Advances in Dynamical Systems and Applications
ISSN 0973-5321, Volume 8, Number 2, pp. 401–412 (2013)
http://campus.mst.edu/adsa

Weighted Sums of Orthogonal Polynomials Related to Birth-Death Processes with Killing

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Abstract

We consider sequences of orthogonal polynomials arising in the analysis of birth-death processes with killing. Motivated by problems in this stochastic setting we discuss criteria for convergence of certain weighted sums of the polynomials.

AMS Subject Classifications: 42C05, 60J80.
Keywords: Birth-death process with killing, orthogonal polynomials, quasi-stationary distribution.

1 Introduction

A *birth-death process with killing* is a continuous-time Markov chain $\mathcal{X} := \{X(t),\ t \geq 0\}$ taking values in $\{0, 1, 2, \ldots\}$, where 0 is an absorbing state, and with transition rates $q_{ij},\ j \neq i$, satisfying

$$\begin{aligned} &q_{i,i+1} = \lambda_i,\ \ q_{i+1,i} = \mu_{i+1},\ \ q_{i0} = \nu_i, \quad i \geq 1, \\ &q_{ij} = 0, \quad i = 0 \ \text{ or } \ |i-j| > 1, \end{aligned} \tag{1.1}$$

where $\lambda_i > 0,\ \mu_{i+1} > 0$ and $\nu_i \geq 0$ for $i \geq 1$. It will be convenient to let $\lambda_0 = \mu_1 = 0$. The parameters λ_i and μ_i are the *birth* rate and *death* rate, respectively, while ν_i is the rate of absorption (or *killing* rate) in state i. We will assume throughout that $\nu_i > 0$ for at least one state $i \geq 1$. When $\nu_1 > 0$ but $\nu_i = 0$ for all $i > 1$, $\mathcal{X}$ is usually referred to as a (pure) *birth-death process*, ν_1 then being interpreted as the death rate in state 1.

Received October 30, 2012; Accepted January 25, 2013
Communicated by Andrea Laforgia

The transition rates of the process $\mathcal{X}$ determine a sequence of polynomials $\{Q_n\}$ through the recurrence relation

$$\begin{aligned}&\lambda_n Q_n(x) = (\lambda_n + \mu_n + \nu_n - x)Q_{n-1}(x) - \mu_n Q_{n-2}(x), \quad n > 1,\\ &\lambda_1 Q_1(x) = \lambda_1 + \nu_1 - x, \quad Q_0(x) = 1.\end{aligned} \tag{1.2}$$

The sequence $\{Q_n\}$ plays an important role in the analysis of the process $\mathcal{X}$ and will be the main object of study in this paper. We will focus in particular on weighted sums

$$\sum_{n=0}^{\infty} w_n Q_n(x) \tag{1.3}$$

for certain nonnegative weights w_n depending on the transition rates of the process and certain values of x, since the existence of *quasi-stationary distributions* (see Section 4) for the corresponding birth-death process with killing requires the convergence of such series. Our aim is to collect and supplement a number of results that have appeared in the stochastic literature, and present them from an orthogonal-polynomial perspective. This will also give us the opportunity to rectify some statements in [11] and to supply some new proofs.

In the special case of a (pure) birth-death process relevant weighted sums of the type (1.3) have been studied in [8] (where the polynomials R_n^* have the role of our Q_n). However, the technique employed there (involving *kernel polynomials*) does not seem to be applicable in the more general setting at hand.

The remainder of this paper is organized as follows. In Section 2 we collect a number of basic properties of the polynomial sequence $\{Q_n\}$. These will enable us to derive in Section 3 some further properties of the polynomials Q_n and, subsequently, to establish criteria for convergence of the series (1.3) for certain values of w_n and x. In Section 4 we will briefly discuss the relevance of our findings for the analysis of birth-death processes with killing, in particular with regard to the existence of quasi-stationary distributions.

