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Approximation Theory and Functional Analysis on Time Scales

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Abstract

Here we start by proving the Riesz representation theorem for positive linear functionals on the space of continuous functions over a time scale. Then we prove further properties for the related Riemann–Stieltjes integral on time scales and we prove the related Hölder's inequality. Next we prove the Hölder's inequality for general positive linear functionals on time scales. We introduce basic concepts of Approximation theory on time scales and we discuss some limitations of the modulus of continuity there. Next we prove the famous Korovkin theorem on time scales, regarding the approximation of unit operator by sequences of positive linear operators on the space of continuity functions defined on a compact interval of a time scale. Then we produce several Shisha–Mond type inequalities related to Korovkin's theorem, putting the convergence of positive linear operators and positive linear functionals in a quantitative form and giving rates of convergence, all operating on Lipschitz functions on a time scale. At the end we present an example of a concrete and genuine positive linear operator on time scales and we give its approximation and interpolation properties over continuous functions.

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1 Introduction

Denote by $C([a,b])$, the space of continuous real valued functions on $[a,b] \subset \mathbb{R}$.

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Definition 1.1. Let L be a linear operator mapping $C([a,b])$ into itself. L is called positive iff whenever $f \geq g$; $f, g \in C([a,b])$, we have that $L(f) \geq L(g)$.

We mention the famous Korovkin theorem.

Theorem 1.2 (See Korovkin [10, 1960, p. 14]). *Let $[a,b]$ be a compact interval in $\mathbb{R}$ and $(L_n)_{n\in\mathbb{N}}$ be a sequence of positive linear operators L_n mapping $C([a,b])$ into itself. Suppose that $(L_n f)$ converges uniformly to f for the three test functions $f = 1, x, x^2$. Then $(L_n f)$ converges uniformly to f on $[a,b]$ for all functions $f \in C([a,b])$.*

We need the following definition.

Definition 1.3. Let $f \in C([a,b])$ and $0 \leq h \leq b-a$. Call

$$\omega_1(f,h) := \sup_{\substack{\text{all } x,y \\ |x-y|\leq h}} |f(x) - f(y)|, \tag{1.1}$$

the first modulus of continuity of f at h.

We also mention the following famous theorem.

Theorem 1.4 (See Shisha and Mond [12, 1968]). *Let $[a,b] \subset \mathbb{R}$ be a compact interval. Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of positive linear operators acting on $C([a,b])$ into itself. For $n = 1, 2, ...$, suppose $L_n(1)$ is bounded. Let $f \in C([a,b])$. Then for $n = 1, 2, ...$, we have*

$$\|L_n f - f\|_\infty \leq \|f\|_\infty \cdot \|L_n 1 - 1\|_\infty + \|L_n(1) + 1\|_\infty \omega_1(f, \mu_n), \tag{1.2}$$

where

$$\mu_n := \left\|L_n\left((t-x)^2\right)(x)\right\|_\infty^{\frac{1}{2}},$$

and $\|\cdot\|_\infty$ stands for the sup-norm over $[a,b]$. In particular, if $L_n(1) = 1$ then (1.2) reduces to

$$\|L_n(f) - f\|_\infty \leq 2\omega_1(f, \mu_n). \tag{1.3}$$

Remark 1.5. (i) In forming μ_n^2, x is kept fixed, however t forms the functions t, t^2 on which L_n acts.

(ii) One can easily see, for $n = 1, 2, ...$

$$\mu_n^2 \leq \left\|\left(L_n\left(t^2\right)\right)(x) - x^2\right\|_\infty + 2c\,\|(L_n(t))(x) - x\|_\infty + c^2\,\|(L_n(1))(x) - 1\|_\infty, \tag{1.4}$$

where $c := \max(|a|, |b|)$.

So if the assumptions of Korovkin's Theorem 1.2 are fulfilled then $\mu_n \to 0$, therefore $\omega_1(f, \mu_n) \to 0$, as $n \to +\infty$, and we obtain from (1.2) that $\|L_n f - f\|_\infty \to 0$, as $n \to +\infty$, which is Korovkin's conclusion!!! I.e., Korovkin's result has been recast in a quantitative form.

Next we mention from [1].

We need the following definition.

Definition 1.6. Let $B : \mathbb{R} \to \mathbb{R}_+$, be a bell-shaped function of compact support $[-T, T]$, $T > 0$. We assume it is even, nondecreasing for $x < 0$ and nonincreasing for $x \geq 0$. Suppose also that $B(0) =: B^* > 0$ is the global maximum of B. The function B may have jump discontinuities and it is measurable. Assume further that $B(\pm T) = 0$.

An example for B can be the hat function

$$\beta(x) := \begin{cases} 1+x, & -1 \leq x \leq 0, \\ 1-x, & 0 < x \leq 1, \\ 0, & \text{elsewhere}, \end{cases} \tag{1.5}$$

etc.

Definition 1.7. Let $f : [a, b] \to \mathbb{R}$, $a, b \in \mathbb{R}$, $a < b$, a bounded and measurable function, $n \in \mathbb{N}$, $h := \frac{b-a}{n}$, $x_k := a + kh$, $k = 0, 1, ..., n$, $x \in [a, b]$.

We define the interpolation neural network operator

$$H_n(f, x) := \frac{\sum_{k=0}^{n} f(x_k) B\left(\frac{Tn(x-x_k)}{b-a}\right)}{\sum_{k=0}^{n} B\left(\frac{Tn(x-x_k)}{b-a}\right)}. \tag{1.6}$$

This is a positive linear operator.

We state the interpolation result.

Theorem 1.8 (See [1]). *Let $f : [a, b] \to \mathbb{R}$ be a bounded and measurable function. Then*

$$H_n(f, x_i) = f(x_i), \ \ i = 0, 1, ..., n, \tag{1.7}$$

where $x_i := a + ih$, $h := \frac{b-a}{n}$, $n \in \mathbb{N}$.

We state the related approximation result at Jackson speed of convergence $\frac{1}{n}$.

Theorem 1.9 (See [1]). *Let $f \in C([a, b])$. Then*

$$\|H_n(f) - f\|_\infty \leq \frac{2B^*}{B\left(\frac{T}{2}\right)} \omega_1\left(f, \frac{b-a}{n}\right), \quad \forall\, n \in \mathbb{N}. \tag{1.8}$$

The above results motivated this paper.

In this article we prove similar to the above approximation theorems on time scales and we expand around, also treating related Functional Analysis topics.

To our knowledge this is the first article of study about classical approximation by positive linear operators on time scales.

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2 Time Scales Basics (see [4])

Since a time scale $\mathbb{T}$ is a closed subset of the real numbers $\mathbb{R}$, it is a complete metric space with the metric (distance)

$$d(t,s) = |t-s|, \text{ for } t,s \in \mathbb{T}. \tag{2.1}$$

Consequently, according to basic theory of general metric spaces we have for $\mathbb{T}$ fundamental concepts such as open ball (intervals), neighborhoods of points, open sets, closed sets, compact sets and so on. In particular, for a given $\delta > 0$, the δ-neighborhood $U_\delta(t)$ of a given point $t \in \mathbb{T}$ is the set of all points $s \in \mathbb{T}$ such that $d(t,s) < \delta$. By a neighborhood of a point $t \in \mathbb{T}$ we mean an arbitrary set in $\mathbb{T}$ containing a δ-neighborhood of the point t. Also we have for functions $f : \mathbb{T} \to \mathbb{R}$ the concepts of limit, continuity, and the properties of continuous functions on general complete metric spaces (note that, in particular, any function $f : \mathbb{Z} \to \mathbb{R}$ is continuous at each point of $\mathbb{Z}$).

Let $\mathbb{T}$ be a time scale. We define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \quad \text{for } t \in \mathbb{T},$$

while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\} \quad \text{for } t \in \mathbb{T}.$$

In this definition we set in addition $\sigma(\max\mathbb{T}) = \max\mathbb{T}$ if there exists a finite $\max\mathbb{T}$, and $\rho(\min\mathbb{T}) = \min\mathbb{T}$ if there exists a finite $\min\mathbb{T}$.

Obviously both $\sigma(t)$ and $\rho(t)$ are in $\mathbb{T}$ when $t \in \mathbb{T}$. This is because $\mathbb{T}$ is a closed subset of $\mathbb{R}$.

Let $t \in \mathbb{T}$. If $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$, we say that t is left-scattered. Also, if $t < \max\mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense, while if $t > \min\mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. Points that are right-dense and left-dense are called dense and points that are right-scattered and left-scattered at the same time are called isolated.

If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = \rho(t) = t$. If $\mathbb{T} = h\mathbb{Z}$, then $\sigma(t) = t + h$ and $\rho(t) = t - h$. But if $\mathbb{T} = q^{\mathbb{N}_0}$ ($q > 1$), $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, then $\sigma(t) = qt$ and $\rho(1) = 1$, $\rho(t) = q^{-1}t$ for $t > 1$.

Let $\mathbb{T}^k$ denote Hilger's truncated above set consisting of $\mathbb{T}$ except for a positive left-scattered maximal point. Now we consider a function $f : \mathbb{T} \to \mathbb{R}$ and define the so-called delta (or Hilger) derivative of f at a point $t \in \mathbb{T}^k$.

Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and $t \in \mathbb{T}^k$. Then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U (in $\mathbb{T}$) of t such that

$$\left| f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s] \right| \le \varepsilon |\sigma(t) - s| \quad \text{for all } s \in U. \tag{2.2}$$

We call $f^\Delta(t)$ the delta (or Hilger) derivative of f at t.

