □

Lemma 2.2. *Suppose that a_n and b_n are nonnegative real sequences and that $\sum_{n=1}^{+\infty} b_n < +\infty$. If $a_{n+1} \leq a_n + b_n$, for all $n \geq 1$, then $lim_n a_n$ exists.*

Lemma 2.3. *Suppose $\{\alpha_n\}$ is a sequence in $[0, 1]$ and ψ_n and δ_n are positive real sequences. Suppose that $\psi_{n+1} \leq (1 - \alpha_n)\psi_n + \alpha_n\psi_{n-1} + \delta_n$ and $\sum_{n=1}^{+\infty} \delta_n < +\infty$. If one of the following conditions holds:*
$(i)\ \lim_n \alpha_n = \alpha < 1$, $(ii)\ \sum_{n=1}^{+\infty}[\alpha_n - \alpha_{n-1}]_+ < +\infty$, where $[\alpha]_+ := \max\{\alpha, 0\}$,
then $\lim_n \psi_n$ exists.

Proof. (i) Set $\phi_n := \max\{\psi_n, \psi_{n-1}\}$. By assumption, we have $\psi_{n+1} \leq \phi_n + \delta_n$. On the other hand $\psi_n \leq \phi_n + \delta_n$, so, $\phi_{n+1} \leq \phi_n + \delta_n$. Using Lemma 2.2, $\lim_n \phi_n$ exists. Taking limsup from both sides of $\psi_{n+1} \leq \phi_n + \delta_n$, we get $\limsup_n \psi_n \leq \lim \phi_n$. On the other hand

$\psi_{n+1} \leq (1-\alpha_n)\psi_n + \alpha_n\phi_n + \delta_n$ and also $\psi_n \leq (1-\alpha_n)\psi_n + \alpha_n\phi_n + \delta_n$. So $\phi_{n+1} \leq (1-\alpha_n)\psi_n + \alpha_n\phi_n + \delta_n$. Taking $\liminf$ as $n \to \infty$, we get $\lim_n \phi_n \leq \liminf \psi_n$. Hence, $\lim_n \psi_n$ exists and $\lim_n \psi_n = \lim_n \phi_n$. (ii) By the proof of part (i), ψ_n is bounded. From the assumption on ψ_n, we have

$$\psi_{n+1} \leq \psi_n + \alpha_{n-1}\psi_{n-1} - \alpha_n\psi_n + [\alpha_n - \alpha_{n-1}]_+\psi_{n-1} + \delta_n.$$

Now the lemma is proved by Lemma 2.2. □

Theorem 2.4. *Suppose that* $A^{-1}(0) \neq \varnothing$ *and* $(E_1)\sum_{n=1}^{+\infty}|e_n| < +\infty$. *Then:*

(1) *The sequence* $\{u_n\}$ *is bounded.*

(2) *If* $(\alpha_1)\ \sum_{n=1}^{+\infty}[\alpha_n - \alpha_{n-1}]_+ < +\infty$, *where* $[\alpha]_+ = \max\{0, \alpha\}$ *or* $(\alpha_2)\ \lim_n \alpha_n = \alpha < 1$*, then* $\lim_n |u_n - p|$ *exists, for each* $p \in A^{-1}(0)$.

Proof. Set $p \in A^{-1}(0)$. First, we prove (1). By the nonexpansivity of the resolvent operator, we have $|u_{n+1} - p| \leq |(1-\alpha_n)(u_n - p) + \alpha_n(u_{n-1} - p) + e_n|$
$\leq (1-\alpha_n)|u_n - p| + \alpha_n|u_{n-1} - p| + |e_n| \leq \max\{|u_n - p|, |u_{n-1} - p|\} + |e_n|$
$\leq ... \leq \max\{|u_1 - p|, |u_0 - p|\} + \sum_{n=1}^{+\infty}|e_n| < \infty.$
Let us prove (2). By the monotonicity of A, we have
$0 \leq <\lambda_{n+1}Au_{n+1}, u_{n+1} - p> = <(1-\alpha_n)u_n + \alpha_n u_{n-1} + e_n - u_{n+1}, u_{n+1} - p>$
$= <(1-\alpha_n)(u_n - p) + \alpha_n(u_{n-1} - p) + e_n, u_{n+1} - p> - |u_{n+1} - p|^2$.
So
$|u_{n+1} - p| \leq |(1-\alpha_n)(u_n - p) + \alpha_n(u_{n-1} - p) + e_n|$
$\leq (1-\alpha_n)|u_n - p| + \alpha_n|u_{n-1} - p| + |e_n|$.
Now the theorem is proved by Lemma 2.3. □

