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Global Exponential Stability of Delay Neural Networks with Impulsive Perturbations

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Abstract

This paper considers the global exponential stability of delay neural networks with impulsive perturbations. By establishing a new impulsive delay inequality which is different from the earlier publication, we obtain some new sufficient conditions ensuring exponential stability of the equilibrium point for such neural networks. The neural networks model considered include the impulsive delay Hopfield neural networks, the impulsive bidirectional associative memory networks and so on. Those conditions ensuring that global exponential stability are simpler and less restrictive versions of some recent results. At last, two numerical examples are given to illustrate the advantages of the results we obtained.

AMS Subject Classifications: 92B20, 34A37.
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1 Introduction and Preliminaries

In the last several years, delay neural networks have received especially considerable attention due to their extensive applications in associative memory, optimization problems, pattern recognition and image processing, see [1–4, 6, 8–25]. Recently, various results for the stability of delay neural networks are obtained via different approaches. In [22], Zhang et al. obtained some global asymptotic stability results by utilizing Lyapunov functional method and the linear matrix inequality approach for delay Hopfield neural networks as follows:

$$u_i'(t) = -c_i u_i(t) + \sum_{j=1}^{n} a_{ij} f_j(u_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(u_j(t-\tau)) + J_i,\ i = 1, 2, \cdots, n.$$

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Furthermore, some sufficient conditions of global robust stability for the above neural networks were presented in [16]. It is well known that the Halanay inequality has been widely applied to stability analysis of various delay neural networks, and it has also proved to be a powerful tool in the investigation of stability properties of delay neural networks, for instance, see [3, 6, 13, 24, 25]. Recently, Zhang and Wang [21] obtained some new criteria concerning global exponential stability by using the Halanay inequality and some physical parameters for generalized neural networks with time-varying delays as follows:

$$x_i'(t) = -d_i(x_i(t)) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t - \tau_{ij}(t))) + I_i,\ i = 1, 2, \cdots, n.$$

Furthermore, all the results in [21] do not require the activation functions satisfy the Lipschitz condition. However, we know that some neural networks are subject to instantaneous perturbations and experience change of the state abruptly, that is, do exhibit impulsive effects [5,7]. Since impulses and delays can affect the dynamical behaviors of the system creating oscillatory and unstable characteristics, it is necessary to investigate impulses and delays effects on the stability of neural networks.

Recently, there have been some papers and monographs on stability analysis of delay neural networks with impulses, see [4, 6, 8, 9, 11, 12, 17, 20, 21, 23, 25] and references therein. Zhang and Sun [23] get a result for the uniform stability of the equilibrium point of the impulsive Hopfield-type neural networks systems with time delays by using Lyapunov functions and analysis technique. However, the conditions on impulses are too restrictive, and the result does only refer to uniform stability of the equilibrium point. In [20], by applying a piecewise continuous vector Lyapunov function, some sufficient conditions were obtained to ensure the global exponential stability of impulsive delay neural networks as follows:

$$\begin{cases} x_i'(t) = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} g_j(c_j x_j(t - \tau_{ij}(t))) + J_i, t \neq t_k, \\ \Delta x_i|_{t=t_k} = x_i(t_k) - x_i(t_k^-), i = 1, 2, \cdots, n;\ k = 1, 2 \cdots. \end{cases}$$

However, the result is invalid for generalized neural networks. Although some stability conditions for impulsive delay neural networks proposed in [6, 20, 21, 23, 25], there still exists open room for further improvement.

In this paper, we present some new sufficient conditions for global exponential stability for delay neural networks with impulsive perturbations by means of establishing a new impulsive delay inequality which is different from the previous work [25]. The results here are discussed from the point of view of its comparison with the earlier results. Our results also improve and generalize some earlier results [20, 23, 25]. In the end, some numerical examples are discussed to illustrate the advantages of our new approach.

2 Preliminaries

Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_+$ the set of nonnegative real numbers, $\mathbb{Z}_+$ denote the set of positive integers and $\mathbb{R}^n$ the n-dimensional real space equipped with the Euclidean norm $\|\cdot\|$. Consider the delay neural network with impulsive perturbations

$$\begin{cases} x_i'(t) = -d_i(x_i(t)) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t - \tau_{ij}(t))) + I_i, t \neq t_k, \\ \Delta x_i|_{t=t_k} = x_i(t_k) - x_i(t_k^-), \;\; i \in \Lambda, \; k \in \mathbb{Z}_+, \end{cases} \tag{2.1}$$

where $\Lambda = \{1, 2, \cdots, n\}$; $n \geq 2$ corresponds to the number of units in a neural network; the impulse times t_k satisfy $0 \leq t_0 < t_1 < \ldots < t_k < \ldots, \lim_{k \to +\infty} t_k = +\infty$; x_i corresponds to the membrane potential of the unit i at time t; f_j, g_j denote, respectively, the measures of response or activation to its incoming potentials of the unit j at time t and $t - \tau_{ij}(t)$; constant a_{ij} denotes the synaptic connection weight of the unit j on the unit i at time t; constant b_{ij} denotes the synaptic connection weight of the unit j on the unit i at time $t - \tau_{ij}(t)$; I_i is the input of the unit i; $\tau_{ij}(t)$ is the transmission delay of the jth neuron on the unit i such that $0 < \tau_{ij}(t) \leq \tau$, $t \geq t_0$, $i, j \in \Lambda$, τ is a constant.

In this paper, we assume that some conditions are satisfied so that the equilibrium point of (2.1) does exist, see [3,20]. Assume that $x^* = (x_1^*, x_2^*, \cdots x_n^*)^T$ is an equilibrium point of (2.1) and impulsive operator is viewed as perturbation of the equilibrium point x^* of system (2.1) without impulses. We assume that the following impulsive condition holds:

(H$_1$) $\Delta x_i|_{t=t_k} = J_{ik}(x_i(t_k^-) - x_i^*), \; |s + J_{ik}(s)| \leq \beta_k^{(i)} |s|, \; \beta_k^{(i)} > 0, \; i \in \Lambda, \; k \in \mathbb{Z}_+.$

If $J_{ik} = 0$, then the model (2.1) becomes a continuous delay neural network which has been investigated intensively in [21].