If $t \in \mathbb{T} \backslash \mathbb{T}^k$, then $f^{\Delta}(t)$ is not uniquely defined, since for such a point t, small neighborhoods U of t consist only of t, and besides, we have $\sigma(t) = t$. Therefore (2.2) holds for an arbitrary number $f^{\Delta}(t)$.

Note that for the calculation of the delta derivative it is convenient to use its definition in the limit form

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}, \tag{2.3}$$

where the limit is taken in $\mathbb{T}$ with the metric (2.1) and in the limit we suppose $s \neq \sigma(t)$.

Using (2.3), we get the following. If $\mathbb{T} = \mathbb{R}$, then $f^{\Delta}(t) = f'(t)$, the ordinary derivative of f at t. If $\mathbb{T} = h\mathbb{Z}$ ($h > 0$), then

$$f^{\Delta}(t) = \frac{f(t+h) - f(t)}{h}.$$

In particular, in the case $\mathbb{T} = \mathbb{Z}$, we have

$$f^{\Delta}(t) = f(t+1) - f(t).$$

If $\mathbb{T} = q^{\mathbb{N}_0}$ ($q > 1$), then

$$f^{\Delta}(t) = \frac{f(qt) - f(t)}{(q-1)t}.$$

For the Riemann–Stieltjes integral on time scales and properties we follow [9]. We write $''f \in \mathcal{R}(\alpha)$ on $[a,b]_{\mathbb{T}}$", if f is Riemann–Stieltjes integrable with respect to a function α on $[a,b]_{\mathbb{T}}$.

We mention the following result.

Proposition 2.1 (See [9]). *If $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is continuous on $[a,b]_{\mathbb{T}}$ and if $\alpha : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is of bounded variation on $[a,b]_{\mathbb{T}}$, then $f \in \mathcal{R}(\alpha)$ on $[a,b]_{\mathbb{T}}$, i.e., it exists the Riemann–Stieltjes (R-S) integral on $[a,b]_{\mathbb{T}}$, denoted by $\int_a^b f(t)\,\Delta\alpha(t)$ or by $\int_a^b f\Delta\alpha$.*

The description of (R-S) integral and properties on time scales parallels to the one on $\mathbb{R}$ of the ordinary case.

Example 2.2. For many interesting examples of time scales see [4].

We give here some important related examples:

(i) Let $0 < r < 1$ fixed. The set $K := \{r^n : n \in \mathbb{N}\} \cup \{0\}$ is a time scale.

(ii) The set $\theta := \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$ is a time scale.

Call $T_m := m + \theta$, where $m \in \mathbb{Z}$, then T_m is a time scale.

Set $W := \cup_{m \in \mathbb{Z}} T_m$, then W is a time scale.

Call $B_m := m - \theta$, $m \in \mathbb{Z}$, which is a time scale.

Set $\Lambda := \cup_{m\in\mathbb{Z}} B_m$, then Λ is a time scale. Also $W \cup \Lambda$ is another time scale.

Notice the above sets $K, \theta, T_m, W, B_m, \Lambda, W \cup \Lambda$ contain all of their limit points, where near them the elements of these sets can be arbitrarily close to each other.

We need the following definition.

Definition 2.3. Let $\mathbb{T}$ be a time scale and $f : \mathbb{T} \to \mathbb{R}$. If

$$|f(x) - f(y)| \leq M |x-y|^{\beta}, \ 0 < \beta \leq 1, \tag{2.4}$$

$M > 0$, $\forall\ x, y \in \mathbb{T}$, we call f a Lipschitz function of order β and we denote it as $f \in Lip(\beta)$.

We make the following remark.

Remark 2.4. Let $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be continuous on $[a,b]_{\mathbb{T}}$, with f^{Δ} existing and bounded on $[a,b)_{\mathbb{T}}$, i.e., $\left|f^{\Delta}(t)\right| \leq A$, $\forall\ t \in [a,b)_{\mathbb{T}}$, where $A > 0$. Let $a \leq c < d \leq b$. By the mean-value theorem on time scales ([7]), we have

$$f^{\Delta}(\tau) \leq \frac{f(d) - f(c)}{d - c} \leq f^{\Delta}(\xi), \tag{2.5}$$

where $\tau, \xi \in [c,d)_{\mathbb{T}}$.

Then it follows from (2.5) that

$$|f(d) - f(c)| \leq A(d - c), \tag{2.6}$$

thus $f \in Lip(1)$ on $[a,b]_{\mathbb{T}}$.

3 More on Riemann–Stieltjes Integral on Time Scales

Denote by $C([a,b]_{\mathbb{T}})$ the Banach space of all functions $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ which are continuous (in the metric of the time scale) on $[a, \rho(b)]_{\mathbb{T}}$ and such that $f(b) = f(\rho(b))$, equipped with the norm

$$\|f\|_{\infty} = \max |f(t)|, \ t \in [a,b]_{\mathbb{T}},$$

where ρ is the backward jump operator in $\mathbb{T}$.

By $B([a,b]_{\mathbb{T}})$ we denote the Banach space of all bounded real-valued functions on $[a,b]_{\mathbb{T}}$ with norm defined by $\|f\|_{\infty} = \sup |f(t)|$, $t \in [a,b]_{\mathbb{T}}$.

We need

Fundamental Lemma A ([5, p. 120]) For every $\delta > 0$ there exists at least one partition $P = \{a = t_0, t_1, ..., t_n = b\}$ of $[a,b]_{\mathbb{T}}$ ($t_0 < t_1 < ... < t_n$) such that for each $k \in \{1, 2, ..., n\}$ either $t_k - t_{k-1} \leq \delta$ or $t_k - t_{k-1} > \delta$ and $\rho(t_k) = t_{k-1}$.

For given $\delta > 0$ we denote by $\mathcal{P}_{\delta}([a,b]_{\mathbb{T}})$ the set of all partitions $P = \{t_0, t_1, ..., t_n\}$ of $[a,b]_{\mathbb{T}}$ that possess the property of Fundamental Lemma A.

We give the Riesz representation theorem for positive linear functionals on $C([a,b]_{\mathbb{T}})$.

Theorem 3.1. *Every positive linear functional F on $C([a,b]_{\mathbb{T}})$ can be represented by a Riemann–Stieltjes Δ-integral in the form*

$$F(f) = \int_a^b f(t)\,\Delta\alpha(t), \tag{3.1}$$

$\forall\ f \in C([a,b]_{\mathbb{T}})$, *where α is an increasing function on $[a,b]_{\mathbb{T}}$ with $\|F\| = F(1) = \alpha(b)$, and $\alpha(a) = 0$.*

Proof. Let F be a positive linear functional on $C([a,b]_{\mathbb{T}})$, which is a subspace of $B([a,b]_{\mathbb{T}})$. By the extension theorem on positive linear functional ([11]), there exists a positive linear functional on $B([a,b]_{\mathbb{T}})$, denoted by $\overline{F}$ such that

$$F = \overline{F}|_{C\left([a,b]_{\mathbb{T}}\right)}. \tag{3.2}$$

We consider the functions χ_t defined for each fixed $t \in [a,b]_{\mathbb{T}}$ by

$$\chi_t(\xi) := \begin{cases} 1, & \text{for } \xi \in [a,t)_{\mathbb{T}}, \\ 0, & \text{for } \xi \in [t,b]_{\mathbb{T}}, \end{cases} \tag{3.3}$$

(for $t \in [a,b)_{\mathbb{T}}$), and

$$\chi_b(\xi) = 1, \text{ for all } \xi \in [a,b]_{\mathbb{T}}.$$

Clearly here $\chi_t \in B([a,b]_{\mathbb{T}})$.

When $t_1 \le t_2$, $t_1, t_2 \in [a,b)_{\mathbb{T}}$, we get $\chi_{t_1}(\xi) \le \chi_{t_2}(\xi)$, $\forall\ \xi \in [a,b]_{\mathbb{T}}$, that is $\chi_{t_1} \le \chi_{t_2} \le \chi_b$. Hence

$$\overline{F}(\chi_{t_1}) \le \overline{F}(\chi_{t_2}) \le \overline{F}(\chi_b) = \overline{F}(1) = F(1). \tag{3.4}$$

We define the function

$$\alpha(t) = \overline{F}(\chi_t), \ \ \forall\, t \in [a,b]_{\mathbb{T}}.$$

Notice also $\alpha(a) = \overline{F}(\chi_a) = \overline{F}(0) = 0$. We have that $\alpha(t_1) \le \alpha(t_2)$, hence α is increasing on $[a,b]_{\mathbb{T}}$, thus α is of bounded variation, with total variation $\alpha(b) = F(1)$.

Next we continue as in [9].

Let $f \in C([a,b]_{\mathbb{T}})$, which is uniformly continuous and which means for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $t', t'' \in [a,b]_{\mathbb{T}}$, $|t' - t''| \le \delta$ implies $|f(t') - f(t'')| < \varepsilon$.

