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Existence of Minimizers for Fractional Variational Problems Containing Caputo Derivatives

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Abstract

We study dynamic minimization problems of the calculus of variations with Lagrangian functionals containing Riemann–Liouville fractional integrals, classical and Caputo fractional derivatives. Under assumptions of regularity, coercivity and convexity, we prove existence of solutions.

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1 Introduction

For the origin of the calculus of variations with fractional operators we should look back to 1996–1997, when Riewe used noninteger order derivatives to better describe nonconservative systems in mechanics [22, 23]. Since then, numerous works on the fractional variational calculus have been written. In particular, we can find a comprehensive literature regarding necessary optimality conditions and Noether's theorem (see, e.g., [1, 3, 5, 9, 11, 12, 14]). For the state of the art on the fractional calculus of variations and respective fractional Euler–Lagrange equations, we refer the reader to the recent book [16]. Here, we remark that results addressed to the existence of solutions for problems of the fractional calculus of variations are rare, being, to the best of our knowledge,

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discussed only in [4, 15]. However, existence theorems are essential ingredients of the deductive method for solving variational problems, which starts with the proof of existence, proceeds with application of optimality conditions, and finishes examining the candidates to arrive to a solution. These arguments make the question of existence an emergent topic, which requires serious attention and more interest [18].

In this note we discuss the problem of existence of solutions for fractional variational problems. We consider functionals with Lagrangians depending on the Riemann–Liouville fractional integral and classical and Caputo fractional derivatives. Necessary optimality conditions for such problems were recently obtained in [17]. Here, inspired by the results given in [4], we prove existence of solutions in an appropriate space of functions and under suitable assumptions of regularity, coercivity and convexity. For the classical methods of existence of minimizers for variational functionals we refer the reader to [7, 8, 10].

The article is organized as follows. In Section 2 we provide the basic definitions and properties for the fractional operators used throughout the text. Main notations are fixed. Our results are then formulated and proved in Section 3: in Section 3.1 we prove existence of minimizers for fractional problems of the calculus of variations with a Lagrangian containing Caputo derivatives; Sections 3.2 and 3.3 are devoted to sufficient conditions implying regularity and coercivity, respectively. Finally, an example is given in Section 4.

2 Preliminaries

We recall here the necessary definitions and present some properties of the fractional operators under consideration. Moreover, we fix our notations for later discussions. The reader interested on fractional analysis is refereed to the books [13, 19, 24].

Let a, b be two real numbers such that $a < b$, let $d \in \mathbb{N}^*$ be the dimension, where $\mathbb{N}^*$ denotes the set of positive integers, and let $\|\cdot\|$ denote the standard Euclidean norm of $\mathbb{R}^d$. For any $1 \leq r \leq \infty$, we denote

- by $\mathrm{L}^r := \mathrm{L}^r(a, b; \mathbb{R}^d)$ the usual space of r-Lebesgue integrable functions endowed with its usual norm $\|\cdot\|_{\mathrm{L}^r}$;
- by $\mathrm{W}^{1,r} := \mathrm{W}^{1,r}(a, b; \mathbb{R}^d)$ the usual r-Sobolev space endowed with its usual norm $\|\cdot\|_{\mathrm{W}^{1,r}}$.

Furthermore, $\mathscr{C} := \mathscr{C}([a, b]; \mathbb{R}^d)$ will be understood as the standard space of continuous functions and $\mathscr{C}_c^\infty := \mathscr{C}_c^\infty([a, b]; \mathbb{R}^d)$ as the standard space of infinitely differentiable functions compactly supported in (a, b). Finally, let us remind that the compact embedding $\mathrm{W}^{1,r} \hookrightarrow\!\!\!\!\rightarrow \mathscr{C}$ holds for $1 < r \leq +\infty$ (see [6] for a detailed proof).

We define the left and the right Riemann–Liouville fractional integrals I_-^α and I_+^α of order $\alpha \in \mathbb{R}$, $\alpha > 0$, by

$$I_-^\alpha[f](t) := \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(y)}{(t-y)^{1-\alpha}} dy, \quad t > a$$

and

$$I_+^\alpha[f](t) := \frac{1}{\Gamma(\alpha)} \int_t^b \frac{f(y)}{(y-t)^{1-\alpha}} dy, \quad t < b,$$

respectively. Here Γ denotes the Euler Gamma function. Note that operators I_-^α and I_+^α are well defined a.e. on (a,b) for any $f \in \mathrm{L}^1$.