Assume that the system (2.1) is supplemented with initial conditions of the form

$$x(s) = \phi(s), \quad s \in [t_0 - \tau, t_0],$$

where $\phi(s) = (\phi_1(s), \phi_2(s), \cdots \phi_n(s))^T \in PC([-\tau, 0], \mathbb{R}^n)$, $PC([-\tau, 0], \mathbb{R}^n) = \{\psi : [-\tau, 0] \to \mathbb{R}^n$, is continuous everywhere except at finite number of points t_k, at which $\psi(t_k^+)$ and $\psi(t_k^-)$ exist and $\psi(t_k^+) = \psi(t_k)\}$.

Since x^* is an equilibrium point of (2.1), one can derive from (2.1) that the transformation $y_i = x_i - x_i^*, i \in \Lambda$ transforms system (2.1) into the system

$$\begin{cases} y_i'(t) = -D_i(y_i(t)) + \sum_{j=1}^{n} a_{ij} \Omega_j(y_j(t)) + \sum_{j=1}^{n} b_{ij} \Gamma_j(y_j(t - \tau_{ij}(t))), t \neq t_k, \; t \geq t_0, \\ y_i(t_k) = y_i(t_k^-) + J_{ik}(y_i(t_k^-)), \;\; i \in \Lambda, \; k \in \mathbb{Z}_+, \end{cases} \tag{2.2}$$

where

$$D_i(y_i(t)) = d_i(x_i^* + y_i(t)) - d_i(x_i^*), \quad \Omega_j(y_j(t)) = f_j(x_j^* + y_j(t)) - f_j(x_j^*),$$
$$\Gamma_j(y_j(t)) = g_j(x_j^* + y_j(t - \tau_{ij}(t))) - g_j(x_j^*).$$

Clearly, x^* is globally exponentially stable for system (2.1) if and only if the trivial solution of system (2.2) is globally exponentially stable. Hence, we only need to prove the stability of the trivial solution of system (2.2).

We also have the following assumptions in later sections:

(H$_2$) There exist positive constants $\Delta_i > 0$, $i \in \Lambda$ such that the inequality

$$\Delta_i \leq \frac{d_i(x_i) - d_i(y_i)}{x_i - y_i} \text{ holds for all } x_i \neq y_i,\ i \in \Lambda.$$

(H$_3$) The functions $\Omega_j(s), \Gamma_j(s)$ satisfy $y_j\Omega_j(y_j) > 0$ and $y_j\Gamma_j(y_j) > 0$ for $y_j \neq 0$, and there exist constants m_j, n_j such that

$$m_j \doteq \sup_{y_j \neq 0} \frac{\Omega_j(y_j)}{y_j} > 0, \quad n_j \doteq \sup_{y_j \neq 0} \frac{\Gamma_j(y_j)}{y_j} > 0.$$

Suppose $\phi_i \in PC([t_0 - \tau, t_0], \mathbb{R})$, $i \in \Lambda$. Let $\Phi = (\phi_1(\cdot), \phi_2(\cdot), \cdots, \phi_n(\cdot))^T$, $X^* = (x_1^*, x_2^*, \cdots, x_n^*)^T$. Define $\|\Phi - X^*\|_\tau^2 = \sup\limits_{t_0 - \tau \leq s \leq t_0} \sum\limits_{i=1}^n |\phi_i(s) - x_i^*|^2$.

Definition 2.1. Assume $X^* = (x_1^*, x_2^*, \cdots, x_n^*)^T \in \mathbb{R}^n$ is an equilibrium point of the system (2.1). The equilibrium point of (2.1) is called globally exponentially stable if there exist constants $\lambda > 0$ and $M \geq 1$ such that for every solution $X = (x_1, x_2, \cdots, x_n)^T$ of system (2.1) with initial value $\Phi = (\phi_1, \phi_2, \cdots, \phi_n)^T$,

$$\sum_{i=1}^n |x_i(t) - x_i^*(t)|^2 \leq M\|\Phi - X^*\|_\tau^2 e^{-\lambda(t - t_0)}.$$

Next, we shall establish a new impulsive delay inequality which is different from the results in [8, 25].

Lemma 2.2. *Assume that there exist* $P, Q > 0$ *and* $m \in PC([t_0 - \tau, \infty), \mathbb{R}_+)$ *such that*

(i) *for* $t = t_k$, $m(t_k) \leq \gamma_k m(t_k^-)$, $\gamma_k > 0$ *are constants with* $\max\limits_{k \in \mathbb{Z}_+} \left\{\frac{1}{\gamma_k}, 1\right\} < \frac{P}{Q}$;

(ii) *for* $t \geq t_0$, $t \neq t_k$, $D^+m(t) \leq -Pm(t) + Q\widetilde{m}(t)$, *where* $\widetilde{m}(t) = \sup\limits_{t - \tau \leq s \leq t} m(s)$;

(iii) $\tau \leq t_k - t_{k-1}$, $k \in \mathbb{Z}_+$.