Remark 2.5. Conditions (α_1) and (α_2) are different as the following examples show.

(1) If $\alpha_n = \begin{cases} 1, & n = k^2 \\ 1 - \frac{1}{n}, & n \neq k^2 \end{cases}$, then $\sum_{n=1}^{+\infty}[\alpha_n - \alpha_{n-1}]_+ < +\infty$, but $\lim \alpha_n = 1$.

(2) If $\alpha_n = \begin{cases} \alpha, & n \text{ is even} \\ \alpha - \frac{1}{n}, & n \text{ is odd} \end{cases}$, where $0 \leq \alpha < 1$, then $\lim_n \alpha_n = \alpha < 1$ and $\sum_{n=1}^{+\infty}[\alpha_n - \alpha_{n-1}]_+ = +\infty$. Therefore, the assumptions (α_1) and (α_2) in Theorem 2.4 are different. Also the condition $\sum_{n=1}^{+\infty}[\alpha_n - \alpha_{n-1}]_+ < +\infty$ is better than the condition

$\sum_{n=1}^{+\infty} |\alpha_n - \alpha_{n-1}| < +\infty$ assumed in [7]. Take $\alpha_n = \begin{cases} \alpha_{n-1} + \frac{1}{n}, & n = k^2 \\ \alpha_{n-1} - \frac{1}{n}, & n \neq k^2 \end{cases}$ for $n \geq 2$ and $\alpha_1 = 1$. Then $\sum_{n=1}^{+\infty} |\alpha_n - \alpha_{n-1}| = +\infty$ but $\sum_{n=1}^{+\infty} [\alpha_n - \alpha_{n-1}]_+ < +\infty$.

Lemma 2.6 (See [8]). *Suppose that $\{\alpha_n\}$ is a nonnegative sequence and $\{\lambda_n\}$ is a positive sequence such that $\sum_{n=1}^{+\infty} \lambda_n = +\infty$. If $\frac{\alpha_n}{\lambda_n} \to 0$ as $n \to +\infty$, then $\frac{\sum_{k=1}^{n} \alpha_k}{\sum_{k=1}^{n} \lambda_k} \to 0$ as $n \to +\infty$.*

Theorem 2.7. *Suppose that $\{u_n\}$ is a bounded sequence given by* (1.8) *and the following conditions are satisfied,*

(i) (Λ_1) $\sum_{n=1}^{+\infty} \lambda_n = +\infty$,

(ii) (α_1) *or* (α_3) $\frac{[\alpha_n - \alpha_{n-1}]_+}{\lambda_{n+1}} \to 0$,

(iii) (E_1) *or* (E_2) $\frac{|e_n|}{\lambda_{n+1}} \to 0$.

Then $A^{-1}(0) \neq \varnothing$ and $\omega_w(w_n) \subset A^{-1}(0)$, where $\omega_w(w_n)$ is the set of weak cluster points of w_n.

