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Dynamics of Rational Systems in the Plane

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Abstract

We investigate the dynamics of some rational systems in the plane.

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1 Introduction

We investigate the dynamics of some rational systems in the plane. For some work on difference equations and systems of difference equations see [1-23].

In [6] it was established that every "positive" solution of the rational system

$$x_{n+1} = \frac{x_n}{y_n} \text{ and } y_{n+1} = x_n + \gamma y_n, \quad n = 0, 1, \ldots, \tag{1.1}$$

that is, every solution with positive initial conditions x_0, y_0, is bounded when $\gamma \in (0, 1)$. We also established that the dynamics of System (1.1) are topologically conjugate with the dynamics of the system

$$x_{n+1} = \frac{x_n w_n}{\gamma + w_n} \text{ and } w_{n+1} = \frac{w_n}{\gamma + x_n}, \quad n = 0, 1, \ldots \tag{1.2}$$

provided that the initial conditions are positive and that $\gamma \in (0, \infty)$.

The dynamics of the two systems (1.1) and (1.2) can be studied simultaneously, through the system

$$x_{n+1} = \frac{x_n}{y_n} = \frac{x_n w_n}{\gamma + x_n}, \; y_{n+1} = x_n + \gamma y_n, \text{ and } w_{n+1} = \frac{w_n}{\gamma + x_n}, \quad n = 0, 1, \ldots . \tag{1.3}$$

The next theorem will be useful in the sequel.

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Theorem 1.1 (See [6]). *Let* $\{x_n, y_n, w_n\}_{n=0}^{\infty}$ *be a positive solution of* (1.3). *Assume that for an infinite sequence of indices* $\{k_i\}_{i=1}^{\infty}$, $\{x_{k_i}\}_{i=1}^{\infty}$ *and* $\{y_{k_i}\}_{i=1}^{\infty}$ *converge to finite limits and that*

$$\lim_{i\to\infty} \frac{x_{k_i} + \gamma}{y_{k_i}} = \lim_{i\to\infty} w_{k_i} = M \in (0, \infty).$$

Then, the double limit and the two iterated limits of the double sequence

$$\phi(k_i, n) = \sum_{k=1}^{n} \gamma^{k-1} x_{k+k_i}, \quad i = 1, 2, \dots, \quad n = 1, 2, \dots,$$

exist and they are equal with M*, that is,*

$$\lim_{i,n\to\infty} \phi(k_i, n) = \lim_{i\to\infty} \lim_{n\to\infty} \phi(k_i, n) = \lim_{n\to\infty} \lim_{i\to\infty} \phi(k_i, n) = M.$$

Furthermore,

$$\lim_{i\to\infty} x_{k_i} \in (0, \infty).$$

Given a solution $\{x_n, y_n, w_n\}_{n=0}^{\infty}$ of (1.3), the paired sequence $\{\gamma^{n-1} x_n, \gamma^n w_n\}_{n=0}^{\infty}$, which converges to (0,0), is a solution of the system

$$\left.\begin{array}{rcl} x_{n+1} & = & \dfrac{x_n w_n}{\gamma^n + x_n} \\[2ex] w_{n+1} & = & \dfrac{\gamma^n w_n}{\gamma^n + x_n} \end{array}\right\}, \quad n = 0, 1, \dots . \tag{1.4}$$

Conversely, given a solution $\{x_n, w_n\}_{n=0}^{\infty}$ of (1.4), the sequence $\{\frac{x_n}{\gamma^{n-1}}, \gamma \cdot \frac{\gamma^n + x_n}{w_n}, \frac{w_n}{\gamma^n}\}_{n=0}^{\infty}$ is a solution of (1.3). Since, every solution of System (1.3) is bounded, in particular the sequence $\{\frac{x_n}{\gamma^{n-1}}, \frac{w_n}{\gamma^n}\}_{n=0}^{\infty}$ is bounded, from which it follows that the solution $\{x_n, w_n\}_{n=0}^{\infty}$ of (1.4) converges to $(0, 0)$.

System (1.4) is interesting on its own right. In Section 2, we study the dynamics of System (1.4) independently from System (1.3) and we establish that every positive solution of System (1.4) converges to $(0, 0)$.

An important feature that was between the lines of the proof of boundedness of positive solutions of System (1.1) and (1.2), which was presented in [6], was that the component $\{x_n\}_{n=0}^{\infty}$ of every positive solution is bounded from below by a positive constant. However, due to the fact that the proof was given in the form of a contradiction it was not actually established. In Section 3 of this paper we will present this result together with the oscillatory behavior of solutions of Systems (1.1) and (1.2) and we establish that the sequence of indices $\{k_i\}_{i=1}^{\infty}$ described in Theorem 1.2 exists.

Finally, in Section 4 we present a detailed analysis of the convergence behavior of solutions of the system

$$x_{n+1} = \frac{x_n}{y_n} = \frac{x_n w_n}{1 + x_n}, \quad y_{n+1} = x_n + y_n, \quad \text{and} \quad w_{n+1} = \frac{w_n}{1 + x_n}, \quad n \in Z$$

when x_0, y_0 and w_0 are positive real numbers.

