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Lyapunov Functionals and Stability in Nonlinear Finite Delay Volterra Discrete Systems

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Abstract

In this research, we utilize Lyapunov functionals and obtain sufficient conditions for the stability of the zero solution of the discrete Volterra system of the form

$$x(t+1) = Px(t) + \sum_{s=t-r}^{t-1} C(t,s)g(x(s)).$$

Due to the nature of the Lyapunov functional, we will be able to show that all solutions are $l\big([t_0,\infty)\cap\mathbb{Z}\big)$.

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1 Introduction

In this report, we explore the use of Lyapunov functionals and obtain conditions for the stability of the zero solution of the nonlinear delay Volterra discrete system

$$x(t+1) = Px(t) + \sum_{s=t-r}^{t-1} C(t,s)g(x(s)), \tag{1.1}$$

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where r is a positive integer, P is a constant $n \times n$ matrix and $C(t,s)$ is an $n \times n$ matrix of functions that are defined on $-r \leq t \leq s < \infty$, where $t, s \in [-r, \infty) \cap \mathbb{Z}$. The nonlinear function $g : \mathbb{R}^n \to \mathbb{R}^n$ is continuous in x. Throughout this paper it is understood that if $x \in \mathbb{R}^n$, then $|x|$ is taken to be the Euclidean norm. Recently, several authors Islam and Raffoul [8], Medina [11], [12], [13], and Raffoul [15], obtained stability and boundedness results of the solutions of discrete Volterra equations by means of representing the solution in terms of the resolvent matrix of the corresponding system of difference Volterra equations. Elaydi et al. [6], and Eloe et al. [7] used the notion of total stability and established results on the asymptotic behavior of the solutions of discrete Volterra system with nonlinear perturbation. Their work heavily depended on the summability of the resolvent matrix. The notion of resolvent can not be used for (1.1) since it is nonlinear. For more results on stability of the zero solution of Volterra discrete system we refer the reader to Agarwal and Pang [1], Elaydi [5] and Murakami and Nagabuchi [14].

As we shall see later, Lyapunov functionals allow us to establish the stability or instability of the system. The advantage of this method is the fact that no information need to be known on the actual solution $x(t)$. On the other hand, one of the most disturbing disadvantages is the fact that there is no general method of constructing such Lyapunov functional. In the particular case of homogeneous autonomous systems with constant coefficients, the Lyapunov function can be found as a quadratic form.

One of the major difficulties that we encountered was relating back the solution $x(t)$ to the Lyapunov functional so that some inequality can be obtained.

In order to write a suitable Lyapunov functional V, we will rewrite our system (1.1) so that $\triangle V$ can be easily calculated along the solutions. Moreover, our constructed Lyapunov functional will require the existence of a positive definite matrix that will depend on the coefficient matrix P. In [10], the authors utilized Lyapunov functionals and obtained sufficient conditions for the exponential stability of the zero solution of the scalar finite delay Volterra difference equation

$$x(t+1) = a(t)x(t) + \sum_{s=t-r}^{t-1} b(t,s)x(s),$$

by allowing $|a(t)| \geq 1$. Also, by displaying a slightly different Lyapunov functional the authors arrived at conditions that guaranteed the instability of the zero solution. However, in this article we could not extend the results of [10] to the vector equation (1.1).

Let $x \in \mathbb{R}^n$ and $U = (u)_{ij}$ be an $n \times n$ matrix. Then we define the norms $|x|$ to be the Euclidean norm and

$$|U| = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |u_{ij}|.$$

It should cause no confusion to denote the norm of a sequence function $\varphi : [-r, \infty) \cap$

$\mathbb{Z} \to \mathbb{R}^n$ with

$$||\varphi|| = \sup_{-r \le s < \infty} |\varphi(s)|.$$

The notation x_t means that $x_t(\tau) = x(t+\tau), \tau \in [-r, 0] \cap \mathbb{Z}$ as long as $x(t+\tau)$ is defined. Thus, x_t is a function mapping an interval $[-r, 0] \cap \mathbb{Z}$ into $\mathbb{R}^n$. We say $x(t) \equiv x(t, t_0, \psi)$ is a solution of (1.1) if $x(t)$ satisfies (1.1) for $t \ge t_0$ and $x_{t_0} = x(t_0 + s) = \psi(s)$, $s \in [-r, 0] \cap \mathbb{Z}$. Throughout this paper it is to be understood that when a function is written without its argument, then the argument is t. We begin with a stability definition. For $t_0 \ge 0$ we define

$$E_{t_0} = [-r, t_0] \cap \mathbb{Z}.$$

Let $C(t)$ denote the set of sequences $\phi : [-r, \infty) \cap \mathbb{Z} \to \mathbb{R}^n$ and $\|\phi\| = \sup\{|\phi(s)| : s \in [-r,\ t] \cap \mathbb{Z}\}$.