Proof. Suppose $[x, y] \in A$. Since $\{u_n\}$ is bounded, there is a subsequence $\{w_{n_j}\}$ of $\{w_n\}$ such that $w_{n_j} \rightharpoonup p \in H$. On the other hand, by the monotonicity of A, we get

$$
\begin{aligned}
&\lambda_{i+1} < x - u_{i+1}, y > \\
&= \lambda_{i+1}\big(< x - u_{i+1}, y - Au_{i+1} > + < x - u_{i+1}, Au_{i+1} > \big) \\
&\geq < x - u_{i+1}, \lambda_{i+1} Au_{i+1} > \\
&= < x - u_{i+1}, (1 - \alpha_i)u_i + \alpha_i u_{i-1} + e_i - u_{i+1} > \\
&= < x - u_{i+1}, (1 - \alpha_i)(u_i - x) + \alpha_i(u_{i-1} - x) + e_i + x - u_{i+1} > \\
&= \big[|u_{i+1} - x|^2 - < u_{i+1} - x, (1 - \alpha_i)(u_i - x) + \alpha_i(u_{i-1} - x) > - < u_{i+1} - x, e_i > \big] \\
&\geq \big[\frac{1}{2}|u_{i+1} - x|^2 - \frac{1}{2}|(1 - \alpha_i)(u_i - x) + \alpha_i(u_{i-1} - x)|^2 - |u_{i+1} - x||e_i| \big] \\
&\geq \big[\frac{1}{2}|u_{i+1} - x|^2 - \frac{1}{2}(1 - \alpha_i)|u_i - x|^2 - \frac{1}{2}\alpha_i|u_{i-1} - x|^2 - |u_{i+1} - x||e_i| \big] \\
&= \big[\frac{1}{2}(|u_{i+1} - x|^2 - |u_i - x|^2) + \\
&\frac{1}{2}\alpha_i(|u_i - x|^2 - |u_{i-1} - x|^2) - |e_i||u_{i+1} - x| \big] \\
&\geq \big[\frac{1}{2}(|u_{i+1} - x|^2 - |u_i - x|^2) + \\
&\frac{1}{2}\alpha_i|u_i - x|^2 - \frac{1}{2}\alpha_{i-1}|u_{i-1} - x|^2 - \frac{1}{2}|u_{i-1} - x|^2[\alpha_i - \alpha_{i-1}]_+ - |e_i||u_{i+1} - x| \big]
\end{aligned}
$$

Summing up from $i=0$ to $i=n_j-1$ and then dividing by $\sum_{i=0}^{n_j-1}\lambda_{i+1}$, we get

$$< x - w_{n_j}, y > = < x - (\sum_{i=0}^{n_j-1}\lambda_{i+1})^{-1}\sum_{i=0}^{n_j-1}\lambda_{i+1}u_{i+1}, y >$$

$$\geq (\sum_{i=0}^{n_j-1}\lambda_{i+1})^{-1}\big[-\frac{1}{2}|u_0-x|^2 - \frac{1}{2}\alpha_{-1}|u_{-1}-x|^2$$

$$-\frac{1}{2}\sum_{i=0}^{n_j-1}|u_{i-1}-x|^2[\alpha_i-\alpha_{i-1}]_+ - \sum_{i=0}^{n_j-1}|e_i||u_{i+1}-x|\big]$$

$$\geq (\sum_{i=0}^{n_j-1}\lambda_{i+1})^{-1}\big[-\frac{1}{2}|u_0-x|^2 - \frac{1}{2}M^2\sum_{i=0}^{n_j-1}[\alpha_i-\alpha_{i-1}]_+ - M\sum_{i=0}^{n_j-1}|e_i|\big],$$

where $M=\sup_n|u_n-x|$ (in the last inequality, we take $\alpha_{-1}=0$ and $u_{-1}=0$). Letting $j\to\infty$, by Lemma 2.6, we get $< x-p, y > \geq 0$. Thus, by the maximality of A, we have $p\in A^{-1}(0)$, as desired. □

3 Weak Convergence

In this section, we prove the weak convergence of the sequence $\{u_n\}$ and its weighted average to a zero of A, which extend the results of [7], under suitable assumptions on the parameters α_n and λ_n.