For every partition $P = \{t_0, t_1, ..., t_n\}$ belonging to $\mathcal{P}_\delta([a,b]_{\mathbb{T}})$ we consider the step function $f^{(\varepsilon)}$ defined on $[a,b]_{\mathbb{T}}$ by

$$f^{(\varepsilon)}(t) := f(t_{k-1}), \text{ if } t \in [t_{k-1}, t_k)_{\mathbb{T}}, \ \ k = 1, ..., n, \tag{3.5}$$

and

$$f^{(\varepsilon)}(b) := f(t_n) = f(b), \tag{3.6}$$

which can be written in the form

$$f^{(\varepsilon)}(t)=\sum_{k=1}^{n} f(t_{k-1})\left[\chi_{t_k}(t)-\chi_{t_{k-1}}(t)\right], \quad \text{for all } t\in[a,b]_{\mathbb{T}}. \tag{3.7}$$

We have that

$$\left|f^{(\varepsilon)}(t)-f(t)\right|<\varepsilon, \ \ \forall\, t\in[a,b]_{\mathbb{T}}. \tag{3.8}$$

We see that as follows: let $t\in[a,b]_{\mathbb{T}}$, then either $t\in[t_{k-1},t_k)_{\mathbb{T}}$ for some $k\in\{1,...,n\}$ or $t=b$. Consider first the case $t\in[t_{k-1},t_k)_{\mathbb{T}}$.

If $t_k-t_{k-1}\le\delta$, then

$$\left|f^{(\varepsilon)}(t)-f(t)\right|=\left|f(t_{k-1})-f(t)\right|<\varepsilon.$$

If $t_k-t_{k-1}>\delta$, then $t_{k-1}=\rho(t_k)$ and hence $[t_{k-1},t_k)_{\mathbb{T}}=\{t_{k-1}\}$ is a singleton, and thus

$$\left|f^{(\varepsilon)}(t)-f(t)\right|=\left|f^{(\varepsilon)}(t_{k-1})-f(t_{k-1})\right|=\left|f(t_{k-1})-f(t_{k-1})\right|=0. \tag{3.9}$$

In the case $t=b$, we get

$$\left|f^{(\varepsilon)}(t)-f(t)\right|=\left|f(b)-f(b)\right|=0.$$

Therefore (22) is true.

Consequently it holds

$$\left\|f^{(\varepsilon)}-f\right\|_{\infty}<\varepsilon. \tag{3.10}$$

By linearity of $\overline{F}$ we find

$$\overline{F}\left(f^{(\varepsilon)}\right)=\sum_{k=1}^{n} f(t_{k-1})\left[\overline{F}(\chi_{t_k})-\overline{F}\left(\chi_{t_{k-1}}\right)\right]=\sum_{k=1}^{n} f(t_{k-1})\left[\alpha(t_k)-\alpha(t_{k-1})\right]. \tag{3.11}$$

Therefore $\overline{F}\left(f^{(\varepsilon)}\right)$ is a Riemann–Stieltjes Δ-sum of f with respect to α, corresponding to the partition P.

Consequently it holds

$$\left|\overline{F}\left(f^{(\varepsilon)}\right)-\int_a^b f(t)\,\Delta\alpha(t)\right|<\varepsilon \tag{3.12}$$

for a sufficiently small $\delta>0$.

Also we find

$$\left|\overline{F}\left(f^{(\varepsilon)}\right)-F(f)\right|=\left|\overline{F}\left(f^{(\varepsilon)}\right)-\overline{F}(f)\right|=$$

$$\left|\overline{F}\left(f^{(\varepsilon)}-f\right)\right|\le\left\|\overline{F}\right\|\left\|f^{(\varepsilon)}-f\right\|_{\infty}\le\left\|\overline{F}\right\|\varepsilon=:(*). \tag{3.13}$$

Let $g \in B([a,b]_{\mathbb{T}})$, we have that $|g| \le \|g\|_\infty < \infty$. Hence $-\|g\|_\infty \le g \le \|g\|_\infty$ and $-\|g\|_\infty \overline{F}(1) \le \overline{F}(g) \le \|g\|_\infty \overline{F}(1)$, i.e.,

$$\left|\overline{F}(g)\right| \le \overline{F}(1)\|g\|_\infty = F(1)\|g\|_\infty, \tag{3.14}$$

so that $\overline{F}$ is a real valued bounded linear functional on $B([a,b]_{\mathbb{T}})$ with $\left\|\overline{F}\right\|_\infty \le F(1) < \infty$.

Similarly F is a real valued bounded linear functional on $C([a,b]_{\mathbb{T}})$, with $\|F\|_\infty \le F(1)$.

Furthermore it holds for $g(t) = 1$, $\forall\, t \in [a,b]_{\mathbb{T}}$, that

$$F(1) = |F(1)| \le \|F\|_\infty \cdot 1 = \|F\|_\infty,$$

along with

$$F(1) = \left|\overline{F}(1)\right| \le \left\|\overline{F}\right\|_\infty \cdot 1 = \left\|\overline{F}\right\|_\infty.$$

We conclude that

$$F(1) = \|F\|_\infty = \left\|\overline{F}\right\|_\infty = \alpha(b). \tag{3.15}$$

So that

$$(*) = \alpha(b)\,\varepsilon.$$

I.e.,

$$\left|\overline{F}\left(f^{(\varepsilon)}\right) - F(f)\right| \le \alpha(b) \cdot \varepsilon. \tag{3.16}$$

Finally we derive

$$\left|F(f) - \int_a^b f(t)\,\Delta\alpha(t)\right| \le \left|F(f) - \overline{F}\left(f^{(\varepsilon)}\right)\right| + \left|\overline{F}\left(f^{(\varepsilon)}\right) - \int_a^b f(t)\,\Delta\alpha(t)\right| \le \tag{3.17}$$

$$\alpha(b) \cdot \varepsilon + \varepsilon = \varepsilon(\alpha(b) + 1),$$

which implies (3.1), because $\varepsilon > 0$ is arbitrary. The proof of the theorem now is complete. □

We make the following remark.

Remark 3.2. Here $\alpha : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is assumed to be increasing and let $f \in C([a,b]_{\mathbb{T}})$. Clearly α is of bounded variation and $f, |f| \in \mathcal{R}(\alpha)$ on $[a,b]_{\mathbb{T}}$. We notice that

$$\left|\int_a^b f(t)\,\Delta\alpha(t)\right| \le \|f\|_\infty (\alpha(b) - \alpha(a)), \tag{3.18}$$

and

$$\left|\int_a^b f(t)\,\Delta\alpha(t)\right| \le \int_a^b |f(t)|\,\Delta\alpha(t). \tag{3.19}$$

Given $f(t) \geq 0, \forall\, t \in [a,b]_{\mathbb{T}}$, we get

$$\int_a^b f(t)\,\Delta\alpha(t) \geq 0. \tag{3.20}$$

Let also $g \in C([a,b]_{\mathbb{T}})$, such that $f(t) \geq g(t), \forall\, t \in [a,b]_{\mathbb{T}}$. Hence $f(t) - g(t) \geq 0$, and

$$\int_a^b f(t)\,\Delta\alpha(t) - \int_a^b g(t)\,\Delta\alpha(t) = \int_a^b (f(t) - g(t))\,\Delta\alpha(t) \geq 0.$$

Thus

$$\int_a^b f(t)\,\Delta\alpha(t) \geq \int_a^b g(t)\,\Delta\alpha(t). \tag{3.21}$$

Clearly if $\alpha = 0$ or a constant, then

$$\int_a^b f(t)\,\Delta\alpha(t) = 0. \tag{3.22}$$

Let now α be strictly increasing on $[a,b]_{\mathbb{T}}$ and $f(t) \geq 0, \forall\, t \in [a,b]_{\mathbb{T}}$ with (3.22). Then $f(t) = 0, \forall\, t \in [a,b]_{\mathbb{T}}$. If $f(t) > 0, \forall\, t \in [a,b]_{\mathbb{T}}$, α is increasing on $[a,b]_{\mathbb{T}}$ and $\int_a^b f(t)\,\Delta\alpha(t) = 0$. Then α is either zero, or a constant different than zero.

Next comes Hölder's inequality for (R-S) integrals.

Theorem 3.3. *Let $f, g \in C([a,b]_{\mathbb{T}})$, α is increasing on $[a,b]_{\mathbb{T}}$, and $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\int_a^b |f(t)\,g(t)|\,\Delta\alpha(t) \leq \left(\int_a^b |f(t)|^p\,\Delta\alpha(t)\right)^{\frac{1}{p}} \left(\int_a^b |g(t)|^q\,\Delta\alpha(t)\right)^{\frac{1}{q}}. \tag{3.23}$$

Equality holds nontrivially, when $|g(x)| = c\,|f(x)|^{p-1}$, $c > 0$, $\forall\, x \in [a,b]_{\mathbb{T}}$.

Proof. For $\widetilde{\alpha}, \widetilde{\beta} \geq 0$, the basic inequality holds

$$\widetilde{\alpha}^{\frac{1}{p}} \widetilde{\beta}^{\frac{1}{q}} \leq \frac{\widetilde{\alpha}}{p} + \frac{\widetilde{\beta}}{q}, \tag{3.24}$$

with equality when $\widetilde{\alpha} = \widetilde{\beta}$.