Then for $t \geq t_0$,

$$m(t) \leq \widetilde{m}(t_0) \left(\prod_{t_0 < t_k \leq t} \gamma_k \right) e^{-\lambda(t-t_0)}, \tag{2.3}$$

where $\lambda > 0$ satisfies the inequality

$$\lambda \leq P - Q \max_{k \in \mathbb{Z}_+} \left\{ \frac{1}{\gamma_k}, 1 \right\} e^{\lambda \tau}. \tag{2.4}$$

Proof. First, condition (i) implies that there exists a constant $\lambda > 0$ such that the inequality (2.4) holds. Next, we shall prove that (2.3) holds for all $t \geq t_0$. First, it is obvious that $m(t) \leq \widetilde{m}(t_0)$ for $t \in [t_0 - \tau, t_0]$. From (2.3), we next prove, for $t \in [t_0, t_1)$,

$$m(t) \leq \widetilde{m}(t_0) e^{-\lambda(t-t_0)}. \tag{2.5}$$

If this is not true, then there exists some $t \in [t_0, t_1)$ such that $m(t) > \widetilde{m}(t_0) e^{-\lambda(t-t_0)}$. For convenience, let

$$W_0(t) = \widetilde{m}(t_0) e^{-\lambda(t-t_0)}, \quad t^\star = \sup\{t | m(s) \leq W_0(s), s \in [t_0, t),\ t \in [t_0, t_1)\}.$$

It is clear that $t^\star \in [t_0, t_1)$ and

(1_a) $m(t^\star) = W_0(t^\star)$;

(2_a) $m(t) \leq W_0(t), t \in [t_0, t^\star]$;

(3_a) for any $\delta > 0$, there exists $t_\delta \in (t^\star, t^\star + \delta)$ such that $m(t_\delta) > W_0(t_\delta)$.

Hence, we get

$$\begin{aligned} D^+ m(t^\star) &\leq -P m(t^\star) + Q \widetilde{m}(t^\star) \\ &\leq -P W_0(t^\star) + Q W_0(t^\star - \tau) \\ &\leq -P W_0(t^\star) + Q \max_{k \in \mathbb{Z}_+} \left\{ \frac{1}{\gamma_k}, 1 \right\} W_0(t^\star - \tau). \end{aligned}$$

On the other hand,

$$\begin{aligned} W_0'(t^\star) &= -\lambda \cdot \widetilde{m}(t_0) e^{-\lambda(t^\star - t_0)} \\ &\geq \left(Q \max_{k \in \mathbb{Z}_+} \left\{ \frac{1}{\gamma_k}, 1 \right\} \cdot e^{\lambda \tau} - P \right) \widetilde{m}(t_0) e^{-\lambda(t^\star - t_0)} \\ &= -P W_0(t^\star) + Q \max_{k \in \mathbb{Z}_+} \left\{ \frac{1}{\gamma_k}, 1 \right\} W_0(t^\star - \tau). \end{aligned}$$

So we obtain $D^+ m(t^\star) \leq W_0'(t^\star)$, which is a contradiction with (3_a). Therefore, (2.5) holds for all $t \in [t_0, t_1)$. From condition (i), we get

$$m(t_1) \leq \gamma_1 m(t_1^-) \leq \gamma_1 \widetilde{m}(t_0) e^{-\lambda(t_1 - t_0)}.$$

For $t \in [t_1, t_2)$, we next show that

$$m(t) \leq \gamma_1 \widetilde{m}(t_0) e^{-\lambda(t-t_0)}. \tag{2.6}$$

If this is not true, there exists $t \in [t_1, t_2)$ such that $m(t) > \gamma_1 \widetilde{m}(t_0) e^{-\lambda(t-t_0)}$. Set

$$W_1(t) = \gamma_1 \widetilde{m}(t_0) e^{-\lambda(t-t_0)}, \quad t^* = \sup\{t | m(s) \leq W_1(s), s \in [t_1, t), t \in [t_1, t_2)\}.$$

Consequently, we get $t^* \in [t_1, t_2)$ and

(1_b) $m(t^*) = W_1(t^*)$;

(2_b) $m(t) \leq W_1(t), t \in [t_1, t^*]$;

(3_b) for any $\delta > 0$, there exists $t_\delta \in (t^*, t^* + \delta)$ such that $m(t_\delta) > W_1(t_\delta)$.

In view of condition (iii), (2.5) and (2_b), we have

$$\begin{aligned}
D^+ m(t^*) &\leq -Pm(t^*) + Q\widetilde{m}(t^*) \\
&= -PW_1(t^*) + Q\widetilde{m}(t^*) \\
&\leq -PW_1(t^*) + Q \cdot \max\{\widetilde{m}(t_0)\gamma_1 e^{-\lambda(t^*-t_0-\tau)}, \widetilde{m}(t_0) e^{-\lambda(t^*-t_0-\tau)}\} \\
&\leq -PW_1(t^*) + Q \max\{\gamma_1, 1\} \widetilde{m}(t_0) e^{-\lambda(t^*-t_0-\tau)} \\
&\leq -PW_1(t^*) + Q \max\left\{\frac{1}{\gamma_1}, 1\right\} \gamma_1 \widetilde{m}(t_0) e^{-\lambda(t^*-t_0-\tau)} \\
&\leq -PW_1(t^*) + Q \max_{k \in \mathbb{Z}_+} \left\{\frac{1}{\gamma_k}, 1\right\} W_1(t^* - \tau).
\end{aligned}$$

On the other hand, we note

$$\begin{aligned}
W_1'(t^*) &= -\lambda \cdot \gamma_1 \widetilde{m}(t_0) e^{-\lambda(t^*-t_0)} \\
&\geq \left(Q \max_{k \in \mathbb{Z}_+} \left\{\frac{1}{\gamma_k}, 1\right\} \cdot e^{\lambda\tau} - P\right) \gamma_1 \widetilde{m}(t_0) e^{-\lambda(t^*-t_0)} \\
&= -PW_1(t^*) + Q \max_{k \in \mathbb{Z}_+} \left\{\frac{1}{\gamma_k}, 1\right\} W_1(t^* - \tau).
\end{aligned}$$

Thus, we arrive at $D^+ m(t^*) \leq W_1'(t^*)$, which contradicts (3_b). So we have proven (2.6) holds for all $t \in [t_1, t_2)$. Furthermore, we can claim that

$$m(t) \leq \gamma_1 \gamma_2 \widetilde{m}(t_0) e^{-\lambda(t-t_0)}, \quad t \in [t_2, t_3).$$