2 Null Sequences

In this section we present the dynamics of solutions of System (1.4). The theorem below is the main result in this section.

Theorem 2.1. *Let* $\{x_n, w_n\}$ *be a positive solution of* (1.4). *Then*

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} w_n = 0. \tag{2.1}$$

Before we present the proof of the theorem we need the following lemma.

Lemma 2.2. *Let* $\{x_n, w_n\}$ *be a positive solution of* (1.4). *Then the following hold:*

$$w_{n+1} < x_{n+1} + w_{n+1} = w_n, \text{ for all } n \geq 0. \tag{2.2}$$

$$x_{n+1} < w_n, \text{ for all } n \geq 0. \tag{2.3}$$

$$w_{n+1} < \left(\frac{1}{x_0}\right)^{\frac{1}{n}} \cdot \gamma^{\frac{n+1}{2}}, \text{ for all } n \geq 2. \tag{2.4}$$

.

Proof. Indeed, for all $n \geq 0$,

$$w_{n+1} < x_{n+1} + w_{n+1} = \frac{x_n w_n}{\gamma^n + x_n} + \frac{\gamma^n w_n}{\gamma^n + x_n} = w_n$$

which establishes (2.2). From the first equation of the system

$$x_{n+1} = \frac{x_n w_n}{\gamma^n + x_n} < \frac{x_n w_n}{x_n} = w_n, \text{ for all } n \geq 0.$$

By dividing the two equations of System (1.4) side by side we have

$$\frac{x_{n+1}}{w_{n+1}} = \frac{x_n}{\gamma^n}, \text{ for all } n \geq 0 \Rightarrow x_n = \gamma^n \frac{x_{n+1}}{w_{n+1}}, \text{ for all } n \geq 0.$$

Thus,

$$x_0 = \frac{1}{w_1} \cdot x_1 = \frac{1}{w_1 w_2} \cdot \gamma x_2$$

$$= \frac{1}{w_1 w_2 w_3} \cdot \gamma \cdot \gamma^2 x_3 = \ldots = \frac{1}{w_1 \cdots w_{n+1}} \cdot \gamma \cdots \gamma^n x_{n+1} = \frac{\gamma^{\frac{n(n+1)}{2}} x_{n+1}}{w_1 \cdots w_{n+1}}.$$

By isolating the term x_{n+1}, and with the use of (2.3) we have

$$w_1 \cdots w_{n-1} w_{n+1} < \frac{1}{x_0} \cdot \gamma^{\frac{n(n+1)}{2}}, \text{ for all } n \geq 2.$$

From this and (2.2) we have

$$w_{n+1}^n < w_1 \cdots w_{n-1} w_{n+1} < \frac{1}{x_0} \cdot \gamma^{\frac{n(n+1)}{2}}, \text{ for all } n \geq 2,$$

from which (2.4) follows. The proof is complete. □

Now we present the proof of the theorem.

Proof. The proof is a consequence of (2.4) and (2.3). The proof is complete. □

Remark 2.3. Given a positive solution $\{x_n, w_n\}_{n=0}^{\infty}$ of System (1.4) and by setting

$$x_{-n-1} = \gamma^{-n-1}\frac{x_{-n}}{w_{-n}} \text{ and } w_{-n-1} = x_{-n} + w_{-n}, \text{ for all } n \geq 0$$

the sequence $\{x_n, w_n\}_{n=-\infty}^{\infty}$ is a positive solution of the system

$$x_{n+1} = \frac{x_n w_n}{\gamma^n + x_n} \text{ and } w_{n+1} = \frac{\gamma^n w_n}{\gamma^n + x_n}, \quad n \in Z. \tag{2.5}$$

Also, it is easy to see that

$$\lim_{n\to-\infty} x_n = \lim_{n\to-\infty} w_n = \infty.$$

3 Oscillation and Boundedness

In this section we present the oscillatory behavior of all positive and bounded solutions of System (1.3) and we also establish that the component $\{x_n\}_{n=0}^{\infty}$ of every positive solution $\{x_n, y_n, w_n\}_{n=0}^{\infty}$ of System (1.3) is bounded from below by a positive constant. The next theorem is the main result in this section.

