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Global Behavior of a Higher Order Rational Difference Equation

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Abstract

The aim of this paper is to investigate the global stability and the existence of unbounded solutions of the difference equation

$$x_{n+1} = \frac{A + Bx_{n-2r-1}}{C + Dx^k_{n-2l}} \quad n = 0, 1, \ldots$$

where the initial values are nonnegative real numbers and A, B are nonnegative real numbers, $C, D > 0$ and r, l, k are nonnegative integers.

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1 Introduction and Preliminaries

The mathematical modeling of a physical or economical problem very often leads to difference equations (for partial review of the theory of difference equations and their applications see [3, 6, 7, 9]).

The study of nonlinear rational difference equations of higher order is of paramount importance, since we still know so little about such equations. It is worthwhile to point out that although several approaches have been developed for finding the global character of difference equations, a relatively a large number of difference equations has not been thoroughly understood yet [5, 7, 10, 11].

In [1], we have investigated the global asymptotic stability of the difference equation

$$x_{n+1} = \frac{Bx_{n-2k-1}}{C + D\prod_{i=l}^{k} x_{n-2i}}, \quad n = 0, 1, \ldots$$

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Also in [4], we have discussed the global asymptotic stability of the difference equation

$$x_{n+1} = \frac{Ax_{n-2r-1}}{B + Cx_{n-2l}x_{n-2k}}, \quad n = 0, 1, \ldots$$

In this paper, we study the global asymptotic stability and the existence of unbounded solutions of the difference equation

$$x_{n+1} = \frac{A + Bx_{n-2r-1}}{C + Dx_{n-2l}^k}, \quad n = 0, 1, \ldots \tag{1.1}$$

where A, B are nonnegative real numbers and $C, D > 0$ and r, l, k are nonnegative integers.

When $r = l = 0$ and $k = 1$, equation (1.1) is reduced to the difference equation

$$x_{n+1} = \frac{A + Bx_{n-1}}{C + Dx_n}, \quad n = 0, 1, \ldots$$

which has been investigated in [10].

When $r = l = 0, k = 2$, equation (1.1) is reduced to the equation

$$x_{n+1} = \frac{A + Bx_{n-1}}{C + Dx_n^2}, \quad n = 0, 1, \ldots$$

which we have discussed in [2], where A, B are nonnegative real numbers and $C, D > 0$.

We give some preliminaries which will be needed in this paper. Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, \ldots \tag{1.2}$$

where $f : R^{k+1} \to R$.

Definition 1.1 (See [9]). An equilibrium point for equation (1.2) is a point $\bar{x} \in R$ such that $\bar{x} = f(\bar{x}, \bar{x}, \ldots, \bar{x})$.

Definition 1.2 (See [9]).

1. An equilibrium point $\bar{x}$ for equation (1.2) is called *locally stable* if for every $\epsilon > 0$, there exists a $\delta > 0$ such that every solution $\{x_n\}$ with initial conditions $x_{-k}, x_{-k+1}, \ldots, x_0 \in]\bar{x} - \delta, \bar{x} + \delta[$ is such that $x_n \in]\bar{x} - \epsilon, \bar{x} + \epsilon[$ for all $n \in \mathbb{N}$. Otherwise $\bar{x}$ is said to be *unstable*.

2. The equilibrium point $\bar{x}$ of equation (1.2) is called *locally asymptotically stable* if it is locally stable and there exists $\gamma > 0$ such that for any initial conditions $x_{-k}, x_{-k+1}, \ldots, x_0 \in]\bar{x} - \gamma, \bar{x} + \gamma[$, the corresponding solution $\{x_n\}$ tends to $\bar{x}$, and a *global attractor* if every solution $\{x_n\}$ converges to $\bar{x}$ as $n \to \infty$.

3. The equilibrium point $\bar{x}$ for equation (1.2) is called *globally asymptotically stable* if it is locally asymptotically stable and global attractor.