2 Preliminaries

By letting

$$P_0(x) := 1 \text{ and } P_n(x) := (-1)^n \lambda_1 \lambda_2 \dots \lambda_n Q_n(x), \quad n \geq 1,$$

we obtain the *monic* polynomials corresponding to $\{Q_n\}$ of (1.2), which satisfy the three-terms recurrence relation

$$\begin{aligned}&P_n(x) = (x - \lambda_n - \mu_n - \nu_n)P_{n-1}(x) - \lambda_{n-1}\mu_n P_{n-2}(x), \quad n > 1,\\ &P_1(x) = x - \lambda_1 - \nu_1, \quad P_0(x) = 1.\end{aligned} \tag{2.1}$$

As a consequence (see, for example, Chihara [3, Theorems I.4.4 and II.3.1]) $\{P_n\}$, and hence $\{Q_n\}$, constitutes a sequence of orthogonal polynomials with respect to a probability measure (a positive Borel measure of total mass 1) on $\mathbb{R}$. That is, there exist a (not necessarily unique) probability measure ψ on $\mathbb{R}$ and constants $\rho_j > 0$ such that

$$\rho_j \int_{-\infty}^{\infty} Q_i(x)Q_j(x)\psi(dx) = \delta_{ij}, \quad i, j \geq 0, \tag{2.2}$$

where δ_{ij} is Kronecker's delta. It can readily be seen that, actually,

$$\rho_0 = 1 \text{ and } \rho_n = \frac{\lambda_1\lambda_2 \dots \lambda_n}{\mu_2\mu_3 \dots \mu_{n+1}}, \quad n > 0. \tag{2.3}$$

The particular form of the parameters in the recurrence relation (2.1) and our assumption $\nu_i > 0$ for at least one state i allow us to draw more specific conclusions on ψ. Namely, by [6, Theorem 1.3] there exists a probability measure ψ on the *open* interval $(0, \infty)$ with finite moment of order -1, that is,

$$\int_{(0,\infty)} \frac{\psi(dx)}{x} < \infty, \tag{2.4}$$

satisfying (2.2). Moreover, by [6, Theorem 4.1] this measure is the unique probability measure ψ satisfying (2.2) if and only if

$$\sum_{n=0}^{\infty} \rho_n Q_n^2(0) = \infty. \tag{2.5}$$

In the terminology of the theory of the moment problem (2.5) is necessary and sufficient for the *Hamburger moment problem* associated with the polynomials $\{Q_n\}$ to be *determined.* By [6, Theorem 4.1] again, (2.5) is also necessary and sufficient for (2.2) to have a unique solution ψ with all its support on the nonnegative real axis, that is, for the *Stieltjes moment problem* associated with $\{Q_n\}$ to be *determined.* We note that these results generalize Karlin and McGregor [7, Theorem 14 and Corollary] (see also Chihara [4, Theorems 2 and 3]), which refer to the pure birth-death case $\nu_1 > 0$ and $\nu_i = 0$ for $i > 1$.

The orthogonality relation (2.2) implies that the *orthonormal* polynomials $\{p_n\}$ corresponding to $\{Q_n\}$ satisfy $p_n(x) = \sqrt{\rho_n}Q_n(x)$ so, by a renowned result from the theory of moments (Shohat and Tamarkin [9, Corollary 2.7]), we actually have

$$\sum_{n=0}^{\infty} \rho_n Q_n^2(x) < \infty \quad \text{for all } x \in \mathbb{R} \tag{2.6}$$

if the Hamburger moment problem associated with $\{Q_n\}$ is *indeterminate*. For later use we recall another famous result from the theory of moments (see [9, Corollary 2.6]), stating that if the Hamburger moment problem is determined, then

$$\psi(\{x\}) = \left(\sum_{n=0}^{\infty} \rho_n Q_n^2(x)\right)^{-1}, \quad x \in \mathbb{R}, \tag{2.7}$$

which is to be interpreted as zero if the sum diverges. Hence, if the Hamburger moment problem is determined we have

$$\sum_{n=0}^{\infty} \rho_n Q_n^2(x) < \infty \iff \psi(\{x\}) > 0, \quad x \in \mathbb{R}. \tag{2.8}$$

It follows that $\psi(\{0\}) = 0$ since determinacy of the Hamburger moment problem is equivalent to (2.5). Evidently, this is consistent with the fact that there must be an orthogonalizing measure on the *open* interval $(0, \infty)$.

If the Hamburger moment problem associated with $\{Q_n\}$ is indeterminate, then, by Chihara [2, Theorem 5], there is a unique orthogonalizing probability measure for which the infimum of its support is maximal. We will refer to this measure (which happens to be discrete) as the *natural* measure. Evidently, the natural measure has all its mass on the positive real axis.