Now suppose, without loss of generality, that

$$\int_a^b |f(t)|^p\,\Delta\alpha(t)\,, \int_a^b |g(t)|^q\,\Delta\alpha(t) \neq 0.$$

Apply inequality (3.24) for

$$\widetilde{\alpha}(t)=\frac{|f(t)|^{p}}{\int_a^b |f(\tau)|^{p}\,\Delta\alpha(\tau)},\quad \text{and } \widetilde{\beta}(t)=\frac{|g(t)|^{q}}{\int_a^b |g(\tau)|^{q}\,\Delta\alpha(\tau)}, \tag{3.25}$$

to get

$$\left(\frac{|f(t)|}{\left(\int_a^b |f(\tau)|^{p}\,\Delta\alpha(\tau)\right)^{\frac{1}{p}}}\right)\left(\frac{|g(t)|}{\left(\int_a^b |g(\tau)|^{q}\,\Delta\alpha(\tau)\right)^{\frac{1}{q}}}\right)\leq$$

$$\frac{|f(t)|^{p}}{p\int_a^b |f(\tau)|^{p}\,\Delta\alpha(\tau)}+\frac{|g(t)|^{q}}{q\int_a^b |g(\tau)|^{q}\,\Delta\alpha(\tau)}. \tag{3.26}$$

Hence by integrating (3.26) it holds

$$\frac{\int_a^b |f(t)|\,|g(t)|\,\Delta\alpha(t)}{\left(\int_a^b |f(\tau)|^{p}\,\Delta\alpha(\tau)\right)^{\frac{1}{p}}\left(\int_a^b |g(\tau)|^{q}\,\Delta\alpha(\tau)\right)^{\frac{1}{q}}}\leq$$

$$\frac{\int_a^b |f(t)|^{p}\,\Delta\alpha(t)}{p\int_a^b |f(\tau)|^{p}\,\Delta\alpha(\tau)}+\frac{\int_a^b |g(t)|^{q}\,\Delta\alpha(t)}{q\int_a^b |g(\tau)|^{q}\,\Delta\alpha(\tau)}=\frac{1}{p}+\frac{1}{q}=1, \tag{3.27}$$

proving inequality (3.23).

Next we assume that $|g(x)|=c\,|f(x)|^{p-1}$, $c>0$. Then

$$\int_a^b |f(t)|\,|g(t)|\,\Delta\alpha(t)=c\left(\int_a^b |f(t)|^{p}\,\Delta\alpha(t)\right). \tag{3.28}$$

On the other hand we can write

$$\left(\int_a^b |f(t)|^{p}\,\Delta\alpha(t)\right)^{\frac{1}{p}}\left(\int_a^b |g(t)|^{q}\,\Delta\alpha(t)\right)^{\frac{1}{q}}=$$

$$\left(\int_a^b |f(t)|^{p}\,\Delta\alpha(t)\right)^{\frac{1}{p}}c\left(\int_a^b |f(t)|^{q(p-1)}\,\Delta\alpha(t)\right)^{\frac{1}{q}}=$$

$$c\left(\int_a^b |f(t)|^{p}\,\Delta\alpha(t)\right)^{\frac{1}{p}}\left(\int_a^b |f(t)|^{p}\,\Delta\alpha(t)\right)^{\frac{1}{q}}=c\left(\int_a^b |f(t)|^{p}\,\Delta\alpha(t)\right). \tag{3.29}$$

By (3.28), (3.29) we have proved that (3.23) is attained.

The proof of the theorem now is complete. □

The special case $p=q=2$ reduces to the Cauchy–Schwarz inequality.

Corollary 3.4 (to Theorem 3.3)**.** *It holds*

$$\int_a^b |f(t)\,g(t)|\,\Delta\alpha(t)\leq\left(\int_a^b f^{2}(t)\,\Delta\alpha(t)\right)^{\frac{1}{2}}\left(\int_a^b g^{2}(t)\,\Delta\alpha(t)\right)^{\frac{1}{2}}. \tag{3.30}$$

4 Approximation Basics on Time Scales

We need the following definition.

Definition 4.1. Let $f \in B\left([a,b]_{\mathbb{T}}\right)$. We define

$$\omega_1^{\mathbb{T}}(f,\delta) := \sup_{\substack{x,y\in[a,b]_{\mathbb{T}}:\\ |x-y|\leq\delta}} |f(x)-f(y)|, \quad 0<\delta\leq b-a, \tag{4.1}$$

and

$$\omega_1^{\mathbb{T}}(f,\delta) = \omega_1(f,b-a), \quad \text{if } \delta > b-a. \tag{4.2}$$

We call $\omega_1^{\mathbb{T}}(f,\cdot)$ the first modulus of continuity of f.

Theorem 4.2. *It holds*

(i) $\omega_1^{\mathbb{T}}(f,\delta) < \infty$,

(ii) $\omega_1^{\mathbb{T}}(f,\cdot)$ *is nonnegative and increasing on* $\mathbb{R}_+$,

(iii)
$$\omega_1^{\mathbb{T}}(f,0) = 0, \tag{4.3}$$

(iv) $\lim_{\delta\downarrow 0}\omega_1^{\mathbb{T}}(f,\delta) = 0$, *iff* f *is uniformly continuous on* $[a,b]_{\mathbb{T}}$.

Proof. (i), (ii) and (iii) are obvious.

(iv) ($\Rightarrow$) Let $\omega_1^{\mathbb{T}}(f,\delta) \to 0$ as $\delta \downarrow 0$. Then $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ with $\omega_1^{\mathbb{T}}(f,\delta) \leq \varepsilon$. I.e., $\forall\, x,y \in [a,b]_{\mathbb{T}} : |x-y| \leq \delta$ we get $|f(x)-f(y)| \leq \varepsilon$. That is f is uniformly continuous on $[a,b]_{\mathbb{T}}$.

($\Leftarrow$) Let f be uniformly continuous on $[a,b]_{\mathbb{T}}$. Then $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$: whenever $|x-y| \leq \delta$, $x,y \in [a,b]_{\mathbb{T}}$, it implies $|f(x)-f(y)| \leq \varepsilon$. I.e., $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$: $\omega_1^{\mathbb{T}}(f,\delta) \leq \varepsilon$. That is $\omega_1^{\mathbb{T}}(f,\delta) \to 0$ as $\delta \downarrow 0$. □

Remark 4.3. (i) $\omega_1^{\mathbb{T}}(f,\cdot)$ fails the subadditivity property and other important properties of usual first modulus of continuity defined on a continuous interval $[a,b]$ or $\mathbb{R}$.

(ii) Let f be continuous on $[a,b]_{\mathbb{T}}$ or $f \in C\left([a,b]_{\mathbb{T}}\right)$. Since f is continuous (in the time scale topology), it is uniformly continuous on the compact set $[a,b]_{\mathbb{T}}$.

We make the following definition.

Definition 4.4. (i) Denote by $C_u\left([a,b]_{\mathbb{T}}\right)$ all the continuous functions from $[a,b]_{\mathbb{T}} \to \mathbb{R}$.

Clearly $C_u\left([a,b]_{\mathbb{T}}\right)$ is a Banach space, and $C\left([a,b]_{\mathbb{T}}\right) \subset C_u\left([a,b]_{\mathbb{T}}\right)$.

(ii) Let $L : C\left([a,b]_{\mathbb{T}}\right) \to C\left([a,b]_{\mathbb{T}}\right)$ or $L : C_u\left([a,b]_{\mathbb{T}}\right) \to C_u\left([a,b]_{\mathbb{T}}\right)$ a linear operator. Let $f,g \in C_u\left([a,b]_{\mathbb{T}}\right)$ or in $C\left([a,b]_{\mathbb{T}}\right)$. The operator L is called positive iff whenever $f \geq g$ on $[a,b]_{\mathbb{T}}$, we have that $L(f) \geq L(g)$ on $[a,b]_{\mathbb{T}}$.

We need (see also [6] in another abstract setting) the following.

Theorem 4.5. *Let T be a positive linear functional from $C_u([a,b]_{\mathbb{T}})$ into $\mathbb{R}$, and $f \in C_u([a,b]_{\mathbb{T}})$ with $f \geq 0$.*

Then the following are equivalent

(i)

$$T(f) = 0 \tag{4.4}$$

(ii)

$$T(fg) = 0, \forall\, g \in C_u([a,b]_{\mathbb{T}}), \tag{4.5}$$

(iii)

$$T(f^m) = 0, \text{ for some } m \in \mathbb{N}. \tag{4.6}$$

Proof. (i)⇒(ii). Let any $n \in \mathbb{N}$ and enough to take a $g \in C_u([a,b]_{\mathbb{T}})$ with $g \geq 0$. If $\min\{g,n\} = g$, i.e., $g \leq n$, clearly it holds then

$$g \leq g + \frac{g^2}{2n}, \tag{4.7}$$

which is always true.

If $\min\{g,n\} = n$, i.e., $n \leq g$, since $(g-n)^2 \geq 0 \Leftrightarrow g^2 - 2gn + n^2 \geq 0 \Leftrightarrow g^2 + n^2 \geq 2gn \Leftrightarrow$

$$g \leq \frac{g^2+n^2}{2n} = \frac{g^2}{2n} + \frac{n}{2} \leq \frac{g^2}{2n} + n. \tag{4.8}$$

So we have proved that

$$0 \leq g - \min\{g,n\} \leq \frac{g^2}{2n}, \quad \forall\, n \in \mathbb{N}. \tag{4.9}$$

Hence

$$0 \leq fg - \min\{fg, nf\} \leq \frac{fg^2}{2n}, \ \forall\, n \in \mathbb{N},$$

so that

$$0 \leq T(fg) - T(\min\{fg, nf\}) \leq \frac{1}{2n} T\left(fg^2\right), \ \forall\, n \in \mathbb{N}. \tag{4.10}$$

However it holds

$$0 \leq T(\min\{fg, fn\}) \leq nT(f) = 0, \forall\, n \in \mathbb{N}. \tag{4.11}$$

That is

$$0 \leq T(fg) \leq \frac{T(fg^2)}{2n}, \quad \forall\, n \in \mathbb{N}, \tag{4.12}$$

producing

$$T(fg) = 0. \tag{4.13}$$

(ii)⇒(iii). obvious.