Suppose that f is continuously differentiable in some open neighborhood of $\bar{x}$. Let

$$a_i = \frac{\partial f}{\partial x_{n-i}}(\bar{x}, \ldots, \bar{x}), \quad \text{for} \quad i = 0, 1, \ldots, k$$

denote the partial derivatives of $f(x_n, x_{n-1}, \ldots, x_{n-k})$ with respect to x_{n-i} evaluated at the equilibrium point $\bar{x}$ of equation (1.2). Then the equation

$$z_{n+1} = \sum_{i=0}^{k} a_i z_{n-i}, \quad n = 0, 1, \ldots \tag{1.3}$$

is called the linearized equation associated with equation (1.2) about the equilibrium point $\bar{x}$, and the equation

$$\lambda^{k+1} - \sum_{i=0}^{k} a_i \lambda^{k-i} = 0 \tag{1.4}$$

is called the characteristic equation associated with equation (1.3) about the equilibrium point $\bar{x}$.

Theorem 1.3 (See [9])**.** *Assume that f is a C^1 function and let $\bar{x}$ be an equilibrium point of equation* (1.2). *Then the following statements are true:*

1. *If all roots of equation* (1.4) *lie in the open disk $|\lambda| < 1$, then $\bar{x}$ is locally asymptotically stable.*
2. *If at least one root of equation* (1.4) *has absolute value greater than one, then $\bar{x}$ is unstable.*

Theorem 1.4 (See [8])**.** *Assume that* $\sum_{i=0}^{k} | a_i | < 1$. *Then every root of equation* (1.4) *has absolute value less than one.*

Definition 1.5 (See [10])**.** A positive semicycle of a solution $\{x_n\}_{n=-k}^{\infty}$ of equation (1.2) consists of a "string" of terms $\{x_l, x_{l+1}, \ldots, x_m\}$, all greater than or equal to the equilibrium $\bar{x}$, with $l \geq -k$ and $m \leq \infty$ and such that

$$\text{either } l = -k, \quad \text{or } l > -k \text{ and } x_{l-1} < \bar{x}$$

and

$$\text{either } m = \infty, \quad \text{or } m < \infty \text{ and } x_{m+1} < \bar{x}.$$

Definition 1.6 (See [10])**.** A negative semicycle of a solution $\{x_n\}_{n=-k}^{\infty}$ of equation (1.2) consists of a "string" of terms $\{x_l, x_{l+1}, \ldots, x_m\}$, all less than or equal to the equilibrium $\bar{x}$, with $l \geq -k$ and $m \leq \infty$ and such that

$$\text{either } l = -k, \quad \text{or } l > -k \text{ and } x_{l-1} \geq \bar{x}$$

and

$$\text{either } m = \infty, \quad \text{or } m < \infty \text{ and } x_{m+1} \geq \bar{x}.$$

When $k = 0$, Equation (1.1) reduces to the linear nonhomogeneous difference equation

$$x_{n+1} = \frac{B}{C+D} x_{n-2r-1} + \frac{A}{C+D}, \quad n = 0, 1, \ldots$$

The equilibrium point $\bar{x} = \dfrac{A}{C+D-B}$ of this equation is globally asymptotically stable when $B < C + D$ and unstable when $B \geq C + D$.

When $k \geq 1$, the change of variables $x_n = \sqrt[k]{\dfrac{C}{D}} y_n$ reduces equation (1.1) to the difference equation

$$y_{n+1} = \frac{p + q y_{n-2r-1}}{1 + y_{n-2l}^k}, \quad n = 0, 1, \ldots \tag{1.5}$$

where $p = \dfrac{A}{C}\sqrt[k]{\dfrac{D}{C}}, q = \dfrac{B}{C}$.

When $k = 1$, equation (1.5) reduces to the difference equation

$$y_{n+1} = \frac{p + q y_{n-2r-1}}{1 + y_{n-2l}}, \quad n = 0, 1, \ldots$$

This equation was discussed in [7].