Similarly, we can define W_2 and $\breve{t}$. We only need to note that

$$\begin{aligned} D^+m(\breve{t}) &\leq -Pm(\breve{t}) + Q\widetilde{m}(\breve{t}) \\ &= -PW_2(\breve{t}) + Q\widetilde{m}(\breve{t}) \\ &\leq -PW_2(\breve{t}) + Q\cdot\max\{\widetilde{m}(t_0)\gamma_1\gamma_2 e^{-\lambda(\breve{t}-t_0-\tau)}, \widetilde{m}(t_0)\gamma_1 e^{-\lambda(\breve{t}-t_0-\tau)}\} \\ &\leq -PW_2(\breve{t}) + Q\gamma_1\max\{\gamma_2, 1\}\widetilde{m}(t_0)e^{-\lambda(\breve{t}-t_0-\tau)} \\ &\leq -PW_2(\breve{t}) + Q\frac{\max\{\gamma_2,1\}}{\gamma_2}\gamma_1\gamma_2\widetilde{m}(t_0)e^{-\lambda(\breve{t}-t_0-\tau)} \\ &\leq -PW_2(\breve{t}) + Q\max_{k\in\mathbb{Z}_+}\left\{\frac{1}{\gamma_k}, 1\right\}W_2(\breve{t}-\tau). \end{aligned}$$

Then, applying exactly the same argument as in the proof of (2.6) yields our desirable contradiction. By induction hypothesis, we may prove, in general, that for $t \in [t_m, t_{m+1})$, $m \geq 0$,

$$m(t) \leq \widetilde{m}(t_0)\left(\prod_{k=1}^{m}\gamma_k\right)\cdot e^{-\lambda(t-t_0)},$$

i.e.,

$$m(t) \leq \widetilde{m}(t_0)\left(\prod_{t_0<t_k\leq t}\gamma_k\right)\cdot e^{-\lambda(t-t_0)}, \quad t \geq t_0,$$

where λ satisfies (2.4). Hence, (2.3) holds for all $t \geq t_0$. The proof is complete. □

Remark 2.3. From the proof of Lemma 2.2, we find that condition $\tau \leq t_k - t_{k-1}$, $k \in \mathbb{Z}_+$ can be removed if $\gamma_k \geq 1$ for all $k \in \mathbb{Z}_+$.

Remark 2.4. Lemma 2.2 is similar to the lemma in [25]. However, our results can be applied to cases not covered in the lemma in [25], which will be shown in Section 4.

3 Main Results

Based on the established extended impulsive delay inequality, the following sufficient conditions for global exponential stability of system (2.1) are established.

Theorem 3.1. *Assume that $\tau \leq t_k - t_{k-1}$, $k \in \mathbb{Z}_+$ and conditions (H_1)–(H_3) hold. Moreover, suppose*

$$(H_4)\quad -\frac{1}{2}\min_{i\in\Lambda}\{\Delta_i\} + \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{a_{jk}^2 m_k^2}{\Delta_j} + \left[\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{b_{jk}^2 n_k^2}{\Delta_j}\right]\cdot\max_{k\in\mathbb{Z}_+}\left\{\frac{1}{(\max_{i\in\Lambda}\beta_k^{(i)})^2}, 1\right\} < 0;$$

(H_5) *there exist constants* $E^\star > 0$ *and* $\delta \geq 0$ *such that* $\delta < \lambda$ *and the inequality*

$$2\sum_{k=1}^{m} \ln\left(\max_{i\in\Lambda} \beta_k^{(i)}\right) - \delta(t_m - t_0) < E^\star \quad \text{for all} \quad m \in \mathbb{Z}_+$$

holds, where $\lambda > 0$ *satisfies the inequality*

$$\lambda \leq P - Q \max_{k\in\mathbb{Z}_+} \left\{ \frac{1}{(\max_{i\in\Lambda} \beta_k^{(i)})^2}, 1 \right\} \cdot e^{\lambda\tau}, \tag{3.1}$$

$$P \doteq \frac{1}{2}\min_{i\in\Lambda}\{\Delta_i\} - \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{a_{jk}^2 m_k^2}{\Delta_j}, \quad Q \doteq \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{b_{jk}^2 n_k^2}{\Delta_j}.$$

Then the equilibrium point of the system (2.1) *is globally exponentially stable with the approximate exponential convergence rate* $\lambda - \delta$.

Proof. Suppose $Y = (y_1, y_2, \cdots, y_n)^T$ is a solution of system (2.2). We shall prove the zero solution of (2.2) is globally exponentially stable. Consider a Lyapunov function defined by

$$V(t) = \frac{1}{2}\sum_{i=1}^{n} |y_i(t)|^2 > 0.$$

Then from conditions (H_2), (H_3), and applying exactly the same argument as in the proof of [21, Theorem 1], yields the Dini derivative of $V(t)$ along the solutions of system (2.2), for $t \in [t_k, t_{k+1})$, $k \in \mathbb{Z}_+$

$$\begin{aligned} D^+V(t)|_{(2.2)} &= \sum_{i=1}^{n}\left[-\frac{1}{2}\Delta_i + \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{a_{jk}^2 m_k^2}{\Delta_j}\right] y_i^2(t) \\ &\quad + \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{b_{jk}^2 n_k^2}{\Delta_j} y_i^2(t - \tau_{ij}(t)) \\ &\leq -\overrightarrow{P}V(t) + \overrightarrow{Q}\widetilde{V}(t), \end{aligned} \tag{3.2}$$

where

$$\widetilde{V}(t) = \sup_{t-\tau\leq s\leq t} V(s), \quad \overrightarrow{P} \doteq \min_{i\in\Lambda}\{\Delta_i\} - 2\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{a_{jk}^2 m_k^2}{\Delta_j}, \quad \overrightarrow{Q} \doteq 2\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{b_{jk}^2 n_k^2}{\Delta_j}.$$

On the other hand, from condition (H_1) we can easily get

$$\begin{aligned} V(t_k) = \frac{1}{2}\sum_{i=1}^{n}|y_i(t_k)|^2 &= \frac{1}{2}\sum_{i=1}^{n}|J_{ik}(y_i(t_k^-)) + y_i(t_k^-)|^2 \\ &\leq \frac{1}{2}\sum_{i=1}^{n}(\beta_k^{(i)})^2|y_i(t_k^-)|^2 \\ &\leq \left(\max_{i\in\Lambda}\beta_k^{(i)}\right)^2 V(t_k^-). \end{aligned} \tag{3.3}$$