Theorem 3.1. *Suppose* $\{u_n\}$ *is a bounded sequence generated by* (1.8) *and conditions* (Λ_1), (α_1) *and* (E_1) *are satisfied. Then* $w_n \rightharpoonup p\in A^{-1}(0)$ *as* $n\to\infty$*, which is also an asymptotic center of* $\{u_n\}$.

Proof. By Theorem 2.7, $A^{-1}(0)\neq\varnothing$ and $\omega_w(w_n)\subset A^{-1}(0)$. Thus by part (2) of Theorem 2.4, $\lim_n|u_n-p|$, exists for each $p\in\omega_w(w_n)$. We show that $\omega_w(w_n)$ is singleton. Suppose $p,q\in\omega_w(w_n)$ and $p\neq q$, then by Theorem 2.4,$\lim_n(|u_n-p|^2-|u_n-q|^2)$ exists and hence $\lim_{n\to+\infty} < u_n, p-q >$ exists. Thus $\lim_{n\to+\infty} < w_n, p-q >$ exists. This implies that $< q, p-q > = < p, p-q >$ and hence $p=q$. So, $w_n\rightharpoonup p\in A^{-1}(0)$ as $n\to+\infty$. Now, we show that p is the asymptotic center of $\{u_n\}$. Suppose that $q\in H$ and $q\neq p$, then

$$|u_n-p|^2 = |u_n-q|^2 + 2 < u_n, q-p > + |p|^2 - |q|^2.$$

Multiplying both sides of the above equality by λ_n, summing up from $n=1$ to $n=m$, dividing by $\sum_{n=1}^{m}\lambda_n$ and taking $\limsup$ as $m\to+\infty$, we get

$$\lim_{n\to+\infty}|u_n-p|^2=\limsup_{m\to+\infty}(\sum_{n=1}^{m}\lambda_n)^{-1}(\sum_{n=1}^{m}\lambda_n|u_n-q|^2)-|q-p|^2<\limsup_{n\to+\infty}|u_n-q|^2.$$

This shows that p is the asymptotic center of the sequence $\{u_n\}$ as desired. □

Theorem 3.2. *Let* $\{u_n\}$ *be a bounded sequence given by* (1.8).
(1) If the following conditions hold:

$$\begin{cases}(\Lambda_2)\sum_{n=1}^{+\infty}\lambda_n^2=+\infty,\\ (\alpha_1)\ \textit{or}\ (\alpha_4)\dfrac{[\alpha_n-\alpha_{n-1}]_+}{\lambda_{n+1}^2}\to 0,\\ (E_1)\ \textit{or}\ (E_3)\dfrac{|e_n|}{\lambda_{n+1}^2}\to 0,\\ (\alpha_5)\sum_{n=1}^{+\infty}\dfrac{\alpha_n^2}{\lambda_{n+1}^2}<+\infty\ \textit{and}\ (E_4)\sum_{n=1}^{+\infty}\dfrac{|e_n|^2}{\lambda_{n+1}^2}<+\infty,\end{cases}\tag{3.1}$$

then $A^{-1}(0)\neq\varnothing$ *and* $\omega_w(u_n)\subset A^{-1}(0)$.
(2) If conditions $(\Lambda_2),(\alpha_1),(\alpha_5),(E_1)$ *and* (E_4) *are satisfied, then* $u_n\rightharpoonup p\in A^{-1}(0)$.