It is well known (see, for example, [3, Section II.4]) that the polynomials Q_n have real zeros $x_{n1} < x_{n2} < \ldots < x_{nn}$, $n \geq 1$, which are closely related to supp(ψ), the support of the orthogonalizing probability measure ψ, where ψ, if not uniquely determined by (2.2), should be interpreted as the natural measure. In particular we have

$$\xi := \lim_{n\to\infty} x_{n1} = \inf \mathrm{supp}(\psi) \geq 0, \tag{2.9}$$

where the limit exists since the sequence $\{x_{n1}\}$ is (strictly) decreasing (see, for example, [3, Theorem I.5.3]). Considering that

$$(-1)^n P_n(x) = \lambda_1\lambda_2 \ldots \lambda_n Q_n(x) = (x_{n1} - x)(x_{n2} - x) \ldots (x_{nn} - x),$$

it now follows that

$$y < x \leq \xi \iff Q_n(y) > Q_n(x) > 0 \text{ for all } n > 0, \tag{2.10}$$

a result that will be used later on.

The quantity ξ (which happens to be the *decay parameter* of the associated birth-death process with killing $\mathcal{X}$) plays an important part in what follows, and it will be useful to relate ξ to the parameters in the recurrence relation (1.2). From [10, Theorem 7] we obtain the bound

$$\xi \geq \inf_{i\geq 1} \left\{ \lambda_i + \mu_i + \nu_i - a_{i+1} - \frac{\lambda_{i-1}\mu_i}{a_i} \right\} \tag{2.11}$$

for any sequence $(a_1, a_2, \ldots)$ of positive numbers. Choosing $a_i = \lambda_{i-1}$ for $i > 1$ it follows in particular that

$$\xi \geq \inf_{i\geq 1} \nu_i. \tag{2.12}$$

In [3, Corollary to Theorem IV.2.1] one finds the simple upper bound

$$\xi \leq \inf_{i\geq 1} \{\lambda_i + \mu_i + \nu_i\}, \tag{2.13}$$

while more refined upper bounds are given in [10]. Similar inequalities hold true for $\sigma := \inf \text{supp}(\psi)'$, the infimum of the *derived* set of the support of the (natural) orthogonalizing measure. (See [3, Section II.4] for the relation between σ and the zeros of the polynomials $\{Q_n\}$.) In particular, by [10, Theorem 9] we have

$$\sigma \geq \liminf_{i\to\infty} \left\{ \lambda_i + \mu_i + \nu_i - a_{i+1} - \frac{\lambda_{i-1}\mu_i}{a_i} \right\} \tag{2.14}$$

for any sequence $(a_1, a_2, \ldots)$ of positive numbers. Again choosing $a_i = \lambda_{i-1}$ for $i > 1$ it follows that

$$\sigma \geq \liminf_{i\to\infty} \nu_i. \tag{2.15}$$

Since ξ must be an isolated point in supp(ψ) if $\xi < \sigma$, we can now conclude the following.

Lemma 2.1. If $\xi < \liminf\limits_{i\to\infty} \nu_i$, then ξ is an isolated point in the support of the (natural) orthogonalizing measure.

We note that as a consequence of this lemma

$$\xi < \liminf_{i\to\infty} \nu_i \implies \xi > 0, \tag{2.16}$$

since ψ is a measure on the positive real axis.

Drawing near the end of our preliminaries we note the useful relation

$$\lambda_n \rho_{n-1}(Q_n(x) - Q_{n-1}(x)) = \sum_{j=0}^{n-1} (\nu_{j+1} - x)\rho_j Q_j(x), \quad n > 0, \tag{2.17}$$

which follows easily by induction from (1.2). Hence we can write, for all $x \in \mathbb{R}$,

$$Q_n(x) = 1 + \sum_{k=0}^{n-1} (\lambda_{k+1}\rho_k)^{-1} \sum_{j=0}^{k} (\nu_{j+1} - x)\rho_j Q_j(x), \quad n > 0, \tag{2.18}$$

and, in particular,

$$Q_n(0) = 1 + \sum_{k=0}^{n-1} (\lambda_{k+1}\rho_k)^{-1} \sum_{j=0}^{k} \nu_{j+1}\rho_j Q_j(0) \geq 1, \quad n > 0. \tag{2.19}$$