In the following, we assume that $k \geq 2$.

2 Linearized Stability Analysis

Now we determine the equilibrium points of equation (1.5) and discuss their local asymptotic behavior. It is clear that the values of the equilibrium points depends on p and q.

The equilibrium points of equation (1.5) are the zeros of the function $f(x) = x^{k+1} + (1-q)x - p$.

When $q > 1$, equation (1.5) has a unique positive equilibrium point $\bar{y} > \sqrt[k]{q-1}$. When $q < 1$, equation (1.5) has a unique positive equilibrium point $\bar{y}$ such that $\bar{y} > \sqrt[k]{\dfrac{1-q}{k-1}}$ if $p > k\left(\dfrac{1-q}{k-1}\right)^{\frac{k+1}{k}}$ and $0 < \bar{y} < \sqrt[k]{\dfrac{1-q}{k-1}}$ if $p < k\left(\dfrac{1-q}{k-1}\right)^{\frac{k+1}{k}}$.

Now assume that $\mathfrak{K} = \max\{2l, 2r+1\}$. Also let t be the largest nonnegative integer such that $0 < 2t+1 \leq \mathfrak{K}$ and s be the largest nonnegative integer such that $0 \leq 2s \leq \mathfrak{K}$.

The linearized equation associated with equation (1.5) about the positive equilibrium $\bar{y}$ is

$$z_{n+1} - \frac{q}{1+\bar{y}^k} z_{n-2r-1} + \frac{k\bar{y}^k}{1+\bar{y}^k} z_{n-2l} = 0, \quad n = 0, 1, \ldots \tag{2.1}$$

The characteristic equation associated with this equation is

$$\lambda^{\mathfrak{K}+1} - \frac{q}{1+\bar{y}^k}\lambda^{\mathfrak{K}-2r-1} + \frac{k\bar{y}^k}{1+\bar{y}^k}\lambda^{\mathfrak{K}-2l} = 0. \tag{2.2}$$

We summarize the results of this section in the following two theorems.

Theorem 2.1. *Assume that* $\mathfrak{K} = 2l$ *and let* $\bar{y}$ *be the unique positive equilibrium point of equation* (1.5). *Then the following statements are true.*

1. *If* $q > 1$, *then* $\bar{y}$ *is an unstable equilibrium point.*
2. *If* $q < 1$, *then we have the following:*
   (a) *If* $p < k\left(\frac{1-q}{k-1}\right)^{\frac{k+1}{k}}$, *then* $\bar{y}$ *is locally asymptotically stable.*
   (b) *If* $p > k\left(\frac{1-q}{k-1}\right)^{\frac{k+1}{k}}$, *then* $\bar{y}$ *is an unstable equilibrium point.*

Proof. When $\mathfrak{K} = 2l$, the associated characteristic equation (2.2) becomes

$$\lambda^{2l+1} - \frac{q}{1+\bar{y}^k}\lambda^{2l-2r-1} + \frac{k\bar{y}^k}{1+\bar{y}^k} = 0. \tag{2.3}$$

1. If $q > 1$, then $\bar{y} > \sqrt[k]{q-1}$. It follows that the characteristic equation (2.3) has a root in the interval $(-\infty, -1)$ and so $\bar{y}$ is an unstable equilibrium point.
2. Assume that $q < 1$ and let $\bar{y}$ be the positive equilibrium point of equation (1.5). Then we have the following:
   (a) If $p < k\left(\frac{1-q}{k-1}\right)^{\frac{k+1}{k}}$, then $\bar{y} < \sqrt[k]{\frac{1-q}{k-1}}$. By Theorem (1.4), we have
   $$\frac{q}{1+\bar{y}^k} + \frac{k\bar{y}^k}{1+\bar{y}^k} = \frac{q+k\bar{y}^k}{1+\bar{y}^k} < 1$$
   and the result follows.
   (b) If $p > k\left(\frac{1-q}{k-1}\right)^{\frac{k+1}{k}}$, then $\bar{y} > \sqrt[k]{\frac{1-q}{k-1}}$. Therefore, equation (2.3) has a root $\lambda < -1$ with $|\lambda| > 1$ and so $\bar{y}$ is an unstable equilibrium point.