Definition 1.1. The zero solution of (1.1) is stable if for each $\varepsilon > 0$ and each $t_0 \ge -r$, there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that $[\phi \in E_{t_0} \to \mathbb{R}^n, \phi \in C(t) : \|\phi\| < \delta]$ implies $|x(t, t_0, \phi)| < \varepsilon$ for all $t_0 \ge 0$.

2 Main Results

In this section, our ultimate goal is to construct a suitable Lyapunov functional, say $V(t, x) := V(t)$ for (1.1), and then under suitable conditions, show that along the solutions of (1.1), $\triangle V(t) \le -\alpha |x|^2$ for some positive α. In order to be able to handle the calculations for $\triangle V(t)$ along the solutions of (1.1), we cleverly rewrite (1.1) by letting

$$A(t, s) := \sum_{u=t-s}^{r} C(u + s, s),\ t, s \in [0,\ \infty) \cap \mathbb{Z}\}.$$

We remark that it immediately follows that $A(t, t - r - 1) = 0$. As a consequence, one can easily verify that (1.1) is equivalent to the new system

$$\triangle x(t) = Qx(t) + A(t + 1, t)g(x(t)) - \triangle_t \sum_{s=t-r-1}^{t-1} A(t, s)g(x(s)), \tag{2.1}$$

where the matrix Q is given by $Q = P - I$ and I is the identity $n \times n$ matrix.

Remark 2.1. Writing (1.1) in the form of (2.1) allows us to obtain stability result regarding the Nonlinear Volterra difference equation

$$x(t + 1) = \sum_{s=t-r}^{t-1} C(t, s)g(x(s)). \tag{2.2}$$

This is remarkable since (1.1) is considered as the perturbed form of $x(t + 1) = Px(t)$, which implies that the stability of the zero solution of (1.1) depends on the stability of linear part; that is, one must require that the magnitude of all the eigenvalues of the matrix A be inside the unit circle.

Before we state and prove our next theorem, we assume there exist a positive definite symmetric and constant $n \times n$ matrix D such that for positive constants λ, μ_1 and μ_2 we have

$$P^T DQ + Q^T D = -\mu_1 I. \tag{2.3}$$

Due to the nonlinearity of the function g, we require that

$$x^T \big(P^T DA(t+1,t) + DA(t+1,t)\big) g(x) \leq -\mu_2 |x|^2 \text{ if } x \neq 0, \tag{2.4}$$

and

$$|g(x)| \leq \lambda |x|. \tag{2.5}$$

It is clear that conditions (2.4) and (2.5) imply that $g(0) = 0$ and hence $x = 0$ is a solution for system (1.1). We note that since D is a positive definite symmetric matrix, then there exists a positive constant k such that

$$k|x|^2 \leq x^T Dx, \quad \text{for all } x. \tag{2.6}$$

Finally, we use the fact that if $W(t)$ and $Z(t)$ are two sequences, then $\triangle W(t)Z(t) = W(t+1)\triangle Z(t) + \big(\triangle W(t)\big)Z(t)$. For more on the calculus of difference equations we refer the reader to [4] and [9].