Let $0 < \alpha < 1$ and $\dot{f}$ denote the usual derivative of f. Then the left and the right Caputo fractional derivatives ${}_cD_-^\alpha$ and ${}_cD_+^\alpha$ of order α are given by

$${}_cD_-^\alpha[f](t) := I_-^{1-\alpha}[\dot{f}](t) \quad \text{and} \quad {}_cD_+^\alpha[f](t) := -I_+^{1-\alpha}[\dot{f}](t)$$

for all $t \in (a,b]$ and $t \in [a,b)$, respectively. Note that the Caputo derivatives of a function $f \in W^{1,1}$ are well defined almost everywhere on (a,b).

We make use of the following property yielding boundedness of Riemann–Liouville fractional integrals in the space L^r.

Proposition 2.1 (See, e.g., [13, 24])**.** *The left Riemann–Liouville fractional integral I_-^α with $\alpha > 0$ is a linear and bounded operator in L^r:*

$$\left\| I_-^\alpha[f] \right\|_{L^r} \leq \frac{(b-a)^\alpha}{\Gamma(1+\alpha)} \left\| f \right\|_{L^r}$$

for all $f \in L^r$, $1 \leq r \leq +\infty$.

3 Main Results

Along the work $1 < p < \infty$. Let p' denote the adjoint of p and let $\alpha \in \mathbb{R}$, $0 < \alpha < 1$. We consider the variational functional

$$\begin{array}{rccl} \mathcal{L}: & \mathrm{E} & \longrightarrow & \mathbb{R} \\ & u & \longmapsto & \displaystyle\int_a^b L(u, I_-^\alpha[u], \dot{u}, {}_cD_-^\alpha[u], t)\, dt \end{array}$$

and our main goal is to prove existence of minimizers for $\mathcal{L}$. We assume that E is a weakly closed subset of $\mathrm{W}^{1,p}$, $\dot{u}$ is the derivative of u and L is a Lagrangian of class $\mathscr{C}^1$:

$$\begin{array}{rccl} L: & (\mathbb{R}^d)^4 \times [a,b] & \longrightarrow & \mathbb{R} \\ & (x_1, x_2, x_3, x_4, t) & \longmapsto & L(x_1, x_2, x_3, x_4, t). \end{array}$$

By $\partial_i L$ we denote the partial derivatives of L with respect to its ith argument.

3.1 A Tonelli-type Theorem

Using general assumptions of regularity, coercivity and convexity, we prove a fractional analog of the classical Tonelli theorem, ensuring the existence of a minimizer for $\mathcal{L}$.

Definition 3.1. We say that L is *regular* if

- $L(u, I^\alpha_-[u], \dot{u}, {}_cD^\alpha_-[u], t) \in \mathrm{L}^1$;
- $\partial_1 L(u, I^\alpha_-[u], \dot{u}, {}_cD^\alpha_-[u], t) \in \mathrm{L}^1$;
- $\partial_2 L(u, I^\alpha_-[u], \dot{u}, {}_cD^\alpha_-[u], t) \in \mathrm{L}^{p'}$;
- $\partial_3 L(u, I^\alpha_-[u], \dot{u}, {}_cD^\alpha_-[u], t) \in \mathrm{L}^{p'}$;
- $\partial_4 L(u, I^\alpha_-[u], \dot{u}, {}_cD^\alpha_-[u], t) \in \mathrm{L}^{p'}$;

for any $u \in \mathrm{W}^{1,p}$.

Definition 3.2. We say that $\mathcal{L}$ is *coercive* on E if

$$\lim_{\substack{\|u\|_{\mathrm{W}^{1,p}} \to \infty \\ u \in \mathrm{E}}} \mathcal{L}(u) = +\infty.$$

Next result gives a Tonelli-type theorem for Lagrangian functionals containing fractional derivatives in the sense of Caputo.

Theorem 3.3 (Tonelli's existence theorem for fractional variational problems)**.** *If*

- *L is regular;*
- *$\mathcal{L}$ is coercive on E;*
- *$L(\cdot, t)$ is convex on $(\mathbb{R}^d)^4$ for any $t \in [a, b]$;*

then there exists a minimizer for $\mathcal{L}$.