□

Theorem 2.2. *Assume that* $\mathfrak{K} = 2r+1$ *and let* $\bar{y}$ *be the unique positive equilibrium point of equation* (1.5). *Then the following statements are true.*

1. *If* $q > 1$, *then* $\bar{y}$ *is a saddle point.*
2. *If* $q < 1$, *then we have the following:*

(a) *If* $p < k\left(\frac{1-q}{k-1}\right)^{\frac{k+1}{k}}$*, then* $\bar{y}$ *is locally asymptotically stable.*

(b) *If* $p > k\left(\frac{1-q}{k-1}\right)^{\frac{k+1}{k}}$*, then* $\bar{y}$ *is a saddle point.*

Proof. When $\mathfrak{K} = 2r+1$, the associated characteristic equation (2.2) becomes

$$\lambda^{2r+2} + \frac{k\bar{y}^k}{1+\bar{y}^k}\lambda^{2r+1-2l} - \frac{q}{1+\bar{y}^k} = 0. \tag{2.4}$$

1. If $q > 1$, then $\bar{y} > \sqrt[k]{q-1}$. It is sufficient to see that the characteristic equation (2.4) has a root in the interval $(0,1)$ and a other root in the interval $(-\infty, -1)$.

2. Assume that $q < 1$ and let $\bar{y}$ be the positive equilibrium point of equation (1.5). Then we have the following:

   (a) If $p < k\left(\frac{1-q}{k-1}\right)^{\frac{k+1}{k}}$, then $\bar{y} < \sqrt[k]{\frac{1-q}{k-1}}$. By Theorem (1.4), we have

   $$\frac{q}{1+\bar{y}^k} + \frac{k\bar{y}^k}{1+\bar{y}^k} = \frac{q+k\bar{y}^k}{1+\bar{y}^k} < 1$$

   and the result follows.

   (b) If $p > k\left(\frac{1-q}{k-1}\right)^{\frac{k+1}{k}}$, then $\bar{y} > \sqrt[k]{\frac{1-q}{k-1}}$. Therefore, the characteristic equation (2.4) has a root in the interval $(0,1)$ and a other root in the interval $(-\infty,-1)$.

□

3 Global Behavior of (1.5)

In this section, we show that under a certain conditions, a solution of equation (1.5) oscillates with semicycles of length one. We study the global stability of the positive equilibrium point $\bar{y}$. Also we show the existence of unbounded solutions.

Theorem 3.1. *Assume that* $\bar{y}$ *denote the unique positive equilibrium of equation* (1.5) *and let* $\{y_n\}_{n=-\mathfrak{K}}^{\infty}$ *be a nontrivial solution of equation* (1.5)*. If either one of the conditions*

$(C_1) y_{-2t-1}, y_{-2t+1}, \ldots, y_{-1} < \bar{y} \le y_{-2s}, y_{-2s+2}, \ldots, y_0$

or

$(C_2)\ y_{-2s}, y_{-2s+2}, \ldots, y_0 < \bar{y} \le y_{-2t-1}, y_{-2t+1}, \ldots, y_{-1}$

is satisfied, then $\{y_n\}_{n=-\mathfrak{K}}^{\infty}$ *oscillates about* $\bar{y}$ *with semicycles of length one.*

Theorem 3.2. *Assume that* $q < 1$ *and* $p < k\left(\frac{1-q}{k-1}\right)^{\frac{k+1}{k}}$. *Then the positive equilibrium point* $0 < \bar{y} < \sqrt[k]{\frac{1-q}{k-1}}$ *is globally asymptotically stable.*