Theorem 2.2. *Let* (2.3)–(2.5) *hold, and suppose there are constants* $\gamma > 0$ *and* $\alpha > 0$ *so that*

$$-\mu_1 - 2\mu_2 + \gamma r \lambda^2 |A(t+1,t)| + \big(\lambda |A^T(t+1,t)D| + |Q^T D|\big) \sum_{s=t-r}^{t-1} |A(t,s)| \leq -\alpha, \tag{2.7}$$

$$-\gamma + \lambda |A^T(t+1,t)D| + |Q^T D| \leq 0, \tag{2.8}$$

and

$$1 - \lambda \sum_{s=t-r-1}^{t-1} |A(t,s)| > 0 \tag{2.9}$$

then, the zero solution of (1.1) *is stable.*

Proof. Define the Lyapunov functional $V(t) = V(t,x)$ by

$$\begin{aligned} V(t) &= \Big(x(t) + \sum_{s=t-r-1}^{t-1} A(t,s) g(x(s))\Big)^T D \Big(x(t) + \sum_{s=t-r-1}^{t-1} A(t,s) g(x(s))\Big) \\ &\quad + \gamma \sum_{s=-r}^{-1} \sum_{z=t+s}^{t-1} |A(t,z)||g(x(z))|^2. \end{aligned} \tag{2.10}$$

First we note that the right side of (2.10) is scalar. Let $x(t) = x(t, t_0, \psi)$ be a solution of (1.1) and define $V(t)$ by (2.10). Then along solutions of (1.1) we have

$$\begin{aligned} \triangle V(t) &= \Big(x(t+1) + \sum_{s=t-r}^{t} A(t+1,s)g(x(s))\Big)^T D \\ &\times \triangle\Big(x(t) + \sum_{s=t-r-1}^{t-1} A(t,s)g(x(s))\Big) \\ &+ \triangle\Big(x(t) + \sum_{s=t-r-1}^{t-1} A(t,s)g(x(s))\Big)^T \\ &\times D\Big(x(t) + \sum_{s=t-r-1}^{t-1} A(t,s)g(x(s))\Big) \\ &+ \gamma r|A(t+1,t)||g(x(t))|^2 - \gamma \sum_{s=-r}^{-1} |A(t,t+s)||g(x(t+s))|^2. \quad (2.11) \end{aligned}$$

Using (2.1) one can easily show that

$$x(t+1) + \sum_{s=t-r}^{t} A(t+1,s)g(x(s)) = Px(t) + \sum_{s=t-r}^{t-1} A(t,s)g(x(s)).$$

With this in mind inequality (2.11) becomes

$$\begin{aligned} \triangle V(t) &= \Big(Px(t) + \sum_{s=t-r}^{t-1} A(t,s)g(x(s))\Big)^T D\Big(Qx(t) + A(t+1,t)g(x(t))\Big) \\ &+ \Big(Qx(t) + A(t+1,t)g(x(t))\Big)^T D\Big(x(t) + \sum_{s=t-r-1}^{t-1} A(t,s)g(x(s))\Big) \\ &+ \gamma r\lambda^2|A(t+1,t)||g(x(t))|^2 - \gamma \sum_{s=-r}^{-1} |A(t,t+s)||g(x(t+s))|^2. \end{aligned}$$

After rearranging terms, the above expression simplifies to

$$\begin{aligned}\triangle V(t) &= x^T(t)\big(P^TDQ+Q^TD\big)x(t)+x^T(t)P^TDA(t+1,t))g(x(t))\\ &+ \big(\sum_{s=t-r}^{t-1}A(t,s)g(x(s))\big)^T DQx(t)\\ &+ \big(\sum_{s=t-r}^{t-1}A(t,s)g(x(s))\big)^T DA(t+1,t)g(x(t))\\ &+ x^T(t)Q^TD\sum_{s=t-r-1}^{t-1}A(t,s)g(x(s))+g^T(x(t))A^T(t+1,t)Dx(t)\\ &+ g^T(x(t))A^T(t+1,t)D\sum_{s=t-r-1}^{t-1}A(t,s)g(x(s))\\ &+ \gamma r|A(t+1,t)||g(x(t))|^2-\gamma\sum_{s=-r}^{-1}|A(t,t+s)||g(x(t+s))|^2. \qquad (2.12)\end{aligned}$$