Theorem 3.1. *Let* $\{x_n, y_n, w_n\}_{n=0}^{\infty}$ *be a nonconstant positive solution of System* (1.3).
1. The sequence $\{x_n\}_{n=0}^{\infty}$ *oscillates about* $1-\gamma$ *and the sequences* $\{y_n\}_{n=0}^{\infty}$ *and* $\{w_n\}_{n=0}^{\infty}$ *oscillate about 1.*

2. Assume that for some integer $N \geq 0$,

$$0 < x_N < 1-\gamma \text{ and } 0 < y_N < 1.$$

Then there exists a positive integer $N_1 > N$, *such that*

$$x_{N_1} \geq 1-\gamma, \quad 0 < y_{N_1} < 1.$$

In addition,

$$x_n \uparrow \text{ from } x_N \text{ to } x_{N_1}, \quad y_n \downarrow \text{ from } y_N \text{ to } y_{N_1-1},$$
$$w_n \uparrow \text{ from } w_N \text{ to } w_{N_1}, \text{and } 0 < y_{N_1-1} \leq y_{N_1} < 1.$$

3. Assume that for some integer $N \geq 0$,

$$x_N \geq 1-\gamma \text{ and } 0 < y_N < 1.$$

Then there exists a positive integer $N_1 > N$, *such that*

$$x_{N_1} > 1 - \gamma, \;\; y_{N_1} \geq 1.$$

In addition,

$$x_n \uparrow \;\text{from}\; x_N \;\text{to}\; x_{N_1}, \;\; y_n \uparrow \;\text{from}\; y_N \;\text{to}\; y_{N_1},$$
$$w_n \downarrow \;\text{from}\; w_N \;\text{to}\; w_{N_1}.$$

4. Assume that for some integer $N \geq 0$,

$$x_N \geq 1 - \gamma \;\text{and}\; y_N \geq 1.$$

Then there exists a positive integer $N_1 > N$, *such that*

$$0 < x_{N_1} < 1 - \gamma, \;\; y_{N_1} > 1.$$

In addition,

$$x_n \downarrow \;\text{from}\; x_N \;\text{to}\; x_{N_1}, \;\; y_n \uparrow \;\text{from}\; y_N \;\text{to}\; y_{N_1 - 1},$$
$$w_n \downarrow \;\text{from}\; w_N \;\text{to}\; w_{N_1}, \;\text{and}\; y_{N_1} < y_{N_1 - 1}.$$

5. Assume that for some integer $N \geq 0$,

$$0 < x_N < 1 - \gamma \;\text{and}\; y_N \geq 1.$$

Then there exists a positive integer $N_1 > N$, *such that,*

$$0 < x_{N_1} < 1 - \gamma, \;\; \gamma < y_{N_1} < 1.$$

In addition,

$$x_n \downarrow \;\text{from}\; x_N \;\text{to}\; x_{N_1}, \;\; y_n \downarrow \;\text{from}\; y_N \;\text{to}\; y_{N_1},$$
$$w_n \uparrow \;\text{from}\; w_N \;\text{to}\; w_{N_1}.$$

6. There exists an infinite sequence of indices $\{k_i\}_{i=1}^{\infty}$, *for which*

$$0 < x_{k_i - 1} < 1 - \gamma, \;\; 1 \leq y_{k_i - 1} < \frac{1}{\gamma}, \;\text{for all}\; i \geq 1, \tag{3.1}$$

and

$$0 < x_{k_i} < 1 - \gamma, \;\; \gamma < y_{k_i} < 1, \;\text{for all}\; i \geq 1. \tag{3.2}$$

In addition,

$$\gamma < w_{k_i} \leq 1$$

and

$$\liminf_{n \to \infty} x_n = \liminf_{i \to \infty} x_{k_i} \in (0, \infty),$$

that is the sequence $\{x_n\}_{n=0}^{\infty}$ *is bounded from below by a positive constant.*

Proof. We establish that $\{x_n\}_{n=0}^{\infty}$ oscillates about $1-\gamma$. The proof that the two sequences $\{y_n\}_{n=0}^{\infty}$ and $\{w_n\}_{n=0}^{\infty}$ oscillate about 1 is similar and the details are omitted. Assume for the sake of contradiction that there exists an integer $N \geq 0$ such that

$$x_n \geq 1-\gamma, \quad \text{for all} \quad n \geq N \tag{3.3}$$

or

$$x_n \leq 1-\gamma, \quad \text{for all} \quad n \geq N. \tag{3.4}$$

We will give the proof when (3.3) holds. The proof when (3.4) holds is similar and the details are omitted. Assuming that (3.3) holds and from

$$y_{n+1} = x_n + \gamma y_n = y_n(\frac{x_n}{y_n} + \gamma) = y_n(x_{n+1} + \gamma), \quad \text{for all} \quad n \geq 0,$$

we have

$$x_{n+1} \geq 1-\gamma, \quad \text{for all} \quad n \geq N \Rightarrow y_{n+1} \geq y_n, \quad \text{for all} \quad n \geq N \tag{3.5}$$

and so we may assume that

$$\lim_{n\to\infty} y_n = y \in (0, \infty].$$

Case 1.

$$y = +\infty.$$

Then there exists an arbitrarily large positive integer $N_1 > N$ such that

$$y_n > \frac{1}{\gamma}, \quad \text{for all} \quad n \geq N_1 > N$$

and consequently

$$x_{n+1} = \frac{x_n}{y_n} < \gamma x_n, \quad \text{for all} \quad n \geq N_1,$$

which implies

$$\lim_{n\to\infty} x_n = 0,$$

that is a contradiction.