Proof. Let $\{y_n\}_{n=-\mathfrak{K}}^{\infty}$ be a solution of equation (1.5). Then

$$y_{n+1} = \frac{p + qy_{n-2r-1}}{1 + y_{n-2l}^k} < p + qy_{n-2r-1}, \quad n = 0, 1, \ldots.$$

Then there exists a real number $\beta > 0$ such that $y_n < \beta$, $n = 1, 2, \ldots$. This implies that

$$y_{n+1} = \frac{p + qy_{n-1}}{1 + y_{n-2l}^k} > \frac{p}{1 + \beta^k}.$$

Let $\lambda = \liminf y_n$ and $\Lambda = \limsup y_n$. Hence we have

$$\frac{p + q\lambda}{1 + \Lambda^k} \leq \lambda \leq \Lambda \leq \frac{p + q\Lambda}{1 + \lambda^k}.$$

This implies that

$$p + q\lambda \leq \lambda + \lambda\Lambda^k$$

and

$$\Lambda + \Lambda\lambda^k \leq p + q\Lambda.$$

Then

$$p\lambda^{k-1} + q\lambda^k \leq \lambda^k + \lambda^k\Lambda^k$$

and

$$\Lambda^k + \Lambda^k\lambda^k \leq p\Lambda^{k-1} + q\Lambda^k.$$

Then we get that

$$p\lambda^{k-1} + \lambda^k(q-1) \leq p\Lambda^{k-1} + \Lambda^k(q-1).$$

That is

$$(1-q)\lambda^k - p\lambda^{k-1} \geq (1-q)\Lambda^k - p\Lambda^{k-1}. \tag{3.1}$$

Consider the function $h(x) = (1-q)x^k - px^{k-1}$. We claim that

$$\frac{p(k-1)}{k(1-q)} < \bar{y}.$$

Proof of the claim: We have that, the equilibrium point $\bar{y}$ is the positive zero of the function $f(x) = x^{k+1} + (1-q)x - p$. Now,

$$\begin{aligned} f(\frac{p(k-1)}{k(1-q)}) &= (\frac{p(k-1)}{k(1-q)})^{k+1} + (1-q)\frac{p(k-1)}{k(1-q)} - p \\ &= (\frac{p(k-1)}{k(1-q)})^{k+1} + \frac{p(k-1)}{k} - p \\ &= (\frac{p(k-1)}{k(1-q)})^{k+1} - \frac{p}{k} \\ &= \frac{p}{k}((\frac{p}{k})^k(\frac{k-1}{1-q})^{k+1} - 1) \\ &< \frac{p}{k}((\frac{1-q}{k-1})^{k+1}(\frac{k-1}{1-q})^{k+1} - 1) = 0, \qquad (p < k\left(\frac{1-q}{k-1}\right)^{\frac{k+1}{k}}). \end{aligned}$$

As the function $f(x)$ is increasing everywhere when $q < 1$, we get

$$\frac{p(k-1)}{k(1-q)} < \bar{y}.$$

The claim is proved.

Now, we have $\frac{p(k-1)}{k(1-q)} < \bar{y} < \sqrt[k]{\frac{1-q}{k-1}}$ and $h(x)$ is increasing on $(\frac{p(k-1)}{k(1-q)}, \infty)$. In view of equation (3.1), we have a contradiction. Therefore $\lambda = \Lambda = \bar{y}$ and $\bar{y}$ is a global attractor. In view of Theorem (2.1) and Theorem (2.2), $\bar{y}$ is globally asymptotically stable. This completes the proof. □

Lemma 3.3. *Assume that $q > 2$ and $p > 0$. Then the following statements are true:*

1. *If $x > \sqrt[k]{q-1} + \frac{p}{\sqrt[k]{q-1}}$, then $\sqrt[k]{q-1} > \frac{p}{x^k - q + 1}$.*

2. *If $x > \sqrt[k]{q-1}$ and $y > \frac{p}{x^k - q + 1}$, then $y > \frac{p + qy}{1 + x^k}$.*