Next we try simplify (2.12) by combining likewise terms. We begin by noting that $g^T(x)A^T(t+1,t)Dx=\big[x^TDA(t+1,t))g(x)\big]^T$, and hence we have

$$\begin{aligned}x^TP^TDA(t+1,t))g(x) &+ g^T(x)A^T(t+1,t)Dx\\ &= x^T\big(P^TDA(t+1,t)+DA(t+1,t)\big)g(x).\end{aligned}$$

$$\begin{aligned}\big(\sum_{s=t-r}^{t-1}A(t,s)g(x(s))\big)^T DQx(t) &+ x^T(t)Q^TD\sum_{s=t-r-1}^{t-1}A(t,s)g(x(s))\\ &= x^T(t)Q^TD\sum_{s=t-r-1}^{t-1}A(t,s)g(x(s))\\ &+ \Big[\big(\sum_{s=t-r}^{t-1}A(t,s)g(x(s))\big)^T DQx(t)\Big]^T\\ &= 2x^T(t)Q^TD\sum_{s=t-r-1}^{t-1}A(t,s)g(x(s))\\ &\le 2|x^T(t)||Q^TD|\sum_{s=t-r}^{t-1}|A(t,s)||g(x(s))|\\ &\le |Q^TD|\sum_{s=t-r}^{t-1}|A(t,s)|\big(|x(t)|^2+|g(x(s))|^2\big),\end{aligned}$$

where have used the inequality $2ab \leq a^2 + b^2$. Similarly,

$$
\begin{aligned}
& \sum_{s=t-r}^{t-1} A(t,s)g(x(s))\Big)^T DA(t+1,t)g(x(t)) \\
+ \quad & g^T(x(t))A^T(t+1,t)D \sum_{s=t-r-1}^{t-1} A(t,s)g(x(s)) \\
= \quad & g^T(x(t))A^T(t+1,t)D \sum_{s=t-r-1}^{t-1} A(t,s)g(x(s)) \\
+ \quad & \Big[\Big(\sum_{s=t-r}^{t-1} A(t,s)g(x(s))\Big)^T DA(t+1,t)g(x(t))\Big]^T \\
= \quad & 2g^T(x(t))A^T(t+1,t)D \sum_{s=t-r-1}^{t-1} A(t,s)g(x(s)) \\
\leq \quad & 2\lambda|x(t)||A^T(t+1,t)D| \sum_{s=t-r}^{t-1} |A(t,s)||g(x(s))| \\
\leq \quad & \lambda|A^T(t+1,t)D| \sum_{s=t-r}^{t-1} |A(t,s)|\big(|x(t)|^2 + |g(x(s))|^2\big).
\end{aligned}
$$

Let $u = t + s$, then

$$
\gamma \sum_{s=-r}^{-1} |A(t,t+s)||g(x(t+s))|^2 = -\gamma \sum_{s=t-r}^{t-1} |A(t,s)||g(x(s))|^2.
$$

By substituting the above four simplified expressions into (2.12) yields

$$
\begin{aligned}
\triangle V(t) \quad \leq \quad & \Big[-\mu_1 - \mu_2 + \gamma r \lambda^2 |A(t+1,t)| \\
+ \quad & \big(\lambda|A^T(t+1,t)D| + |Q^T D|\big) \sum_{t-r}^{t-1} |A(t,s)|\Big] |x(t)|^2 \\
+ \quad & \Big[-\gamma + \lambda|A^T(t+1,t)D| + |Q^T D|\Big] \sum_{t-r}^{t-1} |A(t,s)||g(x(s))|^2. \\
\leq \quad & -\alpha|x(t)|^2. \qquad (2.13)
\end{aligned}
$$

Let $\varepsilon > 0$ be given, we will find $\delta > 0$ so that $|x(t,t_0,\phi)| < \varepsilon$ as long as $[\phi \in E_{t_0} \to \mathbb{R} : \|\phi\| < \delta]$. Let

$$
L^2 = |D|\Big(1 + \lambda \sum_{t_0-r}^{t_0-1} |A(t_0,s)|\Big)^2 + \lambda^2 \nu \sum_{s=-r}^{-1} \sum_{z=t_0+s}^{t_0-1} |A(t_0,z)|.
$$

By (2.13) we have V is decreasing and hence for $t \geq t_0 \geq 0$ we have that

$$\begin{aligned} V(t,x) &\leq V(t_0,\phi) \\ &\leq |D|(\phi(t_0) + \sum_{t_0-r}^{t_0-1} A(t_0,s)g(\phi(s))\Big)^2 \\ &+ \nu\lambda^2 \sum_{s=-r}^{-1} \sum_{z=t_0+s}^{t_0-1} |A(t_0,z)||\phi(z)|^2 \\ &= \delta^2 \, |D|\Big(1 + \lambda \sum_{t_0-r}^{t_0-1} |A(t_0,s)|\Big)^2 \\ &+ \nu\lambda^2 \, \delta^2 \sum_{s=-r}^{-1} \sum_{z=t_0+s}^{t_0-1} |A(t_0,z)| \\ &\leq \delta^2 L^2. \end{aligned} \tag{2.14}$$