Proof. Because the Lagrangian L is regular, $L(u, I^\alpha_-[u], \dot{u}, {}_cD^\alpha_-[u], t) \in \mathrm{L}^1$ and $\mathcal{L}(u)$ exists in $\mathbb{R}$. Let $(u_n)_{n\in\mathbb{N}} \subset \mathrm{E}$ be a minimizing sequence satisfying

$$\mathcal{L}(u_n) \longrightarrow \inf_{u \in \mathrm{E}} \mathcal{L}(u) < +\infty. \tag{3.1}$$

Coercivity of $\mathcal{L}$ implies boundedness of $(u_n)_{n\in\mathbb{N}}$ in $\mathrm{W}^{1,p}$. Moreover, since $\mathrm{W}^{1,p}$ is a reflexive Banach space, there exists $\bar{u}$ and a subsequence of $(u_n)_{n\in\mathbb{N}}$, that we still denote as $(u_n)_{n\in\mathbb{N}}$, such that $u_n \xrightarrow{\mathrm{W}^{1,p}} \bar{u}$. Furthermore, since E is a weakly closed subset of $\mathrm{W}^{1,p}$, $\bar{u} \in$ E. On the other hand, from the convexity of L, we have

$$\begin{aligned}\mathcal{L}(u_n) \geq \mathcal{L}(\bar{u}) + \int_a^b \partial_1 L \cdot (u_n - \bar{u}) + \partial_2 L \cdot (I_-^\alpha[u_n] - I_-^\alpha[\bar{u}]) \\ + \partial_3 L \cdot (\dot{u}_n - \dot{\bar{u}}) + \partial_4 L \cdot ({}_cD_-^\alpha[u_n] - {}_cD_-^\alpha[\bar{u}])\, dt \quad (3.2)\end{aligned}$$

for any $n \in \mathbb{N}$, where $\partial_i L$ is taken in $(\bar{u}, I_-^\alpha[\bar{u}], \dot{\bar{u}}, {}_cD_-^\alpha[\bar{u}], t)$, $i = 1, 2, 3, 4$. Now, because L is regular, $(u_n)_{n\in\mathbb{N}}$ is weakly convergent to $\bar{u}$ in $\mathrm{W}^{1,p}$, I_-^α is linear bounded from L^p to L^p and, since the compact embedding $\mathrm{W}^{1,p} \hookrightarrow\!\!\!\!\rightarrow \mathscr{C}$ holds, one concludes that

- $\partial_3 L(\bar{u}, I_-^\alpha[\bar{u}], \dot{\bar{u}}, {}_cD_-^\alpha[\bar{u}], t) \in \mathrm{L}^{p'}$ and $\dot{u}_n \xrightarrow{\mathrm{L}^p} \dot{\bar{u}}$;
- $\partial_4 L(\bar{u}, I_-^\alpha[\bar{u}], \dot{\bar{u}}, {}_cD_-^\alpha[\bar{u}], t) \in \mathrm{L}^{p'}$ and ${}_cD_-^\alpha[u_n] \xrightarrow{\mathrm{L}^p} {}_cD_-^\alpha[\bar{u}]$;
- $\partial_1 L(\bar{u}, I_-^\alpha[\bar{u}], \dot{\bar{u}}, {}_cD_-^\alpha[\bar{u}], t) \in \mathrm{L}^1$ and $u_n \xrightarrow{\mathrm{L}^\infty} \bar{u}$;
- $\partial_2 L(\bar{u}, I_-^\alpha[\bar{u}], \dot{\bar{u}}, {}_cD_-^\alpha[\bar{u}], t) \in \mathrm{L}^{p'}$ and $I_-^\alpha[u_n] \xrightarrow{\mathrm{L}^p} I_-^\alpha[\bar{u}]$.

Finally, returning to (3.1) and taking $n \to \infty$ in inequality (3.2), we obtain that

$$\inf_{u\in\mathrm{E}} \mathcal{L}(u) \geq \mathcal{L}(\bar{u}) \in \mathbb{R},$$

which completes the proof. □

In order to make the hypotheses of our Theorem 3.3 more concrete, in Sections 3.2 and 3.3 we prove more precise sufficient conditions on the Lagrangian L, that imply regularity and coercivity of functional $\mathcal{L}$. For this purpose we define a family of sets $\mathscr{P}_M$ for any $M \geq 1$.