Evidently, $Q_n(0)$ is increasing in n. Moreover, by [12, Lemma 1], $\lim\limits_{n\to\infty} Q_n(0) = \infty$ if and only if

$$\sum_{k=0}^{\infty} (\lambda_{k+1}\rho_k)^{-1} \sum_{j=0}^{k} \nu_{j+1}\rho_j = \infty, \tag{2.20}$$

which happens to be a necessary and sufficient condition for *absorption* of the associated birth-death process with killing (see [12, Theorem 1]). Another condition on the parameters of the process that will play a role in what follows is

$$\sum_{k=0}^{\infty}(\lambda_{k+1}\rho_k)^{-1}\sum_{j=k+1}^{\infty}\rho_j=\infty. \tag{2.21}$$

This condition is equivalent to the *unkilled* process (the pure birth-death process obtained by setting all killing rates equal to zero) having a *natural* or *exit boundary* at infinity. For interpretations and more information we refer to Anderson [1, Section 8.1].

3 Results

As announced in the Introduction, we will focus in this section on criteria for convergence of the series $\sum w_n Q_n(x)$ for certain weights w_n and certain values of x. Specifically, we will focus on the weights $w_n=\rho_n$ and $w_n=\nu_{n+1}\rho_n$. As far as the argument x is concerned we are primarily interested in the case $x=\xi$, but will present our findings for $x\le\xi$ whenever possible. Concrete results will be obtained conditional on $\xi<\liminf_{i\to\infty}\nu_i$ or $\xi>\limsup_{i\to\infty}\nu_i$. We recall from (2.10) that $Q_n(x)>0$ for all n if $x\le\xi$, a result that will be used repeatedly. Note also that, by (2.10) again, convergence of $\sum\rho_nQ_n(y)$ implies convergence of $\sum\rho_nQ_n(x)$ if $y<x\le\xi$.

We start off by giving some auxiliary lemmas. The first contains a sufficient condition for monotonicity of the sequence $\{Q_n(x)\}_{n\ge N}$ for N sufficiently large, and hence for the existence of $Q_\infty(x):=\lim_{n\to\infty}Q_n(x)$.

Lemma 3.1. Let $x\le\xi$. If $x<\liminf_{i\to\infty}\nu_i$ or $x>\limsup_{i\to\infty}\nu_i$, then the (positive) sequence $\{Q_n(x)\}_{n\ge N}$ is monotone for N sufficiently large.

Proof. If $x\le\xi$ and $x<\liminf_{i\to\infty}\nu_i$ we have $(\nu_{n+1}-x)\rho_nQ_n(x)>0$ for n sufficiently large. Hence, by (2.17),

$$\lambda_{n+1}\rho_n(Q_{n+1}(x)-Q_n(x))>\lambda_n\rho_{n-1}(Q_n(x)-Q_{n-1}(x)),$$

so that

$$Q_n(x)\ge Q_{n-1}(x)\implies Q_m(x)>Q_{m-1}(x),\quad m>n,$$

for n sufficiently large, implying monotonicity of the sequence $\{Q_n(x)\}_{n\ge N}$ for N sufficiently large.

A similar proof leads to the same conclusion if $x>\limsup_{i\to\infty}\nu_i$. □

Our second auxiliary lemma concerns the polynomials

$$D_n(x):=\lambda_n\rho_{n-1}(Q_{n-1}(x)-Q_n(x)),\quad n\ge1. \tag{3.1}$$

Lemma 3.2. Let $x \leq \xi$, and $x < \liminf_{i\to\infty} \nu_i$ or $x > \limsup_{i\to\infty} \nu_i$.

(i) The limit $D_\infty(x) := \lim_{n\to\infty} D_n(x)$ exists (allowing for $\pm\infty$).