Case 2.

$$\lim_{n\to\infty} y_n = y \in (0, \infty).$$

From

$$x_n = y_{n+1} - \gamma y_n, \quad \text{for all} \quad n \geq 0,$$

we get

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} (y_{n+1} - \gamma y_n) = (1-\gamma)y \in (0, \infty).$$

Also,

$$y = \lim_{n\to\infty} y_n = \lim_{n\to\infty} \frac{x_n}{x_{n+1}} = 1 \quad \text{and} \quad \lim_{n\to\infty} x_n = 1-\gamma. \tag{3.6}$$

Combining (3.3) and (3.6) we see that the sequence $\{y_n\}_{n=N}^{\infty}$ is increasing to 1 and so

$$y_n \leq y_{n+1} \leq 1, \quad \text{for all} \;\; n \geq N. \tag{3.7}$$

Consequently,

$$x_{n+1} = \frac{x_n}{y_n} \geq x_n, \quad \text{for all} \;\; n \geq N$$

and so the sequence $\{x_n\}_{n=N}^{\infty}$ is increasing to $1-\gamma$. From this and (3.3) we find,

$$1-\gamma \leq x_n \leq x_{n+1} \leq 1-\gamma, \quad \text{for all} \;\; n \geq N$$

and so

$$x_n = 1-\gamma \;\; \text{and} \;\; y_n = 1, \quad \text{for all} \;\; n \geq N. \tag{3.8}$$

Also,

$$w_n = \frac{\gamma + x_n}{y_n} = 1, \quad \text{for all} \;\; n \geq N.$$

In view of,

$$x_n = \frac{x_{n+1}y_{n+1}}{\gamma + x_{n+1}}, \quad y_n = \frac{y_{n+1}}{\gamma + x_{n+1}}, \quad \text{and} \;\; w_n = x_{n+1} + \gamma w_{n+1}, \quad n = 0, 1, \ldots, N-1,$$

provided, $N \geq 1$, we see that

$$x_n = 1-\gamma \;\; \text{and} \;\; w_n = y_n = 1, \qquad \text{for all} \;\; n \in \{0, 1, \ldots, N-1\}.$$

Therefore, the solution $\{x_n, y_n, w_n\}_{n=0}^{\infty}$ is constant which is a contradiction.
(2) In view of the result established in (1) and the assumption that $x_N < 1-\gamma$, there exists $N_1 > N$ such that

$$x_{N_1} \geq 1-\gamma$$

and also for each $i = N, \ldots, N_1 - 1$,

$$0 < x_i < 1-\gamma. \tag{3.9}$$

In addition, as long as $N < N_1 - 2$, for each $i = N, \ldots, N_1 - 2$ and in view of (3.9),

$$0 < y_{i+1} = y_i(x_{i+1} + \gamma) < y_i < 1$$

and

$$0 < y_{N_1-1} \leq y_{N_1-1}(x_{N_1} + \gamma) = y_{N_1} = x_{N_1-1} + \gamma y_{N_1-1} < 1 - \gamma + \gamma = 1.$$

In addition, for each $i = N, \ldots, N_1 - 2$,

$$0 < x_i < \frac{x_i}{y_i} = x_{i+1} < 1-\gamma \leq x_{N_1}.$$

When $N_1 - 2 = N$ or equivalently $N_1 = N + 2$, we have

$$0 < y_{N+1} = y_N(x_{N+1} + \gamma) < y_N < 1$$

and

$$0 < y_{N+1} \leq y_{N+1}(x_{N+2} + \gamma) = y_{N+2} = x_{N+1} + \gamma y_{N+1} < 1 - \gamma + \gamma = 1.$$

In addition,

$$0 < x_N < \frac{x_N}{y_N} = x_{N+1} < 1 - \gamma \leq x_{N+2}.$$

Finally, when $N_1 - 2 < N$, that is, when $N_1 = N + 1$, clearly

$$x_{N_1} = x_{N+1} \geq 1 - \gamma > x_N > 0 \text{ and } 1 > y_{N_1} = y_{N+1} = y_N(x_{N+1} + \gamma) \geq y_N.$$

The fact that w_n increases from w_N to w_{N_1} follows easily from

$$w_{n+1} = \frac{w_n}{\gamma + x_n} = \frac{\gamma + x_{n+1}}{y_{n+1}} = \frac{1}{y_n}, \quad \text{for all } n \geq 0.$$