From (3.2), (3.3) and (H_5), it is not difficult to check that all conditions in Lemma 2.2 are satisfied. So, by Lemma 2.2, for any $t \in [t_m, t_{m+1})$, $m \geq 0$, we have

$$\begin{aligned} V(t) &\leq \widetilde{V}(t_0)\left(\prod_{k=1}^{m}\left(\max_{i\in\Lambda}\beta_k^{(i)}\right)^2\right)e^{-\lambda(t-t_0)} \\ &\leq \widetilde{V}(t_0)e^{E^\star}\cdot e^{\delta(t_m-t_0)}\cdot e^{-\lambda(t-t_0)} \\ &\leq \widetilde{V}(t_0)e^{E^\star}\cdot e^{\delta(t-t_0)}\cdot e^{-\lambda(t-t_0)} \\ &\leq \frac{1}{2}e^{E^\star}\|\Phi - X^*\|_\tau^2 e^{-(\lambda-\delta)(t-t_0)}, \end{aligned}$$

where λ satisfies the inequality (3.1). Hence, we obtain for any $t \geq t_0$,

$$\sum_{i=1}^{n}|x_i(t)-x^*|^2 \leq M\|\Phi - X^*\|_\tau^2 e^{-(\lambda-\delta)(t-t_0)},$$

where $M = e^{E^\star} \geq 1$, λ satisfies the inequality (3.1). Therefore, the equilibrium point of system (2.1) is globally exponentially stable with the approximate exponential convergence rate $\lambda - \delta$. The proof is thus complete. $\square$

Remark 3.2. In Theorem 3.1, if $\sup_{n\in\mathbb{Z}_+}\left(\prod_{k=1}^{n}\max_{i\in\Lambda}\beta_k^{(i)}\right) < \infty$, then we can choose $\delta = 0$ in condition (H_5).

If we let $\beta_k^{(i)} \in (0,1]$, $i \in \Lambda$, $k = 1, 2, \cdots$ in Theorem 3.1, then from Remark 3.2, we can have the following result.

Corollary 3.3. *Assume that, in addition to conditions* (H_1)–(H_3) *and* $\tau \leq t_k - t_{k-1}$, $k \in \mathbb{Z}_+$, *we have*

$$\left[\min_{k\in\mathbb{Z}_+}\max_{i\in\Lambda}\beta_k^{(i)}\right]^2\left[\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{a_{jk}^2 m_k^2}{\Delta_j} - \frac{1}{2}\min_{i\in\Lambda}\{\Delta_i\}\right] + \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{b_{jk}^2 n_k^2}{\Delta_j} < 0.$$

Then the equilibrium point of the system (2.1) *is globally exponentially stable with the approximate exponential convergence rate* λ, *where* $\lambda > 0$ *satisfies the inequality*

$$\lambda \leq P - Q\frac{e^{\lambda\tau}}{[\min_{k\in\mathbb{Z}_+}\max_{i\in\Lambda}\beta_k^{(i)}]^2},$$

with P *and* Q *defined as in Theorem 3.1.*

Furthermore, if $\beta_k^{(i)} \in [1,\infty)$, $i \in \Lambda$, $k = 1, 2, \cdots$ in Theorem 3.1, then from Remark 2.3, we can obtain the following result.

Corollary 3.4. *Assume that conditions* (H_1)–(H_3) *hold. Moreover, suppose that*

(H_4') $-\frac{1}{2}\min_{i\in\Lambda}\{\Delta_i\} + \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{a_{jk}^2 m_k^2}{\Delta_j} + \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{b_{jk}^2 n_k^2}{\Delta_j} < 0$;

(H_5') *there exist constants* $E^\star > 0$ *and* $\delta \geq 0$ *such that* $\delta < \lambda$ *and the inequality*

$$2\sum_{k=1}^{m}\ln\left(\max_{i\in\Lambda}\beta_k^{(i)}\right) - \delta(t_m - t_0) < E^\star \quad \textit{for all} \quad m \in \mathbb{Z}_+$$

holds, where $\lambda > 0$ *satisfies the inequality*

$$\lambda \leq P - Qe^{\lambda\tau}$$

with P *and* Q *defined as in Theorem 3.1.*

Then the equilibrium point of the system (2.1) *is globally exponentially stable with the approximate exponential convergence rate* $\lambda - \delta$.

Theorem 3.5. *Assume that* $\tau \leq t_k - t_{k-1}$, $k \in \mathbb{Z}_+$ *and conditions* (H_1)–(H_3) *hold. Moreover, suppose that*

(H_6) $$-\min_{i\in\Lambda}\left\{\frac{1}{2}\Delta_i - \sum_{j=1}^{n}\frac{|a_{ji}|m_i^2}{\Delta_j\mu_i}\right\} + \max_{i\in\Lambda}\left\{\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{b_{jk}^2 n_k^2\mu_j}{\Delta_j\mu_i}\right\}\cdot\max_{k\in\mathbb{Z}_+}\left\{\frac{1}{(\max_{i\in\Lambda}\beta_k^{(i)})^2}, 1\right\} < 0, \textit{ where } \sum_{j=1}^{n}\mu_i|a_{ij}| = 1$$
for all $i \in \Lambda$;

(H_7) *there exist constants* $E^\star > 0$ *and* $\delta \geq 0$ *such that* $\delta < \lambda$ *and the inequality*

$$2\sum_{k=1}^{m}\ln\left(\max_{i\in\Lambda}\beta_k^{(i)}\right) - \delta(t_m - t_0) < E^\star \quad \textit{for all} \quad m \in \mathbb{Z}_+$$

holds, where $\lambda > 0$ *satisfies the inequality*

$$\lambda \leq P - Q\max_{k\in\mathbb{Z}_+}\left\{\frac{1}{(\max_{i\in\Lambda}\beta_k^{(i)})^2}, 1\right\}\cdot e^{\lambda\tau}, \tag{3.4}$$

$$P \doteq \min_{i\in\Lambda}\left\{\frac{1}{2}\Delta_i - \sum_{j=1}^{n}\frac{|a_{ji}|m_i^2}{\Delta_j\mu_i}\right\}, \quad Q \doteq \max_{i\in\Lambda}\left\{\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{b_{jk}^2 n_k^2\mu_j}{\Delta_j\mu_i}\right\}.$$

Then the equilibrium point of the system (2.1) *is globally exponentially stable with the approximate exponential convergence rate* $\lambda - \delta$.