(iii)⇒(i). The result is trivial for $m = 1$, so suppose that $m \geq 2$. We proceed by complete induction hypothesis method. If $m = 2$ and $T\left(f^2\right) = 0$, then

$$0 \leq T\left((nf-1)^2\right) = T\left(n^2 f^2 - 2nf + 1\right) =$$

$$n^2 T\left(f^2\right) - 2nT(f) + T(1) = T(1) - 2nT(f). \tag{4.14}$$

Hence

$$0 \leq 2nT(f) \leq T(1), \tag{4.15}$$

and

$$0 \leq T(f) \leq \frac{T(1)}{2n}, \ \forall\, n \in \mathbb{N}. \tag{4.16}$$

That is $T(f) = 0$.

Next, let $m \geq 3$ such that $T(f^m) = 0$, we want to prove $T(f) = 0$. Assume for all m' with $2 \leq m' < m$ that we have true $T\left(f^{m'}\right) = 0$ implies $T(f) = 0$. Let $k = 0$ or $k = 1$ and particular m' as above such that $m + k = 2m'$.

We use the already proved direction (i)⇒(ii).

Since $T(f^m) = 0$ we get

$$T\left(f^{m+k}\right) = T\left(f^m \cdot f^k\right) = 0.$$

I.e.,

$$T\left(f^{m+k}\right) = 0. \tag{4.17}$$

Also it holds

$$T\left(\left(f^{m'}\right)^2\right) = T\left(f^{2m'}\right) = T\left(f^{m+k}\right) = 0, \tag{4.18}$$

i.e.,

$$T\left(\left(f^{m'}\right)^2\right) = 0. \tag{4.19}$$

By what we proved earlier in this direction we find $T\left(f^{m'}\right) = 0$. The last by complete induction hypothesis implies $T(f) = 0$. We are done. □

We prove Hölder's inequality for positive linear operators over time scales.

Theorem 4.6. *Let L be a positive linear operator from $C_u([a,b]_{\mathbb{T}})$ into itself and let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$.*

Then:

(i)

$$(L(|fg|))(x) \leq ((L(|f|^p))(x))^{\frac{1}{p}} ((L(|g|^q))(x))^{\frac{1}{q}}, \tag{4.20}$$

for any $x \in [a,b]_{\mathbb{T}}$ and any $f, g \in C_u([a,b]_{\mathbb{T}})$, i.e.,

$$L(|fg|) \leq (L(|f|^p))^{\frac{1}{p}} (L(|g|^q))^{\frac{1}{q}}, \tag{4.21}$$

which is attained when $|g| = c\,|f|^{p-1}$*,* $c > 0$*,*

and

(ii)

$$\|L(|fg|)\|_\infty \leq (\|L(|f|^p)\|_\infty)^{\frac{1}{p}} (\|L(|g|^q)\|_\infty)^{\frac{1}{q}}. \tag{4.22}$$

Proof. Notice for a particular $x \in [a,b]_{\mathbb{T}}$, the function $(L(\cdot))(x)$ is a positive linear functional on $C_u([a,b]_{\mathbb{T}})$.

For $\widetilde{\alpha}, \widetilde{\beta} \geq 0$, we have

$$\widetilde{\alpha}^{\frac{1}{p}} \widetilde{\beta}^{\frac{1}{q}} \leq \frac{\widetilde{\alpha}}{p} + \frac{\widetilde{\beta}}{q}, \tag{4.23}$$

with equality when $\widetilde{\alpha} = \widetilde{\beta}$.

Now suppose first that

$$(L(|f|^p))(x), (L(|g|^q))(x) \neq 0.$$

Apply inequality (4.23) for

$$\widetilde{\alpha}(t) = \frac{|f(t)|^p}{(L(|f|^p))(x)} \tag{4.24}$$

and

$$\widetilde{\beta}(t) = \frac{|g(t)|^q}{(L(|g|^q))(x)}, \tag{4.25}$$

to get

$$\frac{|f(t)|}{((L(|f|^p))(x))^{\frac{1}{p}}} \frac{|g(t)|}{((L(|g|^q))(x))^{\frac{1}{q}}} \leq \frac{|f(t)|^p}{p(L(|f|^p))(x)} + \frac{|g(t)|^q}{q(L(|g|^q))(x)}, \tag{4.26}$$

$\forall\, t \in [a,b]_{\mathbb{T}}$.

By positivity and linearity of the functional $(L(\cdot))(x)$, where $x \in [a,b]_{\mathbb{T}}$ is fixed and from (4.26) we find

$$\frac{L(|fg|)(x)}{((L(|f|^p))(x))^{\frac{1}{p}} ((L(|g|^q))(x))^{\frac{1}{q}}} \leq$$

$$\frac{(L(|f|^p))(x)}{p(L(|f|^p))(x)} + \frac{(L(|g|^q))(x)}{q(L(|g|^q))(x)} = \frac{1}{p} + \frac{1}{q} = 1, \tag{4.27}$$

that is proving (4.20).

Next we assume $|g| = c\,|f|^{p-1}$, $c > 0$.

Then

$$L(|fg|)(x) = c\,(L(|f|^p)(x)). \tag{4.28}$$

Also it holds

$$((L(|f|^p))(x))^{\frac{1}{p}} ((L(|g|^q))(x))^{\frac{1}{q}} = ((L(|f|^p))(x))^{\frac{1}{p}}\, c \left(\left(L\left(|f|^{q(p-1)}\right)\right)(x)\right)^{\frac{1}{q}}$$

$$= c\left(\left(L\left(|f|^p\right)\right)(x)\right)^{\frac{1}{p}}\left(\left(L\left(|f|^p\right)\right)(x)\right)^{\frac{1}{q}} = c\left(\left(L\left(|f|^p\right)\right)(x)\right), \tag{4.29}$$

proving that (4.21) is attained.

Assume now that

$$\left(L\left(|f|^p\right)\right)(x) = 0,\ \ p > 1 \text{ with } p \notin \mathbb{N}. \tag{4.30}$$

Consider $\lambda := [p] + 1 - p \geq 0$ ($[\cdot]$ is the integral part), then

$$[p] + 1 = \lambda + p > 1. \tag{4.31}$$

We have

$$L\left(|f|^{[p]+1}\right)(x) = L\left(|f|^{\lambda+p}\right)(x) = L\left(|f|^p \cdot |f|^{\lambda}\right)(x) = 0. \tag{4.32}$$

The last is true by Theorem 4.5 from the direction (i)⇒(ii).

I.e.,

$$L\left(|f|^{[p]+1}\right)(x) = 0. \tag{4.33}$$

Again by Theorem 4.5 from the direction (iii)⇒(ii), we derive

$$\left(L\left(|fg|\right)\right)(x) = L\left(|f|\,|g|\right)(x) = 0, \tag{4.34}$$

proving inequality (4.20) is valid as equality with both sides equal to zero.

The case of $p \in \mathbb{N} - \{1\}$ is similar and easier.

The proof of the theorem is now complete. □

5 Approximation on Time Scales

We give Korovkin's theorem on time scales.

Theorem 5.1. *Let $[a,b]_{\mathbb{T}} \subset \mathbb{T}$, $\mathbb{T}$ is a time scale, and $(L_n)_{n\in\mathbb{N}}$ be a sequence of positive linear operators, L_n is mapping $C_u\left([a,b]_{\mathbb{T}}\right)$ into itself. Suppose that $(L_n f)$ converges uniformly to f for the three test functions $f = 1, x, x^2$, $x \in [a,b]_{\mathbb{T}}$. Then $(L_n f)$ converges uniformly to f on $[a,b]_{\mathbb{T}}$ for any function $f \in C_u\left([a,b]_{\mathbb{T}}\right)$.*

Proof. (Similar to Bauer (1978), [2])

Denote $f(x) = x$ by id, $f(x) = x^2$ by id^2, where $x \in [a,b]_{\mathbb{T}}$. Every $f \in C_u\left([a,b]_{\mathbb{T}}\right)$ is bounded:

$$|f(x)| \leq \gamma,\ \forall\, x \in [a,b]_{\mathbb{T}}, \tag{5.1}$$

where $\gamma > 0$.