The proofs of (3), (4), and (5) are along the same lines and the details are omitted.
(6) In view of (1)-(5), we may assume without loss of generality that

$$0 < x_0 < 1 - \gamma, \quad 1 \leq y_0 < \frac{1}{\gamma}.$$

and

$$0 < x_1 < 1 - \gamma, \quad \gamma < y_1 < 1.$$

Set $k_1 = 1$. Furthermore, the results established in (2)-(5) imply that there exist positive integers N_1, N_2, N_3, and N_4 such that

$$N_4 > N_3 > N_2 > N_1 > k_1 = 1$$

and for which

$$x_n \uparrow \text{ from } x_{k_1} < 1 - \gamma \text{ to } x_{N_1} \geq 1 - \gamma, \quad y_n \downarrow \text{ from } y_{k_1} < 1 \text{ to } y_{N_1 - 1} < 1$$

and $y_{N_1} < 1$,

$$x_n \uparrow \text{ from } x_{N_1} \geq 1 - \gamma \text{ to } x_{N_2} > 1 - \gamma, \quad y_n \uparrow \text{ from } y_{N_1} < 1 \text{ to } y_{N_2} \geq 1,$$

$$x_n \downarrow \text{ from } x_{N_2} > 1 - \gamma \text{ to } x_{N_3} < 1 - \gamma, \quad y_n \uparrow \text{ from } y_{N_2} \geq 1 \text{ to } y_{N_3 - 1} \geq 1$$

and $y_{N_3} > 1$,

$$x_n \downarrow \text{ from } x_{N_3} < 1 - \gamma \text{ to } x_{N_4} < 1 - \gamma, \quad y_n \downarrow \text{ from } y_{N_3} > 1 \text{ to } y_{N_4} < 1$$

and $y_{N_4-1} \geq 1$. Also,

$$0 < x_{N_4} < 1-\gamma,\ \gamma < \gamma + x_{N_4-1} \leq \gamma y_{N_4-1} + x_{N_4-1} = y_{N_4} < 1,$$

$$0 < x_{N_4-1} = y_{N_4} - \gamma y_{N_4-1} < 1-\gamma,\ 1 \leq y_{N_4-1} = \frac{y_{N_4}}{\gamma + x_{N_4}} < \frac{1}{\gamma}.$$

and

$$w_{N_4} = \frac{\gamma + x_{N_4}}{y_{N_4}} = \frac{1}{y_{N_4-1}} \in (\gamma, 1].$$

By setting $N_4 = k_2$, we have

$$x_{k_1} = x_1 = \min_{0\leq n\leq N_2} x_n \text{ and } x_{k_2} = x_{N_4} = \min_{N_2\leq n\leq N_4} x_n,$$

from which it follows that

$$\min_{0\leq n\leq k_2} x_n = \min_{0\leq n\leq N_4} x_n = \min\{x_1, x_{N_4}\} = \min\{x_{k_1}, x_{k_2}\}.$$

Inductively, it follows that there exists a sequence of indices $\{k_i\}_{i=1}^{\infty}$, such that (3.1) and (3.2) hold and

$$\min_{0\leq n\leq k_i} x_n = \min_{1\leq m\leq i} x_{k_m}, \quad \text{for all } \ i \geq 1$$

from which it follows that

$$\liminf_{n\to\infty} x_n = \liminf_{i\to\infty} x_{k_i}$$

In view of (3.2) and Theorem 1.2 it follows that every convergent subsequence of $\{x_{k_i}\}_{i=1}^{\infty}$ converges to a positive real number and so

$$\liminf_{n\to\infty} x_n = \liminf_{i\to\infty} x_{k_i} > 0.$$

The proof is complete. □

Remark 3.2. It is important to note that in the two dimensional phase plane the orbit of $\{x_n, y_n\}_{n=0}^{\infty}$ "moves" counterclockwise while the orbit of $\{x_n, w_n\}_{n=0}^{\infty}$ moves clockwise. In addition, the two orbits "meet" along the subsequence $\{k_i\}_{i=1}^{\infty}$ described in Theorem 3.1, along which x_n approaches its lowest accumulation point.

Remark 3.3. Given a positive solution of (1.3), namely $\{x_n, y_n, w_n\}_{n=0}^{\infty}$, we may extend it to a sequence of the form $\{x_n, y_n, w_n\}_{n=-\infty}^{\infty}$, by setting for all $n \geq 0$,

$$x_{-n-1} = \frac{x_{-n}y_{-n}}{\gamma + x_{-n}} = \frac{x_{-n}}{w_{-n}},\quad y_{-n-1} = \frac{y_{-n}}{\gamma + x_{-n}},\quad w_{-n-1} = x_{-n} + \gamma w_{-n}.$$

The sequence $\{x_n, y_n, w_n\}_{n=-\infty}^{\infty}$ is a solution of (1.3) because for all $n \in Z$,

$$x_{n+1} = \frac{x_n}{y_n} = \frac{x_n w_n}{\gamma + x_n},\quad y_{n+1} = x_n + \gamma y_n,\quad w_{n+1} = \frac{w_n}{\gamma + x_n}.$$

Also, all components of the solution $\{x_n, y_n, w_n\}_{n=-\infty}^{\infty}$ are bounded from above and from below by positive constants. Finally, if we choose x_0, y_0, and w_0 such that

$$y_0 = w_0 = \sqrt{\gamma + x_0}, \quad x_0 \in (0, \infty)$$

then for the solution $\{x_n, y_n, w_n\}_{n=-\infty}^{\infty}$ of (1.3) it holds,

$$x_n = x_{-n}, \quad y_n = w_{-n}, \quad \text{and} \quad w_n = y_{-n}, \quad \text{for all} \quad n \in Z.$$