3.2 Sufficient Condition for a Lagrangian L to be Regular

For $M \geq 1$, we define $\mathscr{P}_M$ to be the set of maps $P : (\mathbb{R}^d)^4 \times [a, b] \to \mathbb{R}^+$ such that for any $(x_1, x_2, x_3, x_4, t) \in (\mathbb{R}^d)^4 \times [a, b]$

$$P(x_1, x_2, x_3, x_4, t) = \sum_{k=0}^{N} c_k(x_1, t)\|x_2\|^{d_{2,k}} \|x_3\|^{d_{3,k}} \|x_4\|^{d_{4,k}}$$

with $N \in \mathbb{N}$ and where, for any $k = 0, \ldots, N$, $c_k : \mathbb{R}^d \times [a, b] \longrightarrow \mathbb{R}^+$ is continuous and satisfies $d_{2,k} + d_{3,k} + d_{4,k} \leq p/M$.

The following lemma holds for the family of maps $\mathscr{P}_M$.

Lemma 3.4. *Let $M \geq 1$ and $P \in \mathscr{P}_M$. Then, for any $u \in W^{1,p}$, we have*

$$P(u, I_-^\alpha[u], \dot{u}, {}_cD_-^\alpha[u], t) \in L^M.$$

Proof. Because $c_k(u,t)$ is continuous for any $k = 0, \dots, N$, it is in L^∞. We also have $\|I_-^\alpha[u]\|^{d_{2,k}} \in \mathrm{L}^{p/d_{2,k}}$, $\|\dot{u}\|^{d_{3,k}} \in \mathrm{L}^{p/d_{3,k}}$ and $\|{}_cD_-^\alpha[u]\|^{d_{4,k}} \in \mathrm{L}^{p/d_{4,k}}$. Consequently,

$$c_k(u,t)\|I_-^\alpha[u]\|^{d_{2,k}}\|\dot{u}\|^{d_{3,k}}\|{}_cD_-^\alpha[u]\|^{d_{4,k}} \in \mathrm{L}^r$$

with $r = p/(d_{2,k} + d_{3,k} + d_{4,k}) \geq M$. The proof is complete. □

With the help of Lemma 3.4, it is easy to prove the following sufficient condition on the Lagrangian L, which implies its regularity.

Proposition 3.5. *If there exists $P_0 \in \mathscr{P}_1$, $P_1 \in \mathscr{P}_1$, $P_2 \in \mathscr{P}_{p'}$, $P_3 \in \mathscr{P}_{p'}$ and $P_4 \in \mathscr{P}_{p'}$ such that*

- $|L(x_1,x_2,x_3,x_4,t)| \leq P_0(x_1,x_2,x_3,x_4,t)$*;*
- $\|\partial_1 L(x_1,x_2,x_3,x_4,t)\| \leq P_1(x_1,x_2,x_3,x_4,t)$*;*
- $\|\partial_2 L(x_1,x_2,x_3,x_4,t)\| \leq P_2(x_1,x_2,x_3,x_4,t)$*;*
- $\|\partial_3 L(x_1,x_2,x_3,x_4,t)\| \leq P_3(x_1,x_2,x_3,x_4,t)$*;*
- $\|\partial_4 L(x_1,x_2,x_3,x_4,t)\| \leq P_4(x_1,x_2,x_3,x_4,t)$*;*

for any $(x_1,x_2,x_3,x_4,t) \in (\mathbb{R}^d)^4 \times [a,b]$, then L is regular.

The coercivity assumption in Theorem 3.3 is strongly dependent on the set E. In Section 3.3 we provide an example of such set. Moreover, with such choice for E, we give a sufficient condition on the Lagrangian L implying coercivity of $\mathcal{L}$.

3.3 Sufficient Condition for a Functional $\mathcal{L}$ to be Coercive

Consider $u_0 \in \mathbb{R}^d$ and $\mathrm{E} = \mathrm{W}_a^{1,p}$, where $\mathrm{W}_a^{1,p} := \{u \in \mathrm{W}^{1,p},\ u(a) = u_0\}$. We note that $\mathrm{W}_a^{1,p}$ is a weakly closed subset of $\mathrm{W}^{1,p}$ because of the compact embedding $\mathrm{W}^{1,p} \hookrightarrow\!\!\!\!\rightarrow \mathscr{C}$.

The following lemma is important in the proof of Proposition 3.7.

Lemma 3.6. *There exist A_0, $A_1 \geq 0$ such that*

- $\|u\|_{L^\infty} \leq A_0\|\dot{u}\|_{L^p} + A_1$*;*
- $\|I_-^\alpha[u]\|_{L^p} \leq A_0\|\dot{u}\|_{L^p} + A_1$*;*
- $\|{}_cD_-^\alpha[u]\|_{L^p} \leq A_0\|\dot{u}\|_{L^p} + A_1$*;*

for any $u \in W_a^{1,p}$.