Proof. We suppose $Y = (y_1, y_2, \cdots, y_n)^T$ is a solution of system (2.2). We shall prove the zero solution of (2.2) is globally exponentially stable. Consider a Lyapunov function defined by

$$V(t) = \frac{1}{2}\sum_{i=1}^{n} \mu_i |y_i(t)|^2 > 0.$$

Then from conditions (H_1)–(H_4), applying exactly the same argument as in the proof of [21, Theorem 2], yields for $t \in [t_k, t_{k+1})$, $k = 1, 2, \cdots$,

$$\begin{aligned} D^+V(t)|_{(2.2)} &= \sum_{i=1}^{n}\left[-\frac{1}{2}\Delta_i + \sum_{j=1}^{n}\frac{|a_{ji}|m_i^2}{\Delta_j \mu_i}\right]\mu_i y_i^2(t) \\ &\quad + \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{b_{jk}^2 n_k^2 \mu_j}{\Delta_j \mu_i}\lambda_i y_i^2(t-\tau_{ji}(t)) \\ &\le -\widetilde{P}V(t) + \widetilde{Q}\widetilde{V}(t), \end{aligned}$$

where

$$\widetilde{P} \doteq 2\min_{i\in\Lambda}\left\{\frac{1}{2}\Delta_i - \sum_{j=1}^{n}\frac{|a_{ji}|m_i^2}{\Delta_j\mu_i}\right\}, \quad \widetilde{Q} \doteq 2\max_{i\in\Lambda}\left\{\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{b_{jk}^2 n_k^2\mu_j}{\Delta_j\mu_i}\right\}.$$

From condition (H_1) and (3.3) we can similarly obtain

$$V(t_k) = \frac{1}{2}\sum_{i=1}^{n}\mu_i|y_i^2(t_k)| \le \frac{1}{2}\sum_{i=1}^{n}(\beta_k^{(i)})^2\mu_i|y_i(t_k^-)|^2 \le \left(\max_{i\in\Lambda}\beta_k^{(i)}\right)^2 V(t_k^-).$$

For any $t \ge t_0$, suppose $t \in [t_m, t_{m+1})$, $m \ge 0$. By Lemma 2.2 and condition (H_6), we get

$$\begin{aligned} \frac{1}{2}\left(\min_{i\in\Lambda}\mu_i\right)\sum_{i=1}^{n}|y_i(t)|^2 \le V(t) &\le \widetilde{V}(t_0)\left(\prod_{k=1}^{m}\left(\max_{i\in\Lambda}\beta_k^{(i)}\right)^2\right)e^{-\lambda(t-t_0)} \\ &\le \widetilde{V}(t_0)e^{E^\star}\cdot e^{\delta(t_m-t_0)}\cdot e^{-\lambda(t-t_0)} \\ &\le \widetilde{V}(t_0)e^{E^\star}\cdot e^{\delta(t-t_0)}\cdot e^{-\lambda(t-t_0)} \\ &\le \frac{1}{2}\left(\max_{i\in\Lambda}\mu_i\right)e^{E^\star}\|\Phi - X^*\|_\tau^2 e^{-(\lambda-\delta)(t-t_0)}, \end{aligned}$$

where λ satisfies the inequality (3.4). Hence, we obtain for any $t \ge t_0$

$$\sum_{i=1}^{n}|x_i(t) - x^*|^2 \le M\|\Phi - X^*\|_\tau^2 e^{-(\lambda-\delta)(t-t_0)},$$

where $M = e^{E^\star}\dfrac{\max_{i\in\Lambda}\mu_i}{\min_{i\in\Lambda}\mu_i} \ge 1$, λ satisfies the inequality (3.4). Therefore, the equilibrium point of system (2.1) is globally exponentially stable with the approximate exponential convergence rate $\lambda - \delta$. The proof is thus complete. □

Similarly, we can obtain the following result whose proof is similar to Theorem 3.5. Here, we omit it.

Theorem 3.6. *Assume that* $\tau \leq t_k - t_{k-1}$, $k \in \mathbb{Z}_+$ *and conditions* (H_1)–(H_3) *hold. Moreover, suppose that*

(H_8) $$-\min_{i\in\Lambda}\left\{\frac{1}{2}\Delta_i - \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{a_{jk}^2 m_k^2 \mu_j}{\Delta_j \mu_i}\right\} + \max_{i\in\Lambda}\left\{\sum_{j=1}^{n}\frac{|b_{ji}|n_i^2}{\Delta_j\mu_i}\right\}\cdot\max_{k\in\mathbb{Z}_+}\left\{\frac{1}{(\max\limits_{i\in\Lambda}\beta_k^{(i)})^2},1\right\} < 0,$$ *where* $\sum_{j=1}^{n}\mu_i|b_{ij}| = 1$ *for all* $i \in \Lambda$*;*

(H_9) *there exist constants* $E^\star > 0$ *and* $\delta \geq 0$ *such that* $\delta < \lambda$ *and the inequality*

$$2\sum_{k=1}^{m}\ln\left(\max_{i\in\Lambda}\beta_k^{(i)}\right) - \delta(t_m - t_0) < E^\star \quad \textit{for all} \quad m \in \mathbb{Z}_+$$

holds, where $\lambda > 0$ *satisfies the inequality*

$$\lambda \leq P - Q\max_{k\in\mathbb{Z}_+}\left\{\frac{1}{\left(\max\limits_{i\in\Lambda}\beta_k^{(i)}\right)^2},1\right\}\cdot e^{\lambda\tau},$$

$$P \doteq \min_{i\in\Lambda}\left\{\frac{1}{2}\Delta_i - \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{a_{jk}^2 m_k^2 \mu_j}{\Delta_j\mu_i}\right\}, \quad Q \doteq \max_{i\in\Lambda}\left\{\sum_{j=1}^{n}\frac{|b_{ji}|n_i^2}{\Delta_j\mu_i}\right\}.$$

Then the equilibrium point of the system (2.1) *is globally exponentially stable with the approximate exponential convergence rate* $\lambda - \delta$.