4 Limiting Case $\gamma = 1$

Next, consider the system

$$x_{n+1} = \frac{x_n}{y_n} = \frac{x_n w_n}{1 + x_n}, \quad y_{n+1} = x_n + y_n, \quad \text{and} \quad w_{n+1} = \frac{w_n}{1 + x_n}, \quad n = 0, 1, \ldots$$

where x_0, y_0 and w_0 are positive real numbers. By setting

$$x_{-n-1} = \frac{x_{-n} y_{-n}}{1 + x_{-n}} = \frac{x_{-n}}{w_{-n}}, \quad y_{-n-1} = \frac{y_{-n}}{1 + x_{-n}}, \quad \text{and} \quad w_{-n-1} = x_{-n} + w_{-n},$$

for all $n \geq 0$, we see that $\{x_n, y_n, w_n\}_{n=-\infty}^{\infty}$ is a solution of

$$x_{n+1} = \frac{x_n}{y_n} = \frac{x_n w_n}{1 + x_n}, \quad y_{n+1} = x_n + y_n, \quad \text{and} \quad w_{n+1} = \frac{w_n}{1 + x_n}, \quad n \in Z. \tag{4.1}$$

It is important to mention once again that when

$$y_0 = w_0 = \sqrt{1 + x_0}, \quad x_0 \in (0, \infty)$$

then

$$x_n = x_{-n}, \quad y_n = w_{-n}, \quad \text{and} \quad w_n = y_{-n}, \quad \text{for all} \quad n \in Z.$$

It has been established in [19] that every positive solution of System (4.1) of the form $\{x_n, y_n, w_n\}_{n=0}^{\infty}$ is bounded. The proof was based on the fact that every positive solution of System (4.1), satisfies

$$x_n + y_n + w_n = x_0 + y_0 + w_0, \quad \text{for all} \quad n \geq 0.$$

First, we should mention that the same identity holds for a solution of System (4.1). In this paper, we will establish without the use of the identity that every solution of System (4.1) converges to a limit when $n \to \pm\infty$. Then we will make use of the identity to describe the dynamics of convergent solutions in detail.

Lemma 4.1. *Let* $\{x_n, y_n, w_n\}_{n=-\infty}^{\infty}$ *be a positive solution of System* (4.1)*. Then*
1.

$$0 < w_{n+1} < w_n \text{ and } 0 < y_n < y_{n+1}, \text{ for all } n \geq 0. \tag{4.2}$$

2.

$$w_n = x_{n+1} + w_{n+1}, \ 0 < w_0 - w_n = \sum_{k=1}^{n} x_k, \text{ and } 0 < y_{n+1} - y_0 = \sum_{k=0}^{n} x_k, \tag{4.3}$$

for all $n \geq 0$.
3.

$$w_{n+1} = \frac{w_0}{\prod_{k=0}^{n}(1 + x_k)}, \text{ for all } n \geq 0. \tag{4.4}$$

4.

$$0 < w_{-n} < w_{-n-1} \text{ and } 0 < y_{-n-1} < y_{-n}, \text{ for all } n \geq 0. \tag{4.5}$$

5.

$$y_{-n} = x_{-n-1} + y_{-n-1}, \ 0 < y_0 - y_{-n} = \sum_{k=1}^{n} x_{-k}, \text{ and } 0 < w_{-n-1} - w_0 = \sum_{k=0}^{n} x_{-k}, \tag{4.6}$$

for all $n \geq 0$.
6.

$$y_{-n-1} = \frac{y_0}{\prod_{k=0}^{n}(1 + x_{-k})}, \text{ for all } n \geq 0. \tag{4.7}$$

Proof. The proof is based on straightforward calculations and the details are omitted. □

Theorem 4.2. *Let* $\{x_n, y_n, w_n\}_{n=-\infty}^{\infty}$ *be a positive solution of System* (4.1)*. Then*

$$\lim_{n\to\infty} x_n = \lim_{n\to-\infty} x_n = 0, \ \lim_{n\to\infty} w_n = \lim_{n\to-\infty} y_n = L > 0, \ \lim_{n\to\infty} y_n = \lim_{n\to-\infty} w_n = \frac{1}{L}, \tag{4.8}$$

Proof. Let $x_0 > 0$, $y_0 > 0$ and $w_0 = \dfrac{1 + x_0}{y_0}$ be arbitrary positive real numbers, which generate the positive solution $\{x_n, y_n, w_n\}_{n=-\infty}^{\infty}$ of System (4.1). In view of (4.3) the sequence $\{w_n\}_{n=0}^{\infty}$ of positive numbers decreases to a nonnegative limit. Assume that

$$L = \lim_{n\to\infty} w_n \in [0, \infty).$$

Clearly,

$$0 \leq L < w_n < w_0, \text{ for all } n \geq 0.$$

Set

$$\phi_n = \sum_{k=1}^{n} x_k, \ \ n = 1, 2, \ldots .$$

Then,

$$0 < \phi_n < \phi_{n+1}, \quad \text{for all} \;\; n \geq 0$$

and in view of (4.3), we have

$$0 < \phi_n = w_0 - w_n < w_0 - L, \quad \text{for all} \;\; n \geq 0$$

and so,

$$\lim_{n\to\infty} \phi_n = \sum_{k=1}^{\infty} x_k = w_0 - L.$$

Furthermore, in view of (4.4) and from

$$1 + x < e^x, \quad \text{for all} \;\; x > 0,$$

we have

$$w_{n+1} > \frac{w_0}{e^{x_0} \cdot e^{\phi_n}}, \quad \text{for all} \;\; n \geq 0. \tag{4.9}$$