(ii) If $0 < D_\infty(x) \leq \infty$, then there exist constants $c > 0$ and $N \in \mathbb{N}$ such that

$$Q_n(x) \geq c \sum_{k=n}^{\infty} (\lambda_{k+1}\rho_k)^{-1}, \quad n \geq N, \tag{3.2}$$

and, for any nonnegative sequence $\{\tau_n\}$,

$$\sum_{n=N}^{\infty} \tau_n Q_n(x) \geq c \sum_{n=N}^{\infty} (\lambda_{n+1}\rho_n)^{-1} \sum_{k=N}^{n} \tau_k. \tag{3.3}$$

(iii) If $-\infty \leq D_\infty(x) < 0$, then there exist constants $c > 0$ and $N \in \mathbb{N}$ such that

$$Q_n(x) > c \sum_{k=N}^{n-1} (\lambda_{k+1}\rho_k)^{-1}, \quad n > N, \tag{3.4}$$

and, for any nonnegative sequence $\{\tau_n\}$,

$$\sum_{n=N}^{\infty} \tau_n Q_n(x) \geq c \sum_{n=N}^{\infty} (\lambda_{n+1}\rho_n)^{-1} \sum_{k=n+1}^{\infty} \tau_k. \tag{3.5}$$

Proof. In view of (2.17) $D_n(x)$ can be represented as

$$D_n(x) = \sum_{j=0}^{n-1} (x - \nu_{j+1})\rho_j Q_j(x). \tag{3.6}$$

So, under the conditions of the lemma, the sequence $\{D_n(x)\}_{n\geq N}$ is monotone for N sufficiently large, implying the existence of the limit.

To prove statement (ii) we note that $0 < D_\infty(x) \leq \infty$ implies the existence of constants $c > 0$ and $n \in \mathbb{N}$ such that $D_n(x) > c$ for all $n > N$. Hence

$$Q_n(x) > Q_{n+1}(x) + c(\lambda_{n+1}\rho_n)^{-1}, \quad n \geq N,$$

and (3.2) follows by induction. Multiplying both sides of (3.2) by τ_n, summing over all $n \geq N$ and interchanging summation signs on the right-hand side subsequently yields (3.3).

Statement (iii) is proven similarly. □

Our first theorem gives a sufficient condition for convergence of the series (1.3) with $w_n = \rho_n$.

Theorem 3.3. If $\xi \geq x > \limsup_{i\to\infty} \nu_i$, then

$$\sum_{n=0}^{\infty}(\lambda_{n+1}\rho_n)^{-1} = \infty \implies \sum_{n=0}^{\infty}\rho_n Q_n(x) < \infty. \tag{3.7}$$

Proof. Let $\xi \geq x > \limsup_{i\to\infty} \nu_i$ and suppose $\sum \rho_n Q_n(x) = \infty$. Then, in view of (3.6), $D_n(x) \geq 1$ for n sufficiently large. But by (2.18) and (3.6) we have

$$\sum_{n=0}^{k}(\lambda_{n+1}\rho_n)^{-1}D_{n+1}(x) = 1 - Q_{k+1}(x) < 1$$

for all k, so that $\sum(\lambda_{n+1}\rho_n)^{-1}$ must converge. □

We will see in Section 4 that convergence results for $\sum \rho_n Q_n(\xi)$ are relevant in particular when (2.20) prevails, which happens to be a condition under which we can prove a converse of Theorem 3.3, and more.

Theorem 3.4. Let (2.20) be satisfied. If $\xi \geq x > \limsup_{i\to\infty} \nu_i$, then

$$\sum_{n=0}^{\infty}(\lambda_{n+1}\rho_n)^{-1} < \infty \implies \sum_{n=0}^{\infty}\nu_{n+1}\rho_n Q_n(x) = \sum_{n=0}^{\infty}\rho_n Q_n(x) = \infty. \tag{3.8}$$

Proof. Lemma 3.1 tells us that, under the condition on x, the sequence $\{Q_n(x)\}_{n\geq N}$ is monotone for N sufficiently large, so that $Q_\infty(x)$ exists and $0 \leq Q_\infty(x) \leq \infty$. The conditions (2.20) and $\sum(\lambda_{n+1}\rho_n)^{-1} < \infty$ imply $\sum \nu_{n+1}\rho_n = \infty$. So if $0 < Q_\infty(x) \leq \infty$, then $\sum \nu_{n+1}\rho_n Q_n(x) = \infty$, whence $\sum \rho_n Q_n(x) = \infty$ and we are done. Let us therefore assume that, for n sufficiently large, $Q_n(x)$ decreases to 0 and hence $D_n(x) > 0$. Since $x > \nu_n$ for n sufficiently large, the representation (3.6) shows that $D_n(x)$ is increasing for n sufficiently large, so we must have $0 < D_\infty(x) \leq \infty$. Subsequently choosing $\tau_n = \nu_{n+1}\rho_n$ and applying Lemma 3.2 (ii), we conclude with (2.20) that