Also since f is uniformly continuous on $[a,b]_{\mathbb{T}}$, we have $\forall\, \varepsilon > 0\ \exists\, \delta > 0$: for all $x, y \in [a,b]_{\mathbb{T}}$,

$$|x - y| \leq \sqrt{\delta} \text{ then } |f(x) - f(y)| \leq \varepsilon, \tag{5.2}$$

$\Leftrightarrow$

$$(x-y)^2 \leq \delta \text{ then } |f(x) - f(y)| \leq \varepsilon. \tag{5.3}$$

If $(x-y)^2 \leq \delta$ then

$$|f(x) - f(y)| \leq \varepsilon + \alpha^* (x-y)^2, \tag{5.4}$$

with $\alpha^* := 2\gamma\delta^{-1}$,

and if $(x-y)^2 > \delta$, i.e., $\dfrac{(x-y)^2}{\delta} > 1$ we find that

$$|f(x) - f(y)| \leq 2\gamma \leq \frac{2\gamma}{\delta}(x-y)^2 \tag{5.5}$$

$$= \alpha^* (x-y)^2 \leq \varepsilon + \alpha^* (x-y)^2.$$

So in either case we get

$$|f(x) - f(y)| \leq \varepsilon + \alpha^* (x-y)^2. \tag{5.6}$$

Thus for any $y \in [a,b]_{\mathbb{T}}$ we get

$$|f - f(y)| \leq \varepsilon + \alpha^* (id - y)^2. \tag{5.7}$$

Linearity and positivity of the operators L_n then imply

$$|L_n f - f(y) L_n(1)| \leq \varepsilon L_n(1) + \alpha^* \left[L_n\left(id^2\right) - 2y L_n(id) + y^2 L_n(1)\right]. \tag{5.8}$$

Evaluating the above inequality at $x = y$, we get

$$|L_n f - f \cdot L_n 1| \leq \varepsilon(L_n 1 - 1) + \varepsilon + \alpha^* \left(L_n\left(id^2\right) - 2(id) L_n(id) + (id)^2 L_n(1)\right). \tag{5.9}$$

From the assumption on $(L_n(f))$ for the three functions $f = 1, id, id^2$ and from the triangle inequality it follows that $L_n f$ converges to f uniformly, true for any $f \in C_u([a,b]_{\mathbb{T}})$. $\square$

We give the following quantitative approximation theorem on time scales (see also Theorem 1.4).

Theorem 5.2. *Let $[a,b]_{\mathbb{T}} \subset \mathbb{T}$, $\mathbb{T}$ is a time scale, and $(L_n)_{n\in\mathbb{N}}$ be a sequence of positive linear operators, L_n is mapping $C_u([a,b]_{\mathbb{T}})$ into itself. Here we consider $f \in C_u([a,b]_{\mathbb{T}})$ such that*

$$|f(x) - f(y)| \leq M|x-y|, \ \forall\, x \in [a,b]_{\mathbb{T}}, \tag{5.10}$$

where $M > 0$. Then

$$\|L_n f - f\|_\infty \leq \|f\|_\infty \|L_n(1) - 1\|_\infty + M\left(\sqrt{\|L_n(1) - 1\|_\infty + 1}\right)$$

$$\left(\sqrt{\left\|\left(L_n\left(t^2\right)\right)(x)-x^2\right\|_\infty+2c\left\|\left(L_n(t)\right)(x)-x\right\|_\infty+c^2\left\|L_n(1)-1\right\|_\infty}\right), \quad (5.11)$$

where $c=\max(|a|,|b|)$. *If* $L_n(1)=1$ *we get*

$$\|L_nf-f\|_\infty \le M\sqrt{\left\|\left(L_n\left(t^2\right)\right)(x)-x^2\right\|_\infty+2c\left\|\left(L_n(t)\right)(x)-x\right\|_\infty}. \quad (5.12)$$

If $L_n\left(t^k\right)(x)$ *converges uniformly to* x^k *on* $[a,b]_{\mathbb{T}}$, *for* $k=0,1,2,$ *then by* (5.11), *we get that* L_nf *converges uniformly to* f *over* $[a,b]_{\mathbb{T}}$ *as* $n\to+\infty$.

Given that f^Δ exists and is bounded on $[a,b)_{\mathbb{T}}$, the Lipschitz constant M could have been such that $\left|f^\Delta(t)\right|\le M$, $\forall\, t\in[a,b)_{\mathbb{T}}$, see Remark 2.4.

Proof. Let $x\in[a,b]_{\mathbb{T}}$ be momentarily fixed. We notice the following

$$\begin{aligned}(L_nf)(x)-f(x) &= (L_nf)(x)-f(x)\left(L_n(1)\right)(x)+f(x)\left(L_n(1)\right)(x)-f(x)\\ &= (L_nf)(x)-L_n(f(x))(x)+f(x)\left(\left(L_n(1)\right)(x)-1\right)\\ &= \left(L_n(f-f(x))\right)(x)+f(x)\left(\left(L_n(1)\right)(x)-1\right). \quad (5.13)\end{aligned}$$

Therefore we obtain

$$\begin{gathered}|(L_nf)(x)-f(x)|\le\\ |\left(L_n(f-f(x))\right)(x)|+|f(x)|\left|\left(L_n(1)\right)(x)-1\right|\\ \le \left(L_n(|f-f(x)|)\right)(x)+|f(x)|\left|\left(L_n(1)\right)(x)-1\right| =: (*). \quad (5.14)\end{gathered}$$

Since we assumed

$$|f(y)-f(x)|\le M|y-x|, \quad (5.15)$$

$\forall\, x,y\in[a,b]_{\mathbb{T}}$, we can write

$$|f(\cdot)-f(x)|\le M|id(\cdot)-x|, \quad (5.16)$$

for any fixed $x\in[a,b]_{\mathbb{T}}$. Hence

$$(*)\le M\left(L_n(|id(\cdot)-x|)\right)(x)+|f(x)|\left|\left(L_n(1)\right)(x)-1\right| \overset{(4.20)}{\le} \quad (5.17)$$

$$M\left(\left(L_n(1)\right)(x)\right)^{\frac{1}{2}}\left(\left(L_n\left((id(\cdot)-x)^2\right)\right)(x)\right)^{\frac{1}{2}}+|f(x)|\left|\left(L_n(1)\right)(x)-1\right|\le$$

$$M\left\|L_n(1)\right\|_\infty^{\frac{1}{2}}\left\|L_n\left((id(\cdot)-x)^2\right)(x)\right\|_\infty^{\frac{1}{2}}+\|f\|_\infty\left\|L_n(1)-1\right\|_\infty. \quad (5.18)$$

We have derived that

$$\begin{gathered}\|(L_nf)-f\|_\infty\le\\ \|f\|_\infty\left\|L_n(1)-1\right\|_\infty+M\sqrt{\left\|L_n(1)\right\|_\infty}\sqrt{\left\|\left(L_n\left((t-x)^2\right)\right)(x)\right\|_\infty}\le\\ \|f\|_\infty\left\|L_n(1)-1\right\|_\infty+M\sqrt{\left\|L_n(1)-1\right\|_\infty+1}\\ \sqrt{\left\|\left(L_n\left(t^2\right)\right)(x)-x^2\right\|_\infty+2c\left\|\left(L_n(t)\right)(x)-x\right\|_\infty+c^2\left\|L_n(1)-1\right\|_\infty}, \quad (5.19)\end{gathered}$$

proving the claim. □

When smoothness is present the speed of convergence improves dramatically. We present

Theorem 5.3. *Let $[a,b]_{\mathbb{T}} \subset \mathbb{T}$, $\mathbb{T}$ is a time scale, and $(L_n)_{n\in\mathbb{N}}$ be a sequence of positive linear operators, L_n is mapping $C_u^1([a,b]_{\mathbb{T}})$ (the space of one time continuously differentiable functions on $[a,b]_{\mathbb{T}}$) into $C_u([a,b]_{\mathbb{T}})$. Here we consider $f \in C_u^1([a,b]_{\mathbb{T}})$ such that*

$$\left|f^{\Delta}(x) - f^{\Delta}(y)\right| \leq M|x-y|, \tag{5.20}$$

$\forall\, x,y \in [a,b]_{\mathbb{T}}$, where $M > 0$. Then

$$\|L_n(f) - f\|_{\infty} \leq \|f\|_{\infty}\|L_n(1) - 1\|_{\infty} +$$

$$\left\|f^{\Delta}\right\|_{\infty}\|(L_n(t-x))(x)\|_{\infty} + M\left\|\left(L_n\left((t-x)^2\right)\right)(x)\right\|_{\infty}. \tag{5.21}$$

We have that

$$\|(L_n(t-x))(x)\|_{\infty} \leq$$

$$\left(\sqrt{\|L_n(1)-1\|_{\infty}+1}\right)\sqrt{\left\|\left(\left(L_n(t-x)^2\right)\right)(x)\right\|_{\infty}}, \tag{5.22}$$

and

$$\left\|\left(L_n\left((t-x)^2\right)\right)(x)\right\|_{\infty} \leq \left\|\left(L_n\left(t^2\right)\right)(x) - x^2\right\|_{\infty} +$$

$$2c\|(L_n(t))(x) - x\|_{\infty} + c^2\|L_n(1) - 1\|_{\infty}, \tag{5.23}$$

where $c := \max(|a|,|b|)$.

Clearly by (5.21), (5.22), (5.23), *under the assumptions $\left(L_n\left(t^k\right)\right)(x) \to x^k$, $k = 0,1,2$, converge uniformly on $[a,b]_{\mathbb{T}}$, we obtain $L_n f \to f$, $f \in C_u^1([a,b]_{\mathbb{T}})$, converges uniformly, as $n \to +\infty$.*

If $(L_n(1))(x) = 1$ and $(L_n(t))(x) = x$, we get that

$$\left(L_n(t-x)^2\right)(x) = \left(L_n\left(t^2\right)\right)(x) - x^2, \tag{5.24}$$

and then the speed of uniform convergence of $L_n f \to f$ squares in comparison to Theorem 5.2, under the assumption that $\left(L_n\left(t^2\right)\right)(x) \to x^2$, converges uniformly on $[a,b]_{\mathbb{T}}$, as $n \to +\infty$.

Proof. For the Delta Δ-integral on time scales and properties we refer to [4].

