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Existence of Homoclinic Solutions for a Class of Second-Order Differential Equations with Multiple Lags

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Abstract

This paper is concerned with the existence of homoclinic orbits for second-order differential equations with multiple lags. By using Mawhin's continuation theorem, a nontrivial homoclinic orbit is obtained as a limit of a certain sequence of periodic solutions of the equation.

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1 Introduction

In recent years several authors studied homoclinic orbits for Hamiltonian systems via critical point theory. In particular second-order systems were considered in [1, 2, 4–6,

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12–16, 19] and first-order systems in [3, 7–9, 11, 17, 18]. In this paper we consider the existence of homoclinic orbits for FDE by using Mawhin's continuation theorem. In particular we discuss the existence of homoclinic orbits for the equation

$$x''(t)+a_1(t)x'(t)-a_2(t)x(t)=g(t,x(t-\tau_1),x(t-\tau_2),\cdots,x(t-\tau_n))+f(t), \quad (1.1)$$

where τ_i $(i=1,2,\cdots,n)$ are constants, $a_1(t)$ and $a_2(t)$ are real continuous functions defined on $\mathbb{R}$ with positive period T, $f:\mathbb{R}\to\mathbb{R}$ is a continuous and bounded function, $g(t,x_1,x_2,\cdots,x_n)\in C(\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\cdots\times\mathbb{R},\mathbb{R})$, $g(t,0,0,\cdots,0)=0$, and is T-periodic in t. A solution x of (1.1) is said to be homoclinic (to 0) if $x(t)\to 0$ as $t\to\pm\infty$. In addition, if $x\not\equiv 0$ then x is called a nontrivial homoclinic solution.

This paper is largely motivated by the work of Rabinowitz [15] in which the existence of nontrivial homoclinic solutions for the second-order Hamiltonian system

$$\ddot{q}+V_q(t,q)=0$$

was proved. For the sake of completeness, we first state Mawhin's continuation theorem [10]. Assume X and Y are two Banach spaces, $L:\operatorname{Dom}L\subset X\to Y$ is a linear mapping and $N:X\to Y$ is a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim\operatorname{Ker}L=\operatorname{codim}\operatorname{Im}L<+\infty$ and $\operatorname{Im}L$ is closed in Y. If L is a Fredholm mapping of index zero, then there exist continuous projections $P:X\to X$ and $Q:Y\to Y$ such that $\operatorname{Im}P=\operatorname{Ker}L$ and $\operatorname{Im}L=\operatorname{Ker}Q=\operatorname{Im}(I-Q)$. It follows that $L|_{\operatorname{Dom}L\cap\operatorname{Ker}P}:(I-P)X\to\operatorname{Im}L$ has an inverse which will be denoted by K_P. If Ω is an open and bounded subset of X, the mapping N will be called L-compact on Ω if $QN(\overline{\Omega})$ is bounded and $\overline{K_P(I-Q)N(\overline{\Omega})}$ is compact. Since $\operatorname{Im}Q$ is isomorphic to $\operatorname{Ker}L$, there exists an isomorphism $J:\operatorname{Im}Q\to\operatorname{Ker}L$.

Theorem 1.1 (Mawhin's continuation theorem [10]). *Let L be a Fredholm mapping of index zero, and let N be L-compact on $\overline{\Omega}$. Suppose*

(1) *for each $\lambda\in(0,1)$ and $x\in\partial\Omega$, $Lx\neq\lambda Nx$;*

(2) *for each $x\in\partial\Omega\cap\operatorname{Ker}(L)$, $QNx\neq 0$ and $\deg(QN,\Omega\cap\operatorname{Ker}(L),0)\neq 0$.*

Then the equation $Lx=Nx$ has at least one solution in $\overline{\Omega}\cap D(L)$.

2 Main Result

Now we make the following assumptions on $a_1(t), a_2(t)$ and $f(t)$:

(H_1) $0\le m_1\le|a_1(t)|\le M_1$;

(H_2) $M_2=\max_{t\in[0,T]}a_2(t)\ge a_2(t)\ge m_2=\min_{t\in[0,T]}a_2(t)>0$;

(H$_3$) $f:\mathbb{R}\to\mathbb{R}$ is continuous and bounded, $f\not\equiv 0$ and $\left(\int_{\mathbb{R}}|f(t)|^2\,dt\right)^{\frac{1}{2}}\le\eta$, where $\eta>0$ is a positive constant.

Our main result is the following theorem.