Proof. By Theorem 2.7, $A^{-1}(0)\neq\varnothing$. In order to prove (1), assume $p\in A^{-1}(0)$. By the monotonicity of A, $<\lambda_{n+1}Au_{n+1},p-u_{n+1}>\leq 0$. So by (1.8), $<(1-\alpha_n)u_n+\alpha_n u_{n-1}-u_{n+1}+e_n,p-u_{n+1}>\leq 0$. Thus, we have
$|p-u_{n+1}|^2+\lambda_{n+1}^2|Au_{n+1}|^2\leq|(1-\alpha_n)(u_n-p)+\alpha_n(u_{n-1}-p)+e_n|^2$
$\leq|(1-\alpha_n)(u_n-p)+\alpha_n(u_{n-1}-p)|^2+|e_n|^2+2|e_n||(1-\alpha_n)(u_n-p)+\alpha_n(u_{n-1}-p)|$
$\leq(1-\alpha_n)|u_n-p|^2+\alpha_n|u_{n-1}-p|^2+|e_n|^2+2|e_n||(1-\alpha_n)(u_n-p)+\alpha_n(u_{n-1}-p)|$
$\leq|u_n-p|^2+\alpha_{n-1}|u_{n-1}-p|^2-\alpha_n|u_n-p|^2+|u_{n-1}-p|^2[\alpha_n-\alpha_{n-1}]_++|e_n|^2+2|e_n||(1-\alpha_n)(u_n-p)+\alpha_n(u_{n-1}-p)|$.
Therefore

$$\begin{cases}\lambda_{n+1}^2|Au_{n+1}|^2\leq|u_n-p|^2-|u_{n+1}-p|^2+\alpha_{n-1}|u_{n-1}-p|^2\\ -\alpha_n|u_n-p|^2+M^2[\alpha_n-\alpha_{n-1}]_++|e_n|^2+2M|e_n|,\end{cases}\tag{3.2}$$

where $M=\sup_n|u_n-p|$. On the other hand, by the monotonicity of A, we have

$$0\leq<Au_{n+1}-Au_n,u_{n+1}-u_n>=<Au_{n+1}-Au_n,\alpha_n(u_{n-1}-u_n)+e_n-\lambda_{n+1}Au_{n+1}>,$$

thus

$|Au_{n+1}|^2 \leq < Au_n, Au_{n+1} > + < Au_{n+1} - Au_n, \frac{\alpha_n}{\lambda_{n+1}}(u_{n-1} - u_n) + \frac{e_n}{\lambda_{n+1}} >$
$\leq \frac{1}{2}|Au_n|^2 + \frac{1}{2}|Au_{n+1}|^2 - \frac{1}{2}|Au_n - Au_{n+1}|^2 + \frac{1}{2}|Au_n - Au_{n+1}|^2$
$+ \frac{1}{2}|\frac{\alpha_n}{\lambda_{n+1}}(u_{n-1} - u_n) + \frac{e_n}{\lambda_{n+1}}|^2$
$\leq \frac{1}{2}|Au_n|^2 + \frac{1}{2}|Au_{n+1}|^2 + \frac{\alpha_n^2}{\lambda_{n+1}^2}|u_{n-1} - u_n|^2 + \frac{|e_n|^2}{\lambda_{n+1}^2}.$
Therefore
$|Au_{n+1}|^2 \leq |Au_n|^2 + 2\frac{\alpha_n^2}{\lambda_{n+1}^2}|u_n - u_{n-1}|^2 + 2\frac{|e_n|^2}{\lambda_{n+1}^2}.$
By (α_5) and (E_4), $\lim_{n\to+\infty} |Au_n|$ exists. Sum up both sides of (3.2) from $n = 1$ to $n = k$, next divide by $\sum_{n=1}^{n=k} \lambda_{n+1}^2$ and then letting $k \to \infty$, by Lemma 2.6, we obtain $\lim_k |Au_k| = 0$. Now, if $u_{n_j} \rightharpoonup q$, then by the demiclosedness of A, we have $q \in A^{-1}(0)$. Hence $\omega_w(u_n) \subset A^{-1}(0)$.
(2) The proof is similar to that of Theorem 3.1 and part (2) of Theorem 2.4. □

4 Subdifferential Case

In this section, under different conditions, we present the weak convergence of $\{u_n\}$, when $A = \partial\varphi$, where $\varphi : H \to]-\infty, +\infty]$ is a proper, convex and lower semi-continuous function. Also the rate of convergence of the sequence $\{\varphi(u_n)\}$ to the minimum value of φ is discussed.