By taking limits in (4.9) as $n \to \infty$, we have

$$L \geq \frac{w_0}{e^{x_0 + w_0 - L}}$$

from which it follows that $L > 0$. Along the same lines it can also be shown that

$$\lim_{n\to\infty} y_{-n} = L_1 \in (0, \infty).$$

Now from

$$\lim_{n\to\infty} w_n = L \in (0, \infty) \;\; \text{and} \;\; \lim_{n\to\infty} y_{-n} = L_1 \in (0, \infty)$$

and in view of (4.3) and (4.6) we have that

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} [w_n - w_{n+1}] = L - L = 0$$

and

$$\lim_{n\to\infty} x_{-n-1} = \lim_{n\to\infty} [y_{-n} - y_{-n-1}] = L_1 - L_1 = 0.$$

Finally, from the first equation of System (4.1) we have that

$$y_{n+1} = \frac{1 + x_{n+1}}{w_{n+1}} = \frac{1}{w_{n+1}} + x_n, \quad \text{for all} \;\; n \geq 0$$

and

$$w_{-n-1} = \frac{1 + x_{-n-1}}{y_{-n-1}} = \frac{1}{y_{-n-1}} + x_{-n}, \quad \text{for all} \;\; n \geq 0,$$

from which it follows that

$$\lim_{n\to\infty} y_{n+1} = \frac{1}{L} \text{ and } \lim_{n\to\infty} w_{-n-1} = \frac{1}{L_1}.$$

Finally, in view of (4.3) and (4.6), we have

$$w_0 + x_0 - w_n = y_{n+1} - y_0 = \sum_{k=0}^{n} x_k \text{ and } y_0 + x_0 - y_{-n} = w_{-n-1} - w_0 = \sum_{k=0}^{n} x_{-k}$$

and so

$$y_{n+1} + w_n = w_{-n-1} + y_{-n} = x_0 + y_0 + w_0 > 2, \text{ for all } n \geq 0$$

which implies that

$$L + \frac{1}{L} = L_1 + \frac{1}{L_1} = x_0 + y_0 + w_0 = k > 2$$

and so

$$L = L_1 \text{ or } L = \frac{1}{L_1}.$$

We will establish that $L = L_1$. Assume for the sake of contradiction that $L > L_1$. The proof when $L < L_1$ is along the same lines and is omitted. Then, clearly

$$L_1 = \frac{1}{L}$$

and in view of the established facts that $\{y_n\}_{n=0}^{\infty}$ increases to $\frac{1}{L} = L_1$ and $\{y_{-n}\}_{n=0}^{\infty}$ decreases to L_1 we get a contradiction. The proof is complete. □

Now we will give a more detailed description of the dynamics of a solution. Let $\{x_n, y_n, w_n\}_{n=-\infty}^{\infty}$ be a positive solution of System (4.1) and assume that

$$k = x_0 + y_0 + w_0 > 2.$$

Define the following continuous functions:

$$f : [0, k + 2 - 2\sqrt{k+2}] \to \left[\sqrt{k+2} - 1, \frac{k + \sqrt{k^2 - 4}}{2}\right],$$

with

$$f(x) = \frac{k - x + \sqrt{x^2 - 2(k+2)x + k^2 - 4}}{2}$$

and

$$g : [0, k + 2 - 2\sqrt{k+2}] \to \left[\frac{k - \sqrt{k^2 - 4}}{2}, \sqrt{k+2} - 1\right],$$

with

$$g(x) = \frac{k - x - \sqrt{x^2 - 2(k+2)x + k^2 - 4}}{2}.$$

Also, let C_1 and C_2 be the graphs of the two functions f and g, respectively. Observe that C_1 decays from $(0, \frac{k+\sqrt{k^2-4}}{2})$ to $(k+2-2\sqrt{k+2}, \sqrt{k+2}-1)$ and C_2 rises from $(0, \frac{k-\sqrt{k^2-4}}{2})$ to $(k+2-2\sqrt{k+2}, \sqrt{k+2}-1)$. Furthermore, the two graphs intersect at the point

$$(x_0, y_0) = (x_0, \sqrt{1+x_0}) = (k+2-2\sqrt{k+2}, \sqrt{k+2}-1).$$

The following theorem describes in detail the dynamics of positive solutions of System (4.4). The proof is straightforward and the details are omitted.