Theorem 2.1. *Suppose* (H_1)–(H_3) *and assume*

(H_4) $|g(t,x_1,x_2,\cdots,x_n)|\le r\sum_{i=1}^{n}|x_i|$ *and* $m_2-\frac{M_1^2}{4}-2rn>0.$

Then system (1.1) *possesses a nontrivial homoclinic solution* $x\in C^2(\mathbb{R},\mathbb{R})$ *such that* $x'(t)\to 0$ *as* $t\to\pm\infty$.

In order to prove the main theorem we need some preliminaries. For each $k\in\mathbb{N}$, set

$$X_k:=\{x|x\in C^1(\mathbb{R},\mathbb{R}),x(t+2kT)=x(t),\forall t\in\mathbb{R}\}$$

and $x^{(0)}(t)=x(t)$, define the norm on X_k by

$$\|x\|=\max\left\{\max_{t\in[-kT,kT]}|x(t)|,\max_{t\in[-kT,kT]}|x'(t)|\right\},$$

and set

$$Y_k:=\{y|y\in C(\mathbb{R},\mathbb{R}),y(t+2kT)=y(t),\forall t\in\mathbb{R}\}.$$

We define the norm on Y_k as $\|y\|_0=\max_{t\in[-kT,kT]}|y(t)|$. Thus both $(X_k,\|\cdot\|)$ and $(Y_k,\|\cdot\|_0)$ are Banach spaces.

Remark 2.2. If $x\in X_k$, then it follows that $x^{(i)}(0)=x^{(i)}(2kT)$ ($i=0,1$).

In the works of Izydorek and Janczewska [12] and Tanaka [19], a homoclinic solution of (1.1) is obtained as a limit, as $k\to\pm\infty$, of a certain sequence of functions $x_k\in X_k$. So here we will consider a sequence of systems of functional differential equations

$$x''(t)+a_1(t)x'(t)-a_2(t)x(t)=g(t,x(t-\tau_1),x(t-\tau_2),\cdots,x(t-\tau_n))+f_k(t),\quad(2.1)$$

where for each $k\in\mathbb{N}$, $f_k:\mathbb{R}\to\mathbb{R}$ is a $2kT$-periodic extension of the restriction of f to the interval $[-kT,kT]$ and x_k is a $2kT$-periodic solution of (2.1) obtained via Mawhin's continuation theorem. Define the operators $L_k:X_k\to Y_k$ and $N_k:X_k\to Y_k$ by

$$L_kx(t)=x''(t),\quad t\in\mathbb{R},\quad(2.2)$$

and

$$N_kx(t)=-a_1(t)x'(t)+a_2(t)x(t)+g(t,x(t-\tau_1),x(t-\tau_2),\cdots,x(t-\tau_n))+f_k(t),\quad t\in\mathbb{R}.\quad(2.3)$$

Clearly,

$$\operatorname{Ker} L_k = \{x \in X_k : x(t) = c \in \mathbb{R}\} \tag{2.4}$$

and

$$\operatorname{Im} L_k = \left\{ y \in Y_k : \int_{-kT}^{kT} y(t)dt = 0 \right\} \tag{2.5}$$

is closed in Y_k. Thus L_k is a Fredholm mapping of index zero. Let us define $P_k : X_k \to X_k$ and $Q_k : Y_k \to Y_k / \operatorname{Im}(L_k)$ by

$$P_k x(t) = \frac{1}{2kT} \int_{-kT}^{kT} x(t)dt = x(0), \quad t \in \mathbb{R}, \tag{2.6}$$

for $x = x(t) \in X$ and

$$Q_k y(t) = \frac{1}{2kT} \int_{-kT}^{kT} y(t)dt, \quad t \in \mathbb{R} \tag{2.7}$$

for $y = y(t) \in Y_k$. It is easy to see that $\operatorname{Im} P_k = \operatorname{Ker} L_k$ and $\operatorname{Im} L_k = \operatorname{Ker} Q_k = \operatorname{Im}(I_k - Q_k)$. It follows that $L_k|_{\operatorname{Dom} L_k \cap \operatorname{Ker} P_k} : (I_k - P_k)X_k \to \operatorname{Im} L_k$ has an inverse which will be denoted by K_{P_k}.

Let Ω_k be an open and bounded subset of X_k. We can easily see that $Q_k N_k(\overline{\Omega}_k)$ is bounded and $\overline{K_{P_k}(I_k - Q_k)N_k(\overline{\Omega}_k)}$ is compact. Thus the mapping N_k is L-compact on $\overline{\Omega}_k$. That is, we have the following result.

Lemma 2.3. *Let L_k, N_k, P_k and Q_k be defined by* (2.2), (2.3), (2.6) *and* (2.7) *respectively. Then L_k is a Fredholm mapping of index zero and N_k is L-compact on $\overline{\Omega}_k$, where Ω_k is any open and bounded subset of X_k.*

In order to prove our main result, we need the following lemma [15].