By fundamental property

$$\int_x^t f^{\Delta}(\tau)\,\Delta\tau = f(t) - f(x), \tag{5.25}$$

we have

$$f(t) = f(x) + \int_x^t f^{\Delta}(\tau)\,\Delta\tau, \tag{5.26}$$

and

$$f(t)=f(x)+f^{\Delta}(x)(t-x)+\int_{x}^{t}\left(f^{\Delta}(\tau)-f^{\Delta}(x)\right)\Delta\tau. \tag{5.27}$$

Call

$$\mathcal{R}(t,x):=\int_{x}^{t}\left(f^{\Delta}(\tau)-f^{\Delta}(x)\right)\Delta\tau. \tag{5.28}$$

We estimate first $\mathcal{R}(t,x)$.

(i) If $t\geq x$ we get

$$|\mathcal{R}(t,x)|=\left|\int_{x}^{t}\left(f^{\Delta}(\tau)-f^{\Delta}(x)\right)\Delta\tau\right|\leq\int_{x}^{t}\left|f^{\Delta}(\tau)-f^{\Delta}(x)\right|\Delta\tau$$

$$\leq M\int_{x}^{t}|\tau-x|\,\Delta\tau\leq M(t-x)^{2}. \tag{5.29}$$

(ii) If $t<x$ we obtain

$$|\mathcal{R}(t,x)|=\left|\int_{t}^{x}\left(f^{\Delta}(\tau)-f^{\Delta}(x)\right)\Delta\tau\right|\leq\int_{t}^{x}\left|f^{\Delta}(\tau)-f^{\Delta}(x)\right|\Delta\tau$$

$$\leq M\int_{t}^{x}|\tau-x|\,\Delta\tau\leq M(x-t)^{2}. \tag{5.30}$$

So that, in either case, we have found

$$|\mathcal{R}(t,x)|\leq M(t-x)^{2}. \tag{5.31}$$

Hence we obtain

$$(L_n(f(t)))(x)=(L_n(f(x)))(x)+f^{\Delta}(x)(L_n(t-x))(x)+(L_n(\mathcal{R}(t,x)))(x), \tag{5.32}$$

which gives us

$$(L_n(f(t)))(x)-f(x)=f(x)((L_n(1))(x)-1)+ \tag{5.33}$$
$$f^{\Delta}(x)(L_n(t-x))(x)+(L_n(\mathcal{R}(t,x)))(x).$$

Therefore it holds

$$|(L_n(f(t)))(x)-f(x)|\leq|f(x)|\,|(L_n(1))(x)-1|+ \tag{5.34}$$
$$\left|f^{\Delta}(x)\right|\,|(L_n(t-x))(x)|+(L_n(|\mathcal{R}(t,x)|))(x)\leq$$
$$|f(x)|\,|(L_n(1))(x)-1|+\left|f^{\Delta}(x)\right|\,|(L_n(t-x))(x)|+M\left(\left(L_n\left((t-x)^{2}\right)\right)(x)\right)\leq$$
$$\|f\|_{\infty}\|L_n(1)-1\|_{\infty}+\left\|f^{\Delta}\right\|_{\infty}\|(L_n(t-x))(x)\|_{\infty}+M\left\|\left(L_n\left((t-x)^{2}\right)\right)(x)\right\|_{\infty}. \tag{5.35}$$

Also, as earlier, we have

$$\begin{gathered}\left|\left(L_n\left(t-x\right)\right)\left(x\right)\right| \leq \left(L_n\left(\left|t-x\right|\right)\right)\left(x\right) \leq \\ \sqrt{\left(L_n\left(1\right)\right)\left(x\right)}\sqrt{\left(L_n\left(\left(t-x\right)^2\right)\right)\left(x\right)} \leq \\ \sqrt{\left\|L_n\left(1\right)\right\|_\infty}\sqrt{\left\|\left(\left(L_n\left(t-x\right)^2\right)\right)\left(x\right)\right\|_\infty}.\end{gathered} \tag{5.36}$$

That is

$$\left\|\left(L_n\left(t-x\right)\right)\left(x\right)\right\|_\infty \leq \sqrt{\left\|L_n\left(1\right)\right\|_\infty}\sqrt{\left\|\left(\left(L_n\left(t-x\right)^2\right)\right)\left(x\right)\right\|_\infty}. \tag{5.37}$$

The proof of the theorem now is complete. □

We mention the following result.

Theorem 5.4 (See [3, 4, 8], Taylor's formula)**.** *Assume $\mathbb{T}^k = \mathbb{T}$ and $f \in C_{rd}^n\left(\mathbb{T}\right)$ (the space of n times rd-continuously differentiable functions, see [4]), $n \in \mathbb{N}$ and $s, t \in \mathbb{T}$. Here generally define $h_0\left(t,s\right) = 1$, $\forall\, s,t \in \mathbb{T}$; $k \in \mathbb{N}_0$, and*

$$h_{k+1}\left(t,s\right) = \int_s^t h_k\left(\tau,s\right)\Delta\tau,\ \ \forall\, s,t \in \mathbb{T}. \tag{5.38}$$

(then $h_k^{\Delta}\left(t,s\right) = h_{k-1}\left(t,s\right)$, for $k \in \mathbb{N}$, $\forall\, t \in \mathbb{T}$, for each $s \in \mathbb{T}$ fixed). Then

$$f\left(t\right) = \sum_{k=0}^{n-1} f^{\Delta^k}\left(s\right)h_k\left(t,s\right) + \int_s^t h_{n-1}\left(t,\sigma\left(\tau\right)\right)f^{\Delta^n}\left(\tau\right)\Delta\tau \tag{5.39}$$

(above $f^{\Delta^0}\left(s\right) = f\left(s\right)$).

We need the following definition.

Definition 5.5 (See [4])**.** Let the functions $g_k : \mathbb{T}^2 \to \mathbb{R}$, $k \in \mathbb{N}_0$, defined recursively as follows:

$$g_0\left(t,s\right) = 1,\ \forall\, s,t \in \mathbb{T}, \tag{5.40}$$

and

$$g_{k+1}\left(t,s\right) = \int_s^t g_k\left(\sigma\left(\tau\right),s\right)\Delta\tau,\ \ \forall\, s,t \in \mathbb{T}. \tag{5.41}$$

Notice that

$$g_k^{\Delta}\left(t,s\right) = g_{k-1}\left(\sigma\left(t\right),s\right),\ \ \text{for } k \in \mathbb{N}, t \in \mathbb{T}^k. \tag{5.42}$$

Also it holds

$$g_1\left(t,s\right) = h_1\left(t,s\right) = t-s,\ \forall\, s,t \in \mathbb{T}. \tag{5.43}$$

We need the following result.

Theorem 5.6 (See [4]). *It holds*

$$h_n(t,s) = (-1)^n g_n(s,t), \ \forall\, t \in \mathbb{T} \tag{5.44}$$

and all $s \in \mathbb{T}^{k^n}$ *(*$\mathbb{T}^{k^2} := \left(\mathbb{T}^k\right)^k$*, ...,etc.)*

We need the following corollary.

Corollary 5.7. *Assume* $\mathbb{T}^k = \mathbb{T}$*,* $f \in C_{rd}^n(\mathbb{T})$*,* $n \in \mathbb{N}$ *and* $s,t \in \mathbb{T}$*. Then*

$$f(t) = \sum_{k=0}^{n-1} f^{\Delta^k}(s)\, h_k(t,s) +$$

$$(-1)^n f^{\Delta^n}(s)\, g_n(s,t) + \int_s^t h_{n-1}(t,\sigma(\tau)) \left(f^{\Delta^n}(\tau) - f^{\Delta^n}(s)\right) \Delta\tau. \tag{5.45}$$

Proof. By (5.39) and (5.44). Namely we have

$$\int_s^t h_{n-1}(t,\sigma(\tau))\, \Delta\tau = (-1)^{n-1} \int_s^t g_{n-1}(\sigma(\tau),t)\, \Delta\tau = \tag{5.46}$$

$$(-1)^n \int_t^s g_{n-1}(\sigma(\tau),t)\, \Delta\tau = (-1)^n g_n(s,t).$$

□

We make the following remark.

Remark 5.8 (to Corollary 5.7). One can easily prove inductively that

$$|h_k(t,s)| \leq |t-s|^k, \ \forall\, s,t \in \mathbb{T}, \tag{5.47}$$

$\forall\, k \in \mathbb{N}_0$.

Call

$$\mathcal{R}(t,s) := \int_s^t h_{n-1}(t,\sigma(\tau)) \left(f^{\Delta^n}(\tau) - f^{\Delta^n}(s)\right) \Delta\tau. \tag{5.48}$$

Assume that

$$\left|f^{\Delta^n}(t) - f^{\Delta^n}(s)\right| \leq M |t-s|^\alpha, \ \forall\, s,t \in \mathbb{T}, \tag{5.49}$$

where α is fixed such that $0 < \alpha \leq 1$.

We estimate $\mathcal{R}(t,s)$.