Remark 3.7. If we let $d_i(s) = d_i \cdot s$, $d_i > 0$ in Theorems 3.1–3.6, then Δ_i can be replaced by d_i.

Remark 3.8. If we let $d_i(s) = d_i \cdot s$, $A = (a_{ij})_{n\times n} = 0$ in system (2.1), then the global exponential stability of system (2.1) has been considered in [20]. So, the results in this paper are applicable to more conditions.

Remark 3.9. If $d_i(s) = d_i \cdot s$, $J_{ik}(s) = J_k \cdot s$, and $\tau_{ij}(t)$ may be replaced by constants τ_j in system (2.1), then the uniform stability of system (2.1) has been investigated in [23]. But it is necessary that condition $y(t_k) = Dy(t_k^-)$ holds in [23]. In fact, HNN are subject to sudden and sharp perturbations instantaneously. These perturbations are impossible to control by a constant function. Hence, our results improve the results and can be applied to cases not covered in [23].

Remark 3.10. It should be noted that the stability conditions we obtained contain all the information of neural networks, and only depend on physical parameters of neural networks, which can be checked easily and quickly.

4 Examples

Example 4.1. Consider the simple two-neuron delay neural network with impulsive perturbations

$$\begin{cases} x_1'(t) = -4x_1(t) + 0.3|x_1(t)| + 0.2|x_2(t)| + 0.5|x_1(t-\tau_1)| \\ \qquad\quad + 0.5|x_2(t-\tau_1)|, & t \neq t_k, \\ x_2'(t) = -4x_2(t) - 0.2|x_1(t)| + 0.7|x_2(t)| - 0.4|x_1(t-\tau_2)| \\ \qquad\quad - 0.6|x_2(t-\tau_2)|, & t \geq 0, \\ \Delta x_1|_{t=t_k} = x_1(t_k) - x_1(t_k^-) = J_{1k}(x_1(t_k^-)), \\ \Delta x_2|_{t=t_k} = x_2(t_k) - x_2(t_k^-) = J_{2k}(x_2(t_k^-)), & k \in \mathbb{Z}_+, \end{cases} \tag{4.1}$$

where $\tau_1 = \tau_2 = \ln 3$, $t_k - t_{k-1} = 2$, $t_0 = 0$, $k \in \mathbb{Z}_+$. Let $|s + I_{ik}| \leq \beta_k^{(i)}|s|$, $i = 1, 2$, where

$$\beta_k^{(1)} = \begin{cases} 1.5, & k = 2n-1, \\ 0.4, & k = 2n, n \in \mathbb{Z}_+, \end{cases} \quad \beta_k^{(2)} = \begin{cases} 2.5, & k = 2n-1, \\ 0.2, & k = 2n, n \in \mathbb{Z}_+. \end{cases}$$

Clearly, $\tau = \ln 3$ and

$$\widetilde{D} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, \quad A = \begin{pmatrix} 0.3 & 0.2 \\ -0.2 & 0.7 \end{pmatrix}, \quad B = \begin{pmatrix} 0.5 & 0.5 \\ -0.4 & -0.6 \end{pmatrix}.$$

On the other hand, it is easy to see that system (4.1) has an equilibrium point $x^* = (0,0)^T$ and

$$\max_{k \in \mathbb{Z}_+} \left\{ \frac{1}{\max\limits_{i \in \Lambda} \beta_k^{(i)}}, 1 \right\} = 2.5, \quad \sup_{n \in Z_+} \left(\prod_{k=1}^{n} \max_{i \in \Lambda} \beta_k^{(i)} \right) = 2.5 < \infty,$$

$$P = \frac{1}{2} \min_{i \in \Lambda}\{\Delta_i\} - \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{a_{jk}^2 m_k^2}{\Delta_j} = 1.835, \quad Q = \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{b_{jk}^2 n_k^2}{\Delta_j} = 0.255,$$

$n_k = 1$. Note that $P = 1.835 > 2.5 \times 0.255 = 2.5Q$. From Remark 3.2 we can choose $\delta = 0$, $\lambda = 0.5$ such that $\lambda \leq 1.835 - 0.6375 \times 3^{\lambda}$. It follows from Theorem 3.1 that the equilibrium point $(0,0)^T$ of the system (4.1) is globally exponentially stable with the approximate exponential convergence rate 0.5.

Remark 4.2. From Example 4.1, it is easy to check that the impulsive delay Halanay inequality in [25] is not feasible here. Hence, the results in this paper are applicable to more conditions.