Lemma 2.4 (See Remark 2.2 and [15]). *There is a positive constant ϱ such that for each $k \in \mathbb{N}$ and $x \in X_k$ the following inequality holds:*

$$\max_{t \in [-kT, kT]} |x(t)| \le \varrho \left[\int_{-kT}^{kT} (|x(t)|^2 + |x'(t)|^2)dt \right]^{\frac{1}{2}}.$$

Now, we consider the auxiliary equation

$$\begin{aligned} x''(t) &+ \lambda[a_1(t)x'(t) - a_2(t)x(t)] \\ &= \lambda[g(t, x(t-\tau_1), x(t-\tau_2), \cdots, x(t-\tau_n)) + f_k(t)], \end{aligned} \tag{2.8}$$

where $0 < \lambda < 1$.

Lemma 2.5. *Suppose that the conditions of Theorem 2.1 are satisfied. If* $x_k(t)$ *is a* $2kT$*-periodic solution of Eq.* (2.8)*, then there are positive constants* D_i*,* $i = 0, 1$*, which are independent of* λ*, such that*

$$\|x_k^{(i)}\|_0 \le D_i, \quad t \in [-kT, kT], \quad i = 0, 1. \tag{2.9}$$

Proof. Suppose that x_k is a $2kT$-periodic solution of Eq. (2.8). We have from (2.8) that

$$\begin{aligned} &\int_{-kT}^{kT} [x_k''(t) + \lambda a_1(t) x_k'(t) - \lambda a_2(t) x_k(t)] x_k(t) dt \\ &\qquad = \lambda \int_{-kT}^{kT} [g(t, x_k(t-\tau_1), x_k(t-\tau_2), \cdots, x_k(t-\tau_n)) + f_k(t)] x_k(t) dt. \end{aligned} \tag{2.10}$$

From (2.10), we have

$$\begin{aligned} &\int_{-kT}^{kT} \{|x_k'(t)|^2 + \lambda a_2(t) |x_k(t)|^2\} dt \\ = \;& -\lambda \int_{-kT}^{kT} [g(t, x_k(t-\tau_1), x_k(t-\tau_2), \cdots, x_k(t-\tau_n)) \\ & + f_k(t) - a_1(t) x_k'(t)] x_k(t) dt \\ \le \;& \lambda \left(\int_{-kT}^{kT} |g(t, x_k(t-\tau_1), x_k(t-\tau_2), \cdots, x_k(t-\tau_n))|^2 dt \right)^{\frac{1}{2}} \\ & \times \left(\int_{-kT}^{kT} |x_k(t)|^2 dt \right)^{\frac{1}{2}} + \lambda \left[\left(\int_{-kT}^{kT} |f_k(t)|^2 dt \right)^{\frac{1}{2}} + M_1 \left(\int_{-kT}^{kT} |x_k'(t)|^2 dt \right)^{\frac{1}{2}} \right] \\ & \times \left(\int_{-kT}^{kT} |x_k(t)|^2 dt \right)^{\frac{1}{2}} \\ \le \;& 2rn\lambda \int_{-kT}^{kT} |x_k(t)|^2 dt + \lambda \left[\eta + M_1 \left(\int_{-kT}^{kT} |x_k'(t)|^2 dt \right)^{\frac{1}{2}} \right] \left(\int_{-kT}^{kT} |x_k(t)|^2 dt \right)^{\frac{1}{2}}, \end{aligned}$$

so

$$\begin{aligned} &\lambda \left(m_2 - \frac{M_1^2}{4} \right) \int_{-kT}^{kT} |x_k(t)|^2 dt \\ \le \;& \lambda \left[\left(\int_{-kT}^{kT} |x_k'(t)|^2 dt \right)^{\frac{1}{2}} - \frac{M_1}{2} \left(\int_{-kT}^{kT} |x_k(t)|^2 dt \right)^{\frac{1}{2}} \right]^2 \\ & + \lambda \left(a_2(t) - \frac{M_1^2}{4} \right) \int_{-kT}^{kT} |x_k(t)|^2 dt + \left(\frac{1}{\lambda} - \lambda \right) \int_{-kT}^{kT} |x_k'(t)|^2 dt \\ \le \;& 2rn\lambda \int_{-kT}^{kT} |x_k(t)|^2 dt + \eta\lambda \left(\int_{-kT}^{kT} |x_k(t)|^2 dt \right)^{\frac{1}{2}}, \end{aligned}$$

which gives

$$\left(m_2 - \frac{M_1^2}{4} - 2rn\right) \int_{-kT}^{kT} |x_k(t)|^2 dt \le \eta \left(\int_{-kT}^{kT} |x_k(t)|^2 dt\right)^{\frac{1}{2}}. \tag{2.11}$$