Theorem 4.1. *Let $\{u_n\}$ be a bounded sequence given by* (1.8) *with $A = \partial\varphi$, where $\varphi : H \to]-\infty, +\infty]$ is a proper, convex and lower semi-continuous function. If conditions* (Λ_1), (α_1), (E_1), *and* (E_5) $\sum_{n=1}^{+\infty} \frac{|e_n|^2}{\lambda_{n+1}} < +\infty$ *are satisfied, then* $u_n \rightharpoonup p \in (\partial\varphi)^{-1}(0)$.

Proof. By Theorem 2.7, $A^{-1}(0) \neq \emptyset$. Assume $p \in A^{-1}(0)$. By the subdifferential inequality and (1.8), we get
$\lambda_{n+1}(\varphi(u_{n+1}) - \varphi(p)) \leq < \lambda_{n+1}\partial\varphi(u_{n+1}), u_{n+1} - p >$
$= < (1 - \alpha_n)u_n + \alpha_n u_{n-1} - u_{n+1} + e_n, u_{n+1} - p >$
$= < (1 - \alpha_n)(u_n - p) + \alpha_n(u_{n-1} - p), u_{n+1} - p > + < e_n, u_{n+1} - p > -|u_{n+1} - p|^2$
$\leq \frac{1}{2}|(1 - \alpha_n)(u_n - p) + \alpha_n(u_{n-1} - p)|^2 - \frac{1}{2}|u_{n+1} - p|^2 + |e_n||u_{n+1} - p|$
$\leq \frac{1}{2}(1 - \alpha_n)|u_n - p|^2 + \frac{1}{2}\alpha_n|u_{n-1} - p|^2 - \frac{1}{2}|u_{n+1} - p|^2 + |e_n||u_{n+1} - p|$
$= \frac{1}{2}(|u_n - p|^2 - |u_{n+1} - p|^2) + \frac{1}{2}\alpha_n(|u_{n-1} - p|^2 - |u_n - p|^2) + |e_n||u_{n+1} - p|$

$\leq \frac{1}{2}(|u_n - p|^2 - |u_{n+1} - p|^2) + \frac{1}{2}\alpha_{n-1}|u_{n-1} - p|^2 - \frac{1}{2}\alpha_n|u_n - p|^2$
$+ \frac{1}{2}|u_{n-1} - p|^2[\alpha_n - \alpha_{n-1}]_+ + |e_n||u_{n+1} - p|.$
Hence

$$\sum_{n=1}^{+\infty} \lambda_{n+1}(\varphi(u_{n+1}) - \varphi(p)) < \infty. \tag{4.1}$$

By (4.1), $\liminf_n \varphi(u_n) = \varphi(p)$. On the other hand, by the convexity of φ and the subdifferential inequality, we have
$\varphi(u_{n+1}) - (1-\alpha_n)\varphi(u_n) - \alpha_n\varphi(u_{n-1}) \leq \varphi(u_{n+1}) - \varphi((1-\alpha_n)u_n + \alpha_n u_{n-1})$
$\leq < \partial\varphi(u_{n+1}), u_{n+1} - (1-\alpha_n)u_n - \alpha_n u_{n-1} >$
$= \frac{1}{\lambda_{n+1}} < (1-\alpha_n)u_n + \alpha_n u_{n-1} - u_{n+1} + e_n, u_{n+1} - (1-\alpha_n)u_n - \alpha_n u_{n-1} >$
$= \frac{1}{\lambda_{n+1}} < e_n, u_{n+1} - (1-\alpha_n)u_n - \alpha_n u_{n-1} > - \frac{1}{\lambda_{n+1}}|u_{n+1} - (1-\alpha_n)u_n - \alpha_n u_{n-1}|^2$
$\leq \frac{|e_n|^2}{2\lambda_{n+1}}.$
So

$$\varphi(u_{n+1}) \leq (1-\alpha_n)\varphi(u_n) + \alpha_n\varphi(u_{n-1}) + \frac{|e_n|^2}{2\lambda_{n+1}}. \tag{4.2}$$