Theorem 3.4. *Assume that $q > 2$. Then equation* (1.5) *has solutions which are neither bounded nor persist.*

Proof. Let $\{y_n\}_{n=-\mathfrak{K}}^{\infty}$ be a solution of equation (1.5) with initial conditions

$$\frac{p}{y_{-2t-1}^k - q + 1} < y_{-2s} < y_{-2s+2} < \ldots < y_0 < \sqrt[k]{q-1}$$

and

$$\sqrt[k]{q-1} + \frac{p}{\sqrt[k]{q-1}} < y_{-2t-1} < y_{-2t+1} < \ldots < y_{-1}.$$

Then

$$y_1 = \frac{p + qy_{-2r-1}}{1 + y_{-2l}^k} > \frac{p + qy_{-2r-1}}{q} = \frac{p}{q} + y_{-2r-1},$$

where $y_{-2l} < \sqrt[k]{q-1}$,

$$y_2 = \frac{p + qy_{-2r}}{1 + y_{-2l+1}^k} < y_{-2r}$$

where $y_{-2l+1} > \sqrt[k]{q-1}$ and $y_{-2r} > \dfrac{p}{y_{-2t-1}^k - q + 1} > \dfrac{p}{y_{-2l+1}^k - q + 1}$.

Now consider the subsequences $\{y_{(2r+2)n-2r+2j-1}\}_{n=0}^{\infty}$ and $\{y_{(2r+2)n-2r+2j}\}_{n=0}^{\infty}$, $0 \leq j \leq r$.

We claim that for each integer j such that $0 \leq j \leq r$, $\{y_{(2r+2)n-2r+2j-1}\}_{n=0}^{\infty}$ is monotonically increasing subsequence of $\{y_n\}_{n=-\mathfrak{K}}^{\infty}$ and $\{y_{(2r+2)n-2r+2j}\}_{n=0}^{\infty}$ is monotonically decreasing subsequence of $\{y_n\}_{n=-\mathfrak{K}}^{\infty}$.

Proof of the claim:

Consider the case when $0 \leq r < l$. For $n = 1$, we have for $0 \leq j \leq r$

$$y_{2j+1} = \frac{p + qy_{2j-2r-1}}{1 + y_{2j-2l}^k} > \frac{p + qy_{2j-2r-1}}{q} = \frac{p}{q} + y_{2j-2r-1},$$

where $y_{2j-2l} < \sqrt[k]{q-1}$,

$$y_{2j+2} = \frac{p + qy_{2j-2r}}{1 + y_{2j-2l+1}^k} < y_{2j-2r},$$

where $y_{2j-2l+1} > \sqrt[k]{q-1}$, and $y_{2j-2r} > \dfrac{p}{y_{-2t-1}^k - q + 1} > \dfrac{p}{y_{2j-2l+1}^k - q + 1}$.

Also, we have the following:

$$y_{2l+1} = \frac{p + qy_{2l-2r-1}}{1 + y_0^k} > \frac{p + qy_{2l-2r-1}}{q} = \frac{p}{q} + y_{2l-2r-1},$$

where $y_0 < \sqrt[k]{q-1}$,

$$y_{2l+2} = \frac{p + qy_{2l-2r}}{1 + y_1^k} < \frac{p + qy_{(2l-2r)-2r-2}}{1 + y_{-2r-1}^k} = y_{2l-2r},$$

where $y_1 > y_{-2r-1}$ and $y_{2l-2r} < y_{(2l-2r)-2r-2}$.