Theorem 4.3. *Assume that* $x_0, y_0 \in (0, \infty)$ *and that*

$$w_0 = \frac{1+x_0}{y_0}.$$

Let

$$k = x_0 + y_0 + w_0 \in (2, \infty).$$

Then
1. For every $n \in Z$*,*

$$(y_n = f(x_n) \ \ \textit{and} \ \ w_n = g(x_n)) \ \ \textit{or} \ \ (y_n = g(x_n) \ \ \textit{and} \ \ w_n = f(x_n)).$$

2. If $(x_0, y_0) \in C_1$*, that is* $y_0 = f(x_0)$*, then* $(x_0, w_0) \in C_2$*,*

$$(x_n, y_n) \in C_1, \ \ (x_n, w_n) \in C_2, \ \ \textit{for all} \ \ n \geq 0,$$

$$x_n \downarrow 0, \ \ y_n \uparrow \frac{k+\sqrt{k^2-4}}{2}, \ \ w_n \downarrow \frac{k-\sqrt{k^2-4}}{2}, \ \ \textit{as} \ \ n \to \infty.$$

Also, there exists $N \geq 1$ *such that*

$$(x_{-n}, y_{-n}) \in C_2, \ \ (x_{-n}, w_{-n}) \in C_1, \ \ \textit{for all} \ \ n \geq N,$$

$$x_{-n} \downarrow 0, \ \ y_{-n} \downarrow \frac{k-\sqrt{k^2-4}}{2}, \ \ w_{-n} \uparrow \frac{k+\sqrt{k^2-4}}{2}, \ \ \textit{as} \ \ n \to \infty.$$

3. If $(x_0, y_0) \in C_2$*, that is* $y_0 = g(x_0)$*, then* $(x_0, w_0) \in C_1$*,*

$$(x_{-n}, y_{-n}) \in C_2, \ \ (x_{-n}, w_{-n}) \in C_1, \ \ \textit{for all} \ \ n \geq 0,$$

$$x_{-n} \downarrow 0, \ \ y_{-n} \downarrow \frac{k-\sqrt{k^2-4}}{2}, \ \ w_{-n} \uparrow \frac{k+\sqrt{k^2-4}}{2}, \ \ \textit{as} \ \ n \to \infty.$$

Also, there exists $N \geq 1$ *such that*

$$(x_n, y_n) \in C_1, \quad (x_n, w_n) \in C_2, \quad \textit{for all} \;\; n \geq N,$$

$$x_n \downarrow 0, \quad y_n \uparrow \frac{k + \sqrt{k^2 - 4}}{2}, \quad w_n \downarrow \frac{k - \sqrt{k^2 - 4}}{2}, \quad \textit{as} \;\; n \to \infty.$$

4. In particular, if $(x_0, y_0) \in C_1 \cap C_2$*, that is*

$$x_0 = k + 2 - 2\sqrt{k+2} \;\; \textit{and} \;\; y_0 = \sqrt{1 + x_0} = \sqrt{k+2} - 1$$

then

$$x_n = x_{-n}, \quad y_n = w_{-n}, \quad w_n = y_{-n}, \quad \textit{for all} \;\; n \geq 0,$$

$$(x_n, y_n) = (x_{-n}, w_{-n}) \in C_1 \;\; \textit{for all} \;\; n \geq 0,$$

$$(x_{-n}, y_{-n}) = (x_n, w_n) \in C_2 \;\; \textit{for all} \;\; n \geq 0,$$

and as $n \to \infty$*,*

$$x_n = x_{-n} \downarrow 0, \quad w_n = y_{-n} \downarrow \frac{k - \sqrt{k^2 - 4}}{2}, \quad y_n = w_{-n} \uparrow \frac{k + \sqrt{k^2 - 4}}{2}.$$

5 Extensions and Open Problems

The boundedness result, which is established for all positive solutions of the system

$$x_{n+1} = \frac{x_n}{y_n} = \frac{x_n w_n}{\gamma + x_n}, \quad y_{n+1} = x_n + \gamma_n y_n, \text{and} \;\; w_{n+1} = \frac{w_n}{\gamma + x_n}, \quad n = 0, 1, \ldots \quad (5.1)$$

may be easily extended for positive solutions of the system

$$x_{n+1} = \frac{x_n}{y_n} = \frac{x_n w_n}{\gamma_n + x_n}, \quad y_{n+1} = x_n + \gamma_n y_n, \text{and} \;\; w_{n+1} = \frac{w_n}{\gamma_n + x_n}, \quad n = 0, 1, \ldots \quad (5.2)$$

where $\{\gamma_n\}_{n=0}^{\infty}$ is an arbitrary sequence of positive real numbers for which

$$0 \leq \liminf_{n \to \infty} \gamma_n \leq \limsup_{n \to \infty} \gamma_n < 1.$$

For the limiting case we pose the following open problem.

Open Problem 5.1. Investigate the boundedness character of all positive solutions of System (5.2) when

$$\limsup_{n \to \infty} \gamma_n = 1.$$

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