By (2.10), we have

$$\begin{aligned} V(t,x) &\geq \Big(x(t) + \sum_{s=t-r-1}^{t-1} A(t,s)g(x(s))\Big)^T \\ &\times D\Big(x(t) + \sum_{s=t-r-1}^{t-1} A(t,s)g(x(s))\Big) \\ &\geq k^2\Big(|x| - |\sum_{s=t-r-1}^{t-1} A(t,s)g(x(s))|\Big)^2. \end{aligned} \tag{2.15}$$

Combining the two inequalities (2.14) and (2.15) we arrive at

$$|x(t)| \leq \frac{\delta L}{k} + \lambda \sum_{s=t-r-1}^{t-1} |A(t,s)||x(s)|.$$

So as long as $|x(t)| < \varepsilon$, we have

$$|x(t)| < \frac{\delta L}{k} + \varepsilon\lambda \sum_{s=t-r-1}^{t-1} |A(t,s)|, \text{ for all } t \geq t_0.$$

Thus, we have from the above inequality

$$|x(t)| < \varepsilon \text{ for } \delta < \frac{k}{L}\big(1 - \lambda \sum_{s=t-r-1}^{t-1} |A(t,s)|\big)\varepsilon.$$

Note that by (2.9), the above inequality regarding δ is valid. □

We have the following corollary.

Corollary 2.3. *Assume all the conditions of Theorem 2.2 hold. Let $x(t)$ be any solution of* (1.1). *Then* $|x(t)|^2 \in l\big([t_0, \infty) \cap \mathbb{Z}\big)$.

Proof. We know from Theorem 2.2 that the zero solution is stable. Thus, for the same δ of stability, we take $x(t, t_0, \phi)| < 1$. Since V is decreasing, we have by summing (2.13) from t_0 to $t-1$ and using (2.14) that,

$$V(t,x) \leq V(t_0, \phi) \leq \delta^2 L^2 - \alpha \sum_{t_0}^{t-1} |x(s)|^2.$$

Since,

$$V(t,x) \geq \Big(x + \sum_{s=t-r-1}^{t-1} A(t,s) g(x(s))\Big)^T D\Big(x + \sum_{s=t-r-1}^{t-1} A(t,s) g(x(s))\Big),$$

we have that

$$\begin{aligned} &\Big(x + \sum_{s=t-r-1}^{t-1} A(t,s) g(x(s))\Big)^T D\Big(x + \sum_{s=t-r-1}^{t-1} A(t,s) g(x(s))\Big) \\ \leq\ & \delta^2 L^2 - \alpha \sum_{t_0}^{t-1} |x(s)|^2. \end{aligned} \tag{2.16}$$

Also, using Schwarz inequality one obtains

$$\begin{aligned} \Big(\sum_{s=t-r-1}^{t-1} |A(t,s)||g(x(s))|\Big)^2 &= \Big(\sum_{s=t-r-1}^{t-1} |A(t,s)|^{1/2} |A(t,s)|^{1/2} |g(x(s))|\Big)^2 \\ &\leq \lambda^2 \sum_{s=t-r-1}^{t-1} |A(t,s)| \sum_{s=t-r-1}^{t-1} |A(t,s)||x(s)|^2. \end{aligned}$$

As $\sum_{s=t-r-1}^{t-1} |A(t,s)|$ is bounded by (2.9) and $|x|^2 < 1$, we have $\sum_{s=t-r-1}^{t-1} |A(t,s)||x(s)|^2$ is bounded and hence $\sum_{s=t-r-1}^{t-1} |A(t,s)||g(x(s))|$ is bounded. Therefore, from (2.16), we arrive at

$$\begin{aligned} \alpha \sum_{s=t_0}^{t-1} |x(s)|^2 &\leq \delta^2 L^2 - \Big(x + \sum_{s=t-r-1}^{t-1} A(t,s) g(x(s))\Big)^T \\ &\times D\Big(x + \sum_{s=t-r-1}^{t-1} A(t,s) g(x(s))\Big) \\ &\leq \delta^2 L^2 + |D|\Big(|x| + \sum_{s=t-r-1}^{t-1} A(t,s) g(x(s))|\Big)^2 \leq K, \end{aligned}$$

from which we deduce that $|x(t)|^2 \in l\big([t_0,\infty)\cap\mathbb{Z}\big)$. □