From (H_4) and (2.11), there exists a positive constant C_1 such that

$$\int_{-kT}^{kT} |x_k(t)|^2 dt \le \frac{\eta^2}{(m_2 - \frac{M_1^2}{4} - 2rn)^2} = C_1. \tag{2.12}$$

From (2.8), we have

$$\begin{aligned}\int_{-kT}^{kT} &[x_k''(t) + \lambda a_1(t)x_k'(t) - \lambda a_2(t)x_k(t)]x_k'(t)dt \\ &= \lambda \int_{-kT}^{kT} [g(t, x_k(t-\tau_1), x_k(t-\tau_2), \cdots, x_k(t-\tau_n)) + f_k(t)]x_k'(t)dt,\end{aligned} \tag{2.13}$$

so

$$\begin{aligned} m_1 \int_{-kT}^{kT} |x_k'(t)|^2 dt &\le \int_{-kT}^{kT} |a_1(t)||x_k'(t)|^2 dt \\ &\le \left[(M_2 + 2rn)\left(\int_{-kT}^{kT} |x_k(t)|^2 dt\right)^{\frac{1}{2}} + \eta\right] \left(\int_{-kT}^{kT} |x_k'(t)|^2 dt\right)^{\frac{1}{2}} \\ &\le [(M_2 + 2rn)\sqrt{C_1} + \eta] \left(\int_{-kT}^{kT} |x_k'(t)|^2 dt\right)^{\frac{1}{2}}, \end{aligned}$$

and as a result there exists a positive constant C_2 such that

$$\int_{-kT}^{kT} |x_k'(t)|^2 dt \le C_2. \tag{2.14}$$

Moreover, for $x \in X_k$ and $t, \tau \in [-kT, kT]$, we have

$$|x(t)| \le x(\tau) + \left|\int_{\tau}^{t} x'(s) ds\right|. \tag{2.15}$$

Integration of (2.15) over $\left[t - \dfrac{1}{2}, t + \dfrac{1}{2}\right]$ shows

$$\begin{aligned} |x(t)| &\le \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |x(\tau)| d\tau + \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} \left|\int_{\tau}^{t} x'(s) ds\right| d\tau \\ &\le 2\left(\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} (|x'(\tau)|^2 + |x(t)|^2) d\tau\right)^{\frac{1}{2}}. \end{aligned} \tag{2.16}$$

Hence (2.15) and (2.16) imply

$$\max_{t\in[-kT,kT]}|x(t)|\le \varrho\left(\int_{-kT}^{kT}(|x'(t)|^2+|x(t)|^2)dt\right)^{\frac{1}{2}},\quad x\in X_k, \tag{2.17}$$

where ϱ is given in Lemma 2.4. From Lemma 2.4, (2.12) and (2.14), we have

$$\begin{aligned}\max_{t\in[-kT,kT]}|x_k(t)| &\le \varrho\left(\int_{-kT}^{kT}(|x_k'(t)|^2+|x_k(t))|^2dt\right)^{\frac{1}{2}}\\ &\le \varrho(C_1+C_2)^{\frac{1}{2}}=D_0.\end{aligned} \tag{2.18}$$

On the other hand, we have from (2.8) that

$$\begin{aligned}&\int_{-kT}^{kT}[x_k''(t)+\lambda a_1(t)x_k'(t)-\lambda a_2(t)x_k(t)]x_k''(t)dt\\ &\quad=\lambda\int_{-kT}^{kT}[g(t,x_k(t-\tau_1),x_k(t-\tau_2),\cdots,x_k(t-\tau_n))+f_k(t)]x_k''(t)dt,\end{aligned} \tag{2.19}$$

so we have

$$\begin{aligned}&\int_{-kT}^{kT}|x_k''(t)|^2dt\\ &\le \left(\int_{-kT}^{kT}|x_k''(t)|^2dt\right)^{\frac{1}{2}}\left[M_1\left(\int_{-kT}^{kT}|x_k'(t)|^2dt\right)^{\frac{1}{2}}+M_2\left(\int_{-kT}^{kT}|x_k(t)|^2dt\right)^{\frac{1}{2}}\right]\\ &\quad+\left(\int_{-kT}^{kT}|g(t,x_k(t-\tau_1),x_k(t-\tau_2),\cdots,x_k(t-\tau_n))|^2dt\right)^{\frac{1}{2}}\\ &\quad\times\left(\int_{-kT}^{kT}|x_k''(t)|^2dt\right)^{\frac{1}{2}}+\left(\int_{-kT}^{kT}|f_k(t)dt\right)^{\frac{1}{2}}\left(\int_{-kT}^{kT}|x_k''(t)|^2dt\right)^{\frac{1}{2}}\\ &\le [(2rn+M_2)\sqrt{C_1}+M_1\sqrt{C_2}+\eta]\left(\int_{-kT}^{kT}|x_k''(t)|^2dt\right)^{\frac{1}{2}},\end{aligned}$$