Proof. It is easy to see that boundedness of the left Riemann–Liouville fractional integral I^{α}_{-} implies the last inequality. In the case of the second inequality, we have $\|u\|_{\mathrm{L}^p} \leq \|u-u_0\|_{\mathrm{L}^p}+\|u_0\|_{\mathrm{L}^p} \leq (b-a)\|\dot{u}\|_{\mathrm{L}^p}+(b-a)^{1/p}\|u_0\|$ for any $u \in \mathrm{W}^{1,p}_a$. Therefore, using again the boundedness of I^{α}_{-}, we arrive to the desired conclusion. Finally, let us consider the first inequality. We have

$$\|u\|_{\mathrm{L}^\infty} \leq \|u-u_0\|_{\mathrm{L}^\infty}+\|u_0\| \leq \|\dot{u}\|_{\mathrm{L}^1}+\|u_0\| \leq (b-a)^{1/p'}\|\dot{u}\|_{\mathrm{L}^p}+\|u_0\|$$

for any $u \in \mathrm{W}^{1,p}_a$. The proof is completed by defining A_0 and A_1 as the maximum of the appearing constants. □

Next proposition gives a sufficient condition for the coercivity of $\mathcal{L}$.

Proposition 3.7. *Assume that*

$$L(x_1,x_2,x_3,x_4,t) \geq c_0\|x_3\|^p + \sum_{k=1}^{N} c_k \|x_1\|^{d_{1,k}}\|x_2\|^{d_{2,k}}\|x_3\|^{d_{3,k}}\|x_4\|^{d_{4,k}}$$

for any $(x_1,x_2,x_3,x_4,t) \in (\mathbb{R}^d)^4 \times [a,b]$*, where* $c_0>0$*,* $c_k \in \mathbb{R}$*,* $N \in \mathbb{N}^*$*, and*

$$0 \leq d_{1,k}+d_{2,k}+d_{3,k}+d_{4,k} < p \tag{3.3}$$

for any $k=1,\dots,N$*. Then,* $\mathcal{L}$ *is coercive on* $\mathrm{W}^{1,p}_a$*.*

Proof. First, let us define $r = p/(d_{2,k}+d_{4,k}+d_{3,k}) \geq 1$. Applying Hölder's inequality, one can easily prove that

$$\mathcal{L}(u) \geq c_0\|\dot{u}\|^p_{\mathrm{L}^p} - (b-a)^{1/r'} \sum_{k=1}^{N} |c_k| \|u\|^{d_{1,k}}_{\mathrm{L}^\infty} \|I^{\alpha}_{-}[u]\|^{d_{2,k}}_{\mathrm{L}^p} \|\dot{u}\|^{d_{3,k}}_{\mathrm{L}^p} \|{}_cD^{\alpha}_{-}[u]\|^{d_{4,k}}_{\mathrm{L}^p}$$

for any $u \in \mathrm{W}^{1,p}_a$. Moreover, from Lemma 3.6 and (3.3), we obtain that

$$\lim_{\substack{\|\dot{u}\|_{\mathrm{L}^p}\to\infty \\ u \in \mathrm{W}^{1,p}_a}} \mathcal{L}(u) = +\infty.$$

Finally, applying again Lemma 3.6, we have that

$$\|\dot{u}\|_{\mathrm{L}^p} \to \infty \Longleftrightarrow \|u\|_{\mathrm{W}^{1,p}} \to \infty$$

in $\mathrm{W}^{1,p}_a$. Therefore, $\mathcal{L}$ is coercive on $\mathrm{W}^{1,p}_a$. The proof is complete. □

In the next section we illustrate our results through an example.

4 An Illustrative Example

Consider the following fractional problem of the calculus of variations:

$$\mathcal{L}(u) = \int_a^b \|u\|^2 + \|I_-^\alpha[u]\|^2 + \|\dot{u}\|^2 + \|{}_cD_-^\alpha[u]\|^2 \, dt \longrightarrow \min_{u \in \mathrm{W}^{1,2}} \tag{4.1}$$
$$u(a) = u_0.$$

It is not difficult to verify that the Lagrangian L for this problem is convex and satisfies the hypotheses of Propositions 3.5 and 3.7 with $p = 2$. Therefore, it follows from Theorem 3.3 that there exists a solution for problem (4.1). Such minimizer can be determined using the optimality conditions proved in [2, 16] and approximated by the numerical methods developed in [20, 21].

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