$$\sum_{n=N}^{\infty}\nu_{n+1}\rho_n Q_n(x) \geq c\sum_{k=N}^{\infty}(\lambda_{k+1}\rho_k)^{-1}\sum_{j=N}^{k}\nu_{j+1}\rho_j = \infty,$$

which establishes the theorem. □

We will see in Section 4 that the question of whether the sums $\sum \nu_{n+1}\rho_n Q_n(x)$ and $x\sum \rho_n Q_n(x)$ are equal-answered in the affirmative in the setting of the previous theorem-plays an crucial role in the application we have in mind. Under the additional condition (2.21) we can also prove equality in the setting of Theorem 3.3.

Theorem 3.5. Let (2.21) be satisfied. If $\xi \geq x > \limsup_{i\to\infty} \nu_i$, then

$$\sum_{n=0}^{\infty}(\lambda_{n+1}\rho_n)^{-1} = \infty \implies \sum_{n=0}^{\infty} \nu_{n+1}\rho_n Q_n(x) = x\sum_{n=0}^{\infty} \rho_n Q_n(x) < \infty. \tag{3.9}$$

Proof. If $\sum(\lambda_{n+1}\rho_n)^{-1} = \infty$ then (3.2) cannot prevail, so we must have $-\infty \leq D_\infty(x) \leq 0$ by Lemma 3.2. Assuming $-\infty \leq D_\infty(x) < 0$ we can choose $\tau_n = \rho_n$ and conclude from Lemma 3.2 (iii) that $\sum \rho_n Q_n(x) = \infty$, which, however, contradicts Theorem 3.3. So we must have $D_\infty(x) = 0$, which, together with (3.6) and Theorem 3.3, establishes the result. □

Now turning to the case $\xi < \liminf_{i\to\infty} \nu_i$, we first observe the following. If the Hamburger moment problem associated with $\{Q_n\}$ is determined we have, in view of (2.8) and (2.9),

$$x < \xi \implies \sum_{n=0}^{\infty} \rho_n Q_n^2(x) = \infty. \tag{3.10}$$

However, when $x = \xi$ the sum may be finite. A sufficient condition for finiteness is given in the next lemma.

Lemma 3.6. If $\xi < \liminf_{i\to\infty} \nu_i$, then $\sum_{n=0}^{\infty} \rho_n Q_n^2(\xi) < \infty$.

Proof. If the Hamburger moment problem associated with $\{Q_n\}$ is indeterminate the conclusion is always true in view of the result stated around (2.6). Otherwise, by (2.8), it suffices to show that $\psi(\{\xi\}) > 0$, but this follows from Lemma 2.1. □

Considering that $Q_n(\xi) > 0$ for all n, we can now state a sufficient condition for convergence of the series (1.3) with $w_n = \rho_n$ and $x = \xi$.

Theorem 3.7. If $\xi < \liminf_{i\to\infty} \nu_i$, then $\sum_{n=0}^{\infty} \rho_n Q_n(\xi) < \infty$.

Proof. Let $\xi < \liminf_{i\to\infty} \nu_i$ and suppose $\sum \rho_n Q_n(\xi) = \infty$. Then

$$\sum_{j=0}^{n}(\nu_{j+1} - \xi)\rho_j Q_j(\xi) \to \infty \text{ as } n \to \infty,$$

so that, by (2.17), $Q_n(\xi)$ increases in n for n sufficiently large. But then we would also have $\sum \rho_n Q_n^2(\xi) = \infty$, which is impossible in view of Lemma 3.6. So $\sum \rho_n Q_n(\xi)$ must converge. □

With a view to the application described in the next section we are, as before, interested in the question of whether $\sum \nu_{n+1}\rho_n Q_n(x)$ and $x\sum \rho_n Q_n(x)$ are equal. Our final result gives a sufficient condition.