Due to our previous remark, it is straight forward to extend Theorem 2.2 and Corollary 2.3 to (2.2) by taking the coefficient matrix $P = 0$.

Theorem 2.4. *Let* (2.3) *and* (2.4) *hold for* $P = 0$ *matrix. Assume* (2.5) *and suppose there are constants* $\gamma > 0$ *and* $\alpha > 0$ *so that*

$$\begin{aligned} -\mu_1 - \mu_2 \;&+\; \gamma r\lambda^2|A(t+1,t)| + \big(\lambda|A^T(t+1,t)D| \\ &+\; |D|\big)\sum_{t-r}^{t-1}|A(t,s)| \le -\alpha, \end{aligned} \tag{2.17}$$

$$-\gamma + \lambda|A^T(t+1,t)D| + |D| \le 0, \tag{2.18}$$

and

$$1 - \lambda \sum_{s=t-r-1}^{t-1} |A(t,s)| > 0 \tag{2.19}$$

then, the zero solution of (2.2) *is stable and* $|x(t)|^2 \in l\big([t_0,\infty)\cap\mathbb{Z}\big)$

Proof. The proof is immediate consequence of Theorem 2.2 and Corollary 2.3 by taking the matrix P to be the zero matrix which implies that $Q = I$. □

Next, we resort to fundamental matrix solution to characterize solutions of (1.1) and then compare both results. We begin by considering the homogenous system,

$$x(t+1) = A(t)x(t) \tag{2.20}$$

where $A(t) = (a_{ij}(t)$ is $n\times n$ nonsingular matrix function. Then if $\Phi(t)$ is a matrix that is nonsingular for all $t \ge t_0$ and satisfies (2.20), then it is said to be a fundamental matrix for (2.20). Also, it is known that if all eigenvalues of $A(t)$ reside inside the unit circle, then there exist positive constants l and $\eta \in (0,1)$ such that $|\Phi(t)\Phi^{-1}(t_0)| \le l\eta^{t-t_0}$. For more on linearization of systems of the form of (2.20), we refer the reader to [4]. Suppose the function g is Lipschitz. That is, there exists a positive constant L such that

$$|g(x) - g(y)| \le L|x-y| \tag{2.21}$$

for all x and y. Then (2.21) along with $g(0) = 0$ imply that $|g(x)| \le L|x|$. Now we remind the reader of the meaning of exponentially stable.

Definition 2.5. The zero solution of (1.1) is said to be exponentially stable if any solution $x(t,t_0,\phi)$ of (1.1) satisfies

$$|x(t,t_0,\phi)| \le B\Big(||\phi||, t_0\Big)\zeta^{\gamma(t-t_0)}, \quad \text{for all } t \ge t_0,$$

where ζ is constant with $0 < \zeta < 1$, $B : \mathbb{R}^+\times\mathbb{Z}^+ \to \mathbb{R}^+$, and γ is a positive constant. The zero solution of (1.1) is said to be uniformly exponentially stable if B is independent of t_0.

Theorem 2.6. *Assume all eigenvalues of $A(t)$ of system* (2.20) *reside inside the unit circle. Also, assume* (2.21) *along with* $g(0) = 0$. *In addition we ask that for some positive constant* R

$$\sum_{s=-r}^{\infty} |C(u,s)| \le R, \tag{2.22}$$

then the zero solution of (1.1) *is exponentially stable provided that* $RL < \frac{1-\eta}{l}$.