Now suppose that for a certain n we have

$$\sqrt[k]{q-1} + \frac{p}{\sqrt[k]{q-1}} < y_{(2r+2)(n-1)-2r+2j-1} < y_{(2r+2)n-2r+2j-1}$$

and

$$y_{(2r+2)n-2r+2j} < y_{(2r+2)(n-1)-2r+2j} < \sqrt[k]{q-1}.$$

Then

$$\begin{aligned} y_{(2r+2)(n+1)-2r+2j-1} =& \frac{p+qy_{(2r+2)n-2r+2j-1}}{1+y_{(2r+2)n+2j-2l}} \\ &> \frac{p+qy_{(2r+2)(n-1)-2r+2j-1}}{1+y_{(2r+2)(n-1)+2j-2l}} \\ &= y_{(2r+2)n-2r+2j-1}, \end{aligned}$$

and

$$\begin{aligned} y_{(2r+2)(n+1)-2r+2j} =& \frac{p+qy_{(2r+2)n-2r+2j}}{1+y_{(2r+2)n+2j-2l+1}} \\ &< \frac{p+qy_{(2r+2)(n-1)-2r+2j}}{1+y_{(2r+2)(n-1)+2j-2l+1}} \\ &= y_{(2r+2)n-2r+2j}. \end{aligned}$$

In the case when $0 \leq l < r$, there exist integers i and m with $m \geq 1$ and $0 \leq i < l$ such that $r = ml + i$. The proof of the claim in this case is similar and will be omitted. This completes the proof of the claim.

It follows by induction from the claim that for each nonnegative integer j with $0 \leq j \leq r$

$$y_{(2r+2)n-2r+2j-1} > \frac{p}{q} + y_{(2r+2)(n-1)-2r+2j-1} > \cdots > n\frac{p}{q} + y_{-2r+2j-1}.$$

This implies that

$$\lim_{n\to\infty} y_{(2r+2)n-2r+2j-1} = \infty, \qquad 0 \leq j \leq r,$$

and so

$$\lim_{n\to\infty} y_{(2r+2)n-2r+2j} = 0, \qquad 0 \leq j \leq r.$$

Therefore

$$\lim_{n\to\infty} y_{2n-1} = \infty \quad \text{and} \quad \lim_{n\to\infty} y_{2n} = 0.$$

This completes the proof. □

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References

[1] R. Abo-Zeid, *Global asymptotic stability of a higher order difference equation*, Bull. Allahabad Math. Soc. 25 (2) (2010), 341 − 351.

[2] R. Abo-Zeid, *Global asymptotic stability of a second order rational difference equation*, J. Appl. Math. & Inform. 28 (3) (2010), 797 − 804.

[3] R.P. Agarwal, *Dfference equations and inequalities, First edition*, Marcel Dekker, 1992.

[4] M.A. Al-Shabi, R. Abo-Zeid, *Global asymptotic stability of a higher order difference equation*, Appl. Math. Sci., 4 (17) (2010), 839 − 847.

[5] E, Camouzis and G. Ladas, *Dynamics of third-order rational Difference Equations; With Open Problems and Conjectures*, Chapman and Hall/HRC Boca Raton, 2008.

[6] S. N. Elaydi, *An Introduction to Difference Equations, Undergraduate Texts in Mathematics*, Springer, New York, NY, USA, 1996.

[7] E.A. Grove and G. Ladas, *Periodicities in Nonlinear Difference Equations*, Chapman and Hall/CRC, 2005.

[8] E.A. Grove, G. Ladas, M. Predescu, and M. Radin, *On the global character of the difference equation* $x_{n+1} = \dfrac{\alpha + \gamma x_{n-2k+1} + \delta x_{n-2l}}{A + x_{n-2l}}$ J. Diff. Eq. App., 9 (2) (2003), 171 − 199.

[9] V. L. Kocic, G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, Kluwer Academic, Dordrecht, 1993.

[10] M.R.S. Kulenović and G. Ladas, *Dynamics of second order rational Difference Equations; With Open Problems and Conjectures*, Chapman and Hall/HRC Boca Raton, 2002.

[11] H. Sedaghat, *Nonlinear Difference Equations, Theory and Applications to Social Science Models*, Kluwer Academic Publishers, Dordrecht, 2003.