Theorem 3.8. Let (2.20) and (2.21) be satisfied. If $\xi < \liminf_{i\to\infty} \nu_i$, then

$$\xi \sum_{n=0}^{\infty} \rho_n Q_n(\xi) = \sum_{n=0}^{\infty} \nu_{n+1}\rho_n Q_n(\xi) < \infty.$$

Proof. Theorem 3.7 tells us that $\sum \rho_n Q_n(\xi) < \infty$ under the conditions of the theorem. Assuming $0 < D_\infty(\xi) \le \infty$, we can choose $\tau_n = \nu_{n+1}\rho_n$ and conclude from Lemma 3.2 (ii) that $\sum \nu_{n+1}\rho_n Q_n(\xi) = \infty$, as a consequence of (2.20). But this is impossible, since it would imply $D_\infty(\xi) = -\infty$, in view of (3.6).

Next assuming $-\infty \le D_\infty(\xi) < 0$, we can choose $\tau_n = \rho_n$ and apply Lemma 3.2 (iii). But in view of (2.21) this would lead us to the false conclusion that $\sum \rho_n Q_n(\xi) = \infty$. So we must have $D_\infty(\xi) = 0$ and the result follows by (3.6). □

4 Application

A *quasi-stationary distribution* for the birth-death process with killing $\mathcal{X}$ of the Introduction is a proper probability distribution $m := (m_j,\ j \ge 1)$ over the nonabsorbing states such that the state probabilities at time t, conditional on the process being in one of the nonabsorbing states at time t, do not vary with t when m is chosen as initial distribution. It is known (see, e.g., [5]) that a quasi-stationary distribution can only exist when eventual absorption at state 0 is certain, that is, (2.20) is satisfied, and $\xi > 0$. Under these circumstances a necessary and sufficient condition for a probability distribution to be a quasi-stationary distribution for $\mathcal{X}$ is given in the next theorem.

Theorem 4.1 (See [5, Theorem 6.2])**.** *Let $\mathcal{X}$ be a birth-death process with killing satisfying* (2.20) *and $\xi > 0$. Then the distribution $(m_j,\ j \ge 1)$ is a quasi-stationary distribution for $\mathcal{X}$ if and only if there is a real number x, $0 < x \le \xi$, such that both*

$$m_j = \frac{\rho_{j-1}Q_{j-1}(x)}{\sum_{n=0}^{\infty} \rho_n Q_n(x)}, \quad j \ge 1, \tag{4.1}$$

and

$$x \sum_{n=0}^{\infty} \rho_n Q_n(x) = \sum_{n=0}^{\infty} \nu_{n+1}\rho_n Q_n(x) < \infty. \tag{4.2}$$

Combining this result with Theorems 3.4, 3.5, 3.7 and 3.8 of the previous section yields the following two theorems.

Theorem 4.2. Let $\mathcal{X}$ be a birth-death process with killing satisfying (2.20), (2.21) and $\xi > \limsup_{i\to\infty} \nu_i$. Then a quasi-stationary distribution for $\mathcal{X}$ exists if and only if $\sum(\lambda_{n+1}\rho_n)^{-1} = \infty$, in which case $(m_j,\ j \geq 1)$ defined by (4.1) constitutes a quasi-stationary distribution for every $x,\ 0 < x \leq \xi$.

Theorem 4.3. Let $\mathcal{X}$ be a birth-death process with killing satisfying (2.20), (2.21) and $\xi < \liminf_{i\to\infty} \nu_i$. Then $(m_j,\ j \geq 1)$ defined by (4.1) with $x = \xi$ constitutes a quasi-stationary distribution for $\mathcal{X}$.

These theorems should be compared with [11, Theorem 2 and Theorem 1]. The proofs of the latter results use the equality

$$\xi \sum_{n=0}^{\infty} \rho_n Q_n(\xi) = \sum_{n=0}^{\infty} \nu_{n+1}\rho_n Q_n(\xi), \tag{4.3}$$

which is claimed in [12, Theorem 2] to be true under all circumstances (allowing for the value ∞). Unfortunately, there is a gap in the proof of [12, Theorem 2], which raises doubts on the unconditional validity of (4.3), and therefore on the conclusions that have been drawn in [11, Theorem 1 and Theorem 2] on the basis of (4.3). Theorems 4.2 and 4.3 show, however, that adding the (mild) condition (2.21) is sufficient for these conclusions to remain valid. Moreover, while [11, Theorem 2] states only the existence of a quasi-stationary distribution under the conditions of Theorems 4.2, the latter theorem actually establishes the existence of an infinite family of quasi-stationary distributions.

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