Proof. Let $\Phi(t)$ be the fundamental matrix for (2.20). For a given initial function $\phi : [-r,\infty) \cap \mathbb{Z} \to \mathbb{R}^n$, by using the Variation of parameters, we have that

$$x(t) = \Phi(t)\Phi^{-1}(t_0)\phi(t_0) + \sum_{u=t_0}^{t-1} \Phi(t)\Phi^{-1}(u+1) \sum_{s=u-r}^{u-1} C(u,s)g(x(s)). \tag{2.23}$$

Then $x(t)$ given by (2.23) is a solution of (1.1), see [4] or [9]. Hence, for $t \ge t_0$, we have

$$|x(t)| \le l\eta^{t-t_0}|\phi(t_0)| + RLl\eta^{t-1} \sum_{u=t_0}^{t-1} \eta^{-u}|x(u)|.$$

The rest of the proof follows along the line of Theorem 4.35 of [4], by invoking Gronwall's inequality. □

Theorem 2.6 gives stronger type of stability since it requires the zero solution of (2.20) to be exponentially stable.

3 Example

In this section, we furnish an example in which we show the power of our obtained results. Let

$P = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$ and $C(t,s) = \begin{pmatrix} 1/3 & 0 \\ 0 & 1/3 \end{pmatrix}$, then

$A(t,s) = \begin{pmatrix} \frac{1}{3}(r-t+s+1) & 0 \\ 0 & \frac{1}{3}(r-t+s+1) \end{pmatrix}$ and

$A(t+1,t) = \begin{pmatrix} \frac{1}{3}r & 0 \\ 0 & \frac{1}{3}r \end{pmatrix}$.

From $P^T DQ + Q^T D = -\mu_1 I$, we obtain

$$D = \begin{pmatrix} \frac{4}{3}\mu_1 & 0 \\ 0 & \frac{4}{3}\mu_1 \end{pmatrix}. \text{ Let } g(x) = \begin{pmatrix} \frac{-9\mu_2}{8\mu_1 r} x_1 \\ \\ \frac{-9\mu_2}{8\mu_1 r} x_2 \end{pmatrix}.$$

Then

$$x^T\big(P^T DA(t+1,t) + DA(t+1,t)\big)g(x) = -\mu_2\left(x_1^2 + x_2^2\right).$$

Hence (2.4) is satisfied. By letting $\frac{9\mu_2}{8r\mu_1} \le \lambda < \frac{3}{r(r+1)}$ we have that $|g(x)| \le \lambda\,|x|$. For the sake of verifying (2.9), we note that

$$|A(t,s)| \le \frac{1}{3}|r - t + s + 1| \le \frac{r}{3}, \text{ for } s \in [t-r, t-1].$$

Thus,

$$\sum_{s=t-r-1}^{t-1} |A(t,s)| \le \sum_{s=t-r-1}^{t-1} \frac{r}{3} \le \frac{r(r+1)}{3}.$$

Thus, $1 - \lambda \sum_{s=t-r-1}^{t-1} |A(t,s)| > 0$ for $\lambda < \frac{3}{r(r+1)}$. Left to verify conditions (2.7) and (2.8). As before, by simple calculations one can easily show that (2.7) and (2.8) correspond to

$$-\mu_1 - 2\mu_2 + \gamma r\lambda^2\frac{r}{3} + \frac{4\lambda r\mu_1}{9} + \frac{2}{3}\mu_1(\frac{r}{3}) \le -\alpha, \tag{3.1}$$

and

$$-\gamma + \frac{4\lambda r\mu_1}{9} + \frac{2}{3}\mu_1 \le 0, \tag{3.2}$$

respectively. Now inequalities (3.1) and (3.2) can be satisfied by the choice of appropriate μ_1, μ_2 and r. Thus we have shown that the zero solution of

$$x(t+1) = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} x(t) - \sum_{s=t-r}^{t-1} \begin{pmatrix} 1/3 & 0 \\ 0 & 1/3 \end{pmatrix} \begin{pmatrix} \frac{-9\mu_2}{8\mu_1 r} x_1 \\ \\ \frac{-9\mu_2}{8\mu_1 r} x_2 \end{pmatrix}$$

is stable by invoking Theorem 2.2.

4 Open Problem

Extend the method of [10] to (1.1) to obtain exponential stability of the zero solution.

The paper of [10] dealt with scalar equations, unlike this paper which deals with systems. Therefore, constructing a Lyapunov functional that is based on the idea of [10] is not obvious and serious thinking has to be put in to it. The authors strongly believe that solving this Open Problem will be very rewarding.

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