Hence, by (4.2), we get
$\varphi(u_{n+1}) \leq \varphi(u_n) + \alpha_{n-1}\varphi(u_{n-1}) - \alpha_n\varphi(u_n) + [\alpha_n - \alpha_{n-1}]_+\varphi(u_{n-1}) + \frac{|e_n|^2}{2\lambda_{n+1}}.$
Thus, by Lemma 2.3, $\lim_n \varphi(u_n)$ exists and hence $\lim_n \varphi(u_n) = \varphi(p)$. Now, if $u_{n_j} \rightharpoonup q$, then $\varphi(q) \leq \liminf_k \varphi(u_k) = \varphi(p)$ implies $q \in (\partial\varphi)^{-1}(0)$. So, $\omega_w(u_n) \subset (\partial\varphi)^{-1}(0)$. In order to prove $u_n \rightharpoonup p$, we show $\omega_w(u_n)$ is singleton. By part (2) of Theorem 2.4, $\lim_n |u_n - p|$, exists for each $p \in \omega_w(u_n)$. Let $p, q \in \omega_w(u_n)$ and $p \neq q$, then $\lim_{n\to+\infty} < u_n, p - q >$ exists. So $p = q$ and hence $\omega_w(u_n)$ is singleton. □

Lemma 4.2 (See [8])**.** *Suppose that $\{a_n\}$ and $\{b_n\}$ are two positive real sequences such that $\{a_n\}$ is non-increasing and convergent to zero and $\sum_{n=1}^{+\infty} a_n b_n < +\infty$. Then $(\sum_{k=1}^{n} b_k)a_n \to 0$ as $n \to +\infty$.*

Theorem 4.3. *Suppose that $\{u_n\}$ is a bounded sequence given by* (1.8) *with $e_n \equiv 0$ and $A = \partial\varphi$, where $\varphi : H \to]-\infty, +\infty]$ is a proper, convex and lower semi-continuous* *function. If* $\sum_{n=1}^{+\infty} \min\{\lambda_n, \lambda_{n+1}\} = +\infty$ *and* (α_1) *are satisfied, then*

$$\varphi(u_n) - \varphi(p) = o\big((\sum_{i=1}^{n+1} \min\{\lambda_i, \lambda_{i+1}\})^{-1}\big),$$

where p *is a minimum point of* φ.

Proof. Set $y_n = \max\{\varphi(u_n) - \varphi(p), \varphi(u_{n-1}) - \varphi(p)\}$. By Theorem 4.1, $y_n \to 0$. By (4.2), we get $\varphi(u_{n+1}) - \varphi(p) \le y_n$, therefore $y_{n+1} \le y_n$. On the other hand, by (4.1), we get

$$\sum_{n=1}^{+\infty} \min\{\lambda_n, \lambda_{n+1}\} y_{n+1} \le \sum_{n=1}^{+\infty} \lambda_n(\varphi(u_n) - \varphi(p)) + \sum_{n=1}^{+\infty} \lambda_{n+1}(\varphi(u_{n+1}) - \varphi(p)) < \infty.$$

So, by Lemma 4.2, $y_{n+1} = o\big((\sum_{i=1}^{n+1} \min\{\lambda_i, \lambda_{i+1}\})^{-1} \big)$. Since $\varphi(u_n) - \varphi(p) \le y_{n+1}$. Hence, $\varphi(u_n) - \varphi(p) = o\big((\sum_{i=1}^{n+1} \min\{\lambda_i, \lambda_{i+1}\})^{-1}\big)$. □

5 Strong Convergence

In this section, the strong convergence of $\{u_n\}$ is obtained by additional conditions on the maximal monotone operator A.