Example 4.3. Consider [21, Example 3] with impulses

$$|s + I_{ik}| < \beta_k^{(i)}|s|, \quad t_k = 2k, \quad k \in \mathbb{Z}_+, \tag{4.2}$$

where

$$\beta_k^{(1)} = 1 + \frac{1}{2k^2}, \quad \beta_k^{(2)} = 1 + \frac{1}{k^2}, \;\; k \in \mathbb{Z}_+.$$

Since $\sup_{n\in\mathbb{Z}_+}\left(\prod_{k=1}^{n}\max_{i\in\Lambda}\beta_k^{(i)}\right) < \infty$, one may choose $\delta = 0$. Note that

$$\max_{k\in\mathbb{Z}_+}\left\{\frac{1}{(\max_{i\in\Lambda}\beta_k^{(i)})^2}, 1\right\} = 1,$$

we only need check $P > Q$. By a straightforward calculation, we get

$$P = \frac{1}{2}\min_{i\in\Lambda}\{d_i\} - \sum_{j=1}^{2}\sum_{k=1}^{2}\frac{a_{jk}^2 m_k^2}{d_j} = 0.1380, \quad Q = \sum_{j=1}^{2}\sum_{k=1}^{2}\frac{b_{jk}^2 n_k^2}{d_j} = 0.1376 < 0.1380.$$

By Theorem 3.1, we obtain that the equilibrium point of the example 3 with impulses (4.2) is globally exponentially stable. In [21, Example 3], the authors get the equilibrium point with $J_{ik} = 0$ is globally exponentially stable. Using Theorem 3.1, we obtain that the equilibrium point of [21, Example 3] is globally exponentially stable under impulsive effects ($|s + I_{ik}| < \beta_k^{(i)}|s|$). Furthermore, (let $y = x - x^*$ in above example) we note that Theorem 3.1 allows for significant increases (decreases) in $y_2(y_1)$ at impulse times as long as the decreases (increases) of $y_2(y_1)$ between impulses balance it properly.

5 Conclusions

In this work, a class of generalized delay neural networks with impulsive perturbations is considered. We obtain some new sufficient conditions ensuring exponential stability of the equilibrium point for such neural networks by means of establishing a new delay inequality with impulses. Our results show delays and impulsive effects on the stability of neural networks. The results here are discussed from the point of view of its comparison with the earlier results. Furthermore, our results can be applied to cases not covered in some earlier results.

References

[1] S. Arik, Z. Orman, Global stability analysis of Cohen-Grossberg neural networks with time varying delays, *Phys. Lett. A* 342 (2005) 410–421.

[2] C. Bai, Global exponential stability and existence of periodic solution of Cohen-Grossberg type neural networks with delays and impulses, *Nonlinear Anal.: Real World Appl.*, 9 (2008) 747–761.

[3] Z. Chen, J. Ruan, Global stability analysis of impulsive Cohen–Grossberg neural networks with delay, *Phys. Lett. A* 345 (2005) 101–111.

[4] X. Fu, X. Li, Global exponential stability and global attractivity of impulsive Hopfield neural networks with time delays, *J. Comput. Appl. Math.*, 231 (2009) 187–199.

[5] X. Fu, B. Yan, Y. Liu, Introduction of Impulsive Differential Systems, *Science Press*, Beijing, 2005.

[6] Z. Huang, Y. Xia, Global exponential stability of BAM neural networks with transmission delays and nonlinear impulses, *Chaos Solitons Fract.*, 38 (2008) 489–498.

[7] V. Lakshmikantham, X. Liu, Stability Analysis in Terms of Two Measures, Singapore: *World Scientific*, 1993.

[8] K. Li, Stability analysis for impulsive Cohen-Grossberg neural networks with time-varying delays and distributed delays, *Nonlinear Anal.: Real World Appl.*, 10 (2009) 2784–2798.

[9] X. Li, Existence and global exponential stability of periodic solution for impulsive Cohen-Grossberg-type BAM neural networks with continuously distributed delays, *Appl. Math. Comput.*, 215 (2009) 292–307.

[10] X. Li, Global exponential stability for a class of neural networks, *Appl. Math. Lett.*, 22 (2009) 1235–1239.

[11] X. Li, Global exponential stability of Cohen-Grossberg-type BAM neural networks with time-varying delays via impulsive control, *Neurocomputing* 73 (2009) 525–530.

[12] X. Li, Z. Chen, Stability properties for Hopfield neural networks with delays and impulsive perturbations, *Nonlinear Anal.: Real World Appl.*, 10 (2009) 3253–3265.

[13] X. Liao, D. Xiao, Globally Exponential Stability of Hopfield Neural Networks with Time varying Delays,*Acta Elec. Sin.*, 28 (2000) 87–90.

[14] M. Liu, Global asymptotic stability analysis of discrete-time Cohen-Grossberg neural networks based on interval systems, *Nonlinear Anal.: Theory, Methods and Appl.*, 69 (2008) 2403–2411.

[15] X. Lou, B. Cui, Novel global stability criteria for high-order Hopfield-type neural networks with time-varying delays, *J. Math. Anal. Appl.*, 330 (2007) 144–158.

[16] Vimal Singh, On global robust stability of interval Hopfield neural networks with delay, *Chaos Solitons Fract.*, 33 (2007) 1183–1188.

[17] Gani Tr. Stamov, Ivanka M. Stamova, Almost periodic solutions for impulsive neural networks with delay, *Math. Comput. Model.,* 31 (2007) 1263–1270.

[18] J. Wei, C. Zhang, Bifurcation analysis of a class of neural networks with delays, *Nonlinear Anal.: Real World Appl.,* 9 (2008) 2234–2252.

[19] S. Xu, J. Lam, A new approach to exponential stability analysis of neural networks with time-varying delays, *Neural Netw.,* 19 (2006) 76 – 83.

[20] Z. Yang, D. Xu, Global exponential stability of hopfield neural networks with variable delays and impulsive effects, *Appl. Math. Mecha.,* 27 (2006) 1517–1522.

[21] H. Zhang, G. Wang, New criteria of global exponential stability for a class of generalized neural networks with time-varying delays, *Neurocomputing* 70 (2007) 2486–2494.

[22] Q. Zhang, X. Wei J. Xu, Delay-dependent global stability results for delayed Hopfield neural networks, *Chaos Solitons Fract.,* 34 (2007) 662–668.

[23] Y. Zhang, J. Sun, Stability of impulsive neural networks with time delays, *Phys. Lett. A* 348 (2005) 44–50.

[24] D. Zhou, J. Cao, Globally exponential stability conditions for cellular neural networks with time-varying delays, *Appl. Math. Comput.,* 131 (2002) 487–496.

[25] J. Zhou, L. Xiang, Z. Liu, Synchronization in complex delayed dynamical networks with impulsive effects, *Phys. A* 384 (2007) 684–692.