(i) Assume $t \geq s$. Then

$$|\mathcal{R}(t,s)| \leq \int_s^t |h_{n-1}(t,\sigma(\tau))| \left|f^{\Delta^n}(\tau) - f^{\Delta^n}(s)\right| \Delta\tau \leq \tag{5.50}$$

$$M \int_s^t |t-\sigma(\tau)|^{n-1} |\tau - s|^\alpha\, \Delta\tau \leq M(\sigma(t)-s)^{n-1} \int_s^t |\tau - s|^\alpha\, \Delta\tau \leq$$

$$M(\sigma(t)-s)^{n-1}(t-s)^{\alpha+1}. \tag{5.51}$$

We proved, if $t \geq s$ then

$$|\mathcal{R}(t,s)| \leq M(\sigma(t)-s)^{n-1}(t-s)^{\alpha+1} \leq M(\sigma(t)-s)^{n+\alpha}. \tag{5.52}$$

(ii) Assume $t < s$. Then

$$|\mathcal{R}(t,s)| \leq \int_t^s |h_{n-1}(t,\sigma(\tau))| \left|f^{\Delta^n}(\tau)-f^{\Delta^n}(s)\right| \Delta\tau \leq \tag{5.53}$$

$$M\int_t^s |t-\sigma(\tau)|^{n-1}|\tau-s|^{\alpha}\Delta\tau \leq M(\sigma(s)-t)^{n-1}(s-t)^{\alpha+1}$$
$$\leq M(\sigma(s)-t)^{n+\alpha}.$$

We proved, if $t < s$ then

$$|\mathcal{R}(t,s)| \leq M(\sigma(s)-t)^{n+\alpha}. \tag{5.54}$$

I.e., we have found

$$|\mathcal{R}(t,s)| \leq M\varphi(t,s), \tag{5.55}$$

where

$$\varphi(t,s) := \begin{cases} (\sigma(t)-s)^{n+\alpha}, & \text{if } t \geq s, \\ (\sigma(s)-t)^{n+\alpha}, & \text{if } t < s, \end{cases} \tag{5.56}$$

notice that $\varphi(\cdot,s) \in C_{rd}(\mathbb{T}), \forall\, s \in \mathbb{T}$.

We give the following related result.

Theorem 5.9. *Assume $\mathbb{T}^k = \mathbb{T}$, $\mathbb{T}$ is a time scale and $s,t \in [a,b]_{\mathbb{T}} \subset \mathbb{T}$. Let $(L_N)_{N\in\mathbb{N}}$ be a sequence of positive linear operators, L_N is mapping $C_{rd}([a,b]_{\mathbb{T}})$ into itself, such that $L_N(1)=1$. Consider $f \in C_u^n([a,b]_{\mathbb{T}})$, $n \in \mathbb{N}$, such that*

$$\left|f^{\Delta^n}(t)-f^{\Delta^n}(s)\right| \leq M|t-s|^{\alpha},\ \forall\, t,s \in [a,b]_{\mathbb{T}}, \tag{5.57}$$

α is fixed such that $0<\alpha\leq 1$, $M>0$. Let also $\varphi(t,s)$ defined by (5.56)*. Then*

(i)

$$\Bigg|(L_N(f(t)))(s)-f(s)-\sum_{k=1}^{n-1} f^{\Delta^k}(s)(L_N(h_k(t,s)))(s)$$
$$+(-1)^{n+1}f^{\Delta^n}(s)(L_N(g_n(s,t)))(s)\Big|$$
$$\leq M(L_N(\varphi(t,s)))(s),\ \forall\, s \in [a,b]_{\mathbb{T}}, \tag{5.58}$$

(ii) additionally assume that $f^{\Delta^k}(s)=0$, $k=1,...,n$, for a fixed $s \in [a,b]_{\mathbb{T}}$, to derive

$$|(L_N(f(t)))(s)-f(s)| \leq M(L_N(\varphi(t,s)))(s), \tag{5.59}$$

and

(iii)

$$\left|\left(L_N\left(f\left(t\right)\right)\right)\left(s\right)-f\left(s\right)\right|\leq\sum_{k=1}^{n-1}\left|f^{\Delta^k}\left(s\right)\right|\left|\left(L_N\left(h_k\left(t,s\right)\right)\right)\left(s\right)\right|+$$

$$\left|f^{\Delta^n}\left(s\right)\right|\left|\left(L_N\left(g_n\left(s,t\right)\right)\right)\left(s\right)\right|+M\left(L_N\left(\varphi\left(t,s\right)\right)\right)\left(s\right),\ \forall\, s\in[a,b]_{\mathbb{T}}. \tag{5.60}$$

If $\left(L_N\left(\varphi\left(t,s\right)\right)\right)\left(s\right)\to 0$*, by (ii), we get*

$$\left(L_N\left(f\left(t\right)\right)\right)\left(s\right)\to f\left(s\right),\ \text{as } N\to+\infty.$$

Other useful convergence consequences follow by (i) and (iii).

Proof. By (5.45), we derive

$$\left(\Delta_N\left(f\left(t\right)\right)\right)\left(s\right):=\left(L_N\left(f\left(t\right)\right)\right)\left(s\right)-f\left(s\right)-\sum_{k=1}^{n-1}f^{\Delta^k}\left(s\right)\left(L_N\left(h_k\left(t,s\right)\right)\right)\left(s\right)$$

$$+\left(-1\right)^{n+1}f^{\Delta^n}\left(s\right)\left(L_N\left(g_n\left(s,t\right)\right)\right)\left(s\right)=\left(L_N\left(\mathcal{R}\left(t,s\right)\right)\right)\left(s\right). \tag{5.61}$$

Hence it holds

$$\left|\left(\Delta_N\left(f\left(t\right)\right)\right)\left(s\right)\right|\leq\left(L_N\left(\left|\mathcal{R}\left(t,s\right)\right|\right)\right)\left(s\right)\overset{(5.55)}{\leq}M\left(L_N\left(\varphi\left(t,s\right)\right)\right)\left(s\right), \tag{5.62}$$

$\forall\, s\in[a,b]_{\mathbb{T}}$, proving the claim. □

Remark 5.10. Let $[a,b]_{\mathbb{T}}\subset\mathbb{T}$, $\mathbb{T}$ is a time scale, such that $\rho\left(b\right)=b$. Hence $C\left([a,b]_{\mathbb{T}}\right)=C_u\left([a,b]_{\mathbb{T}}\right)$. Let $f\in C\left([a,b]_{\mathbb{T}}\right)$, and $N\in\mathbb{N}$. Take $\varepsilon=\frac{1}{N}$, by uniform continuity of f on the compact set $[a,b]_{\mathbb{T}}$, there exists a $\delta=\delta\left(\varepsilon\right)>0$ such that $t',t''\in[a,b]_{\mathbb{T}}$, $\left|t'-t''\right|\leq\delta$ implies

$$\left|f\left(t'\right)-f\left(t''\right)\right|<\frac{1}{N}. \tag{5.63}$$

Define again

$$\chi_t\left(\xi\right):=\begin{cases}1, & \text{for } \xi\in[a,t)_{\mathbb{T}},\\ 0, & \text{for } \xi\in[t,b]_{\mathbb{T}}\end{cases}, \tag{5.64}$$

for $t\in[a,b)_{\mathbb{T}}$, and

$$\chi_b\left(\xi\right)=1 \text{ for all } \xi\in[a,b]_{\mathbb{T}}. \tag{5.65}$$

Notice that $\chi_a\left(\xi\right)=0$, $\xi\in[a,b]_{\mathbb{T}}$.

For a partition $P=\{t_0,t_1,...,t_n\}$ ($t_0=a<t_1<...<t_{n-1}<t_n=b$) belonging to $\mathcal{P}_\delta\left([a,b]_{\mathbb{T}}\right)$ we consider the step function $f^{\left(\frac{1}{N}\right)}$ defined on $[a,b]_{\mathbb{T}}$ by

$$f^{\left(\frac{1}{N}\right)}\left(t\right):=f\left(t_{k-1}\right),\ \text{if } t\in[t_{k-1},t_k)_{\mathbb{T}},$$

$k = 1, ..., n$, and

$$f^{\left(\frac{1}{N}\right)}(b) := f(t_n) = f(b),$$

which can be written in the form

$$f^{\left(\frac{1}{N}\right)}(t) = \sum_{k=1}^{n} f(t_{k-1}) \left[\chi_{t_k}(t) - \chi_{t_{k-1}}(t)\right], \tag{5.66}$$

for all $t \in [a, b]_{\mathbb{T}}$.

In the proof of Theorem 3.1 we saw that for $t_{k-1} < t_k$ we get $0 \leq \chi_{t_{k-1}} \leq \chi_{t_k} \leq ... \leq \chi_b$. Thus it holds

$$\chi_{t_k}(t) - \chi_{t_{k-1}}(t) \geq 0, \ \forall \, t \in [a, b]_{\mathbb{T}}. \tag{5.67}$$

We define the following linear operator on $C([a, b]_{\mathbb{T}})$:

$$(L_N(f))(t) := f^{\left(\frac{1}{N}\right)}(t) = \sum_{k=1}^{n} f(t_{k-1}) \left[\chi_{t_k}(t) - \chi_{t_{k-1}}(t)\right], \tag{5.68}$$

$\forall \, t \in [a, b]_{\mathbb{T}}$.

Clearly L_N is a positive linear operator from $C_u([a, b]_{\mathbb{T}})$ into $B([a, b]_{\mathbb{T}})$. Notice that $(L_n(f))(t_k) = f^{\left(\frac{1}{N}\right)}(t_k) = f(t_k)$, for $k = 0, 1, ..., n$, that is $(L_N(f))$ has the interpolation property over P.

As in the proof of Theorem 3.1 we get

$$\|L_N f - f\|_{\infty} = \left\| f^{\left(\frac{1}{N}\right)} - f \right\|_{\infty} \leq \frac{1}{N}. \tag{5.69}$$

The last proves uniform convergence of $L_N f$ to f, as $N \to +\infty$.

The above genuine example on time scales proves that our theory of approximation by positive linear operators on time scales is not trivial and not only valid on continuous intervals $[a, b]$ of $\mathbb{R}$.

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