Theorem 5.1. *Assume that* $(I + A)^{-1}$ *is a compact operator and conditions* $(\Lambda_2), (\alpha_1)$ *and* (E_1) *are satisfied. Then* $u_n \to p \in A^{-1}(0)$ *if and only if* $\{u_n\}$ *is bounded.*

Proof. By (3.2) and the assumptions, we get $\liminf_n |Au_n| = 0$. Therefore, there exists a subsequence $\{Au_{n_j}\}$ of $\{Au_n\}$ such that $|Au_{n_j}| \to 0$ and $\{u_{n_j} - Au_{n_j}\}$ is bounded. Since $(I + A)^{-1}$ is compact, $\{u_{n_j}\}$ has a strongly convergent subsequence (we denote this again by $\{u_{n_j}\}$) to $p \in H$. By the monotonicity of A, we have $< Au_n - Au_{n_j}, u_n - u_{n_j} >\ge 0$, so $< Au_n, u_n - p >\ge 0$ as $j \to \infty$. The maximality of A implies $p \in A^{-1}(0)$. On the other hand, by part (2) of Theorem 2.4, $\lim_n |u_n - p|$ exists. Hence $u_n \to p \in A^{-1}(0)$. □

Lemma 5.2 (See [8]). *Assume* $\{y_n\}$ *is a positive real sequence satisfying the following inequality:*

$$b_n y_n \le y_{n-1} - y_n + a_n,$$

where $\{b_n\}$ *and* $\{a_n\}$ *are positive sequences.*
(i) If $\{\frac{a_n}{b_n}\}$ *is bounded, then the sequence* $\{y_n\}$ *is bounded.*

(ii) If $\lim_n \frac{a_n}{b_n} = 0$*, then* $\lim_n y_n$ *exists.*

(iii) If $\lim_n \frac{a_n}{b_n} = 0$ *and* $\sum_{n=1}^{+\infty} b_n = +\infty$*, then* $\lim_n y_n = 0$*.*

Theorem 5.3. *Let* $\{u_n\}$ *be bounded and* A *be a maximal monotone and strongly monotone operator. If the conditions*

$$\big[(\Lambda_1), (E_1) \textit{ and } (\alpha_1)\big] \textit{ or } \Big[(\Lambda_1), (E_2) \textit{ and } (\alpha_6)\frac{\alpha_n}{\lambda_{n+1}} \to 0\Big]$$

are satisfied, then $u_n \to p$*, where* p *is the unique element of* $A^{-1}(0)$*.*

Proof. By Theorem 2.7, $A^{-1}(0) \neq \varnothing$. Assume that p is the single element of $A^{-1}(0)$. By the strong monotonicity of A and (1.8), we get

$$2\alpha\lambda_{n+1}|u_{n+1} - p|^2 \leq 2 < u_n - u_{n+1} - \alpha_n(u_n - u_{n-1}) + e_n, u_{n+1} - p > .$$

It follows that

$$2\alpha\lambda_{n+1}|u_{n+1}-p|^2 \leq |u_n-p|^2-|u_{n+1}-p|^2+\alpha_n(|u_{n-1}-p|^2-|u_n-p|^2)+2|e_n||u_{n+1}-p|. \tag{5.1}$$

If the conditions $\Big[(\Lambda_1), (E_2), (\alpha_6)\frac{\alpha_n}{\lambda_{n+1}} \to 0\Big]$ are satisfied, then by Lemma 5.2, the theorem follows.

If the conditions $\big[(\Lambda_1), (E_1)$ and $(\alpha_1)\big]$ are satisfied, then from (5.1), we have $2\alpha\lambda_{n+1}|u_{n+1}-p|^2 \leq |u_n-p|^2-|u_{n+1}-p|^2+\alpha_{n-1}|u_{n-1}-p|^2-\alpha_n|u_n-p|^2+|u_{n-1}-p|^2[\alpha_n-\alpha_{n-1}]_+ + |e_n||u_{n+1}-p|$.

So, $\sum_{n=1}^{+\infty} \lambda_{n+1}|u_{n+1} - p|^2 < \infty$. Thus $\liminf_n |u_{n+1} - p|^2 = 0$. Since, by part (2) of Theorem 2.4, $\lim_n |u_n - p|$ exists, it is $\lim_n |u_n - p| = 0$. □

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