and as a result there exists a positive constant C_4 such that

$$\int_{-kT}^{kT}|x_k''(t)|^2dt\le[(2rn+M_2)\sqrt{C_1}+M_1\sqrt{C_2}+\eta]^2=C_4. \tag{2.20}$$

From Lemma 2.4 and (2.20), we have

$$\begin{aligned}\max_{t\in[-kT,kT]}|x_k'(t)| &\le \varrho\left(\int_{-kT}^{kT}(|x_k'(t)|^2+|x_k''(t)|^2)dt\right)^{\frac{1}{2}}\\ &\le \varrho(C_2+C_4)^{\frac{1}{2}}=D_1.\end{aligned} \tag{2.21}$$

The proof is complete. □

Lemma 2.6. *Let* $k \in \mathbb{N}$. *If* (H_1)–(H_4) *hold, then the system* (2.1) *possesses a* $2kT$-*periodic solution.*

Proof. Suppose that x is a $2kT$-periodic solution of Eq. (2.8). By Lemma 2.5, there exist positive constants D_i ($i = 0, 1$) which are independent of λ such that (2.9) is true. Consider any positive constant $\overline{\alpha}_k > \max\limits_{0 \le i \le 1}\{D_i\} + \xi$, where $\xi = \max\limits_{t \in \mathbb{R}} |f(t)|$. Set

$$\Omega_k := \{x \in X_k : \|x\| < \overline{\alpha}_k\}.$$

We know that L_k is a Fredholm mapping of index zero and N_k is L-compact on $\overline{\Omega}_k$ (see [2]). Recall

$$\mathrm{Ker}(L_k) = \{x \in X_k : x(t) = c \in \mathbb{R}\}$$

and the norm on X_k is

$$\|x\| = \max\left\{\max_{t \in [-kT, kT]} |x(t)|, \max_{t \in [-kT, kT]} |x'(t)|\right\}.$$

Then we have

$$x = \overline{\alpha}_k \quad \text{or} \quad x = -\overline{\alpha}_k \quad \text{for} \quad x \in \partial\Omega_k \cap \mathrm{Ker}(L_k). \tag{2.22}$$

From (H_4), we have (if $\overline{\alpha}_k$ is chosen large enough)

$$a_2(t)\overline{\alpha}_k + g(t, \overline{\alpha}_k, \overline{\alpha}_k, \cdots, \overline{\alpha}_k) - \|f_k\|_0 > 0 \quad t \in [-kT, kT] \tag{2.23}$$

and

$$x'(t) = 0, \quad \forall x \in \partial\Omega_k \cap \mathrm{Ker}(L_k). \tag{2.24}$$

Finally from (2.3), (2.7) and (2.22)–(2.24), we have

$$\begin{aligned}(Q_k N_k x) &= \frac{1}{2kT}\int_{-kT}^{kT}[-a_1(t)x'(t) + a_2(t)x(t) \\ &\quad + g(t, x(t-\tau_1), x(t-\tau_2), \cdots, x(t-\tau_n)) + f_k(t)]dt \\ &= \frac{1}{2kT}\int_{-kT}^{kT}[a_2(t)x(t) + g(t, x(t-\tau_1), x(t-\tau_2), \cdots, x(t-\tau_n)) + f_k(t)]dt \\ &\neq 0, \quad \forall x \in \partial\Omega_k \cap \mathrm{Ker}(L_k).\end{aligned}$$

Then, for any $x \in \mathrm{Ker}\, L_k \cap \partial\Omega_k$ and $\eta \in [0, 1]$, we have

$$\begin{aligned}xH(x, \eta) &= -\eta x^2 - \frac{x}{2kT}(1-\eta)\int_{-kT}^{kT}[-a_1(t)x'(t) + a_2(t)x(t) + f_k(t) \\ &\quad + g(t, x(t-\tau_1), x(t-\tau_2), \cdots, x(t-\tau_n))]dt \\ &\neq 0.\end{aligned}$$

Thus

$$\begin{aligned}
&\deg\{Q_kN_k, \Omega_k \cap \operatorname{Ker}(L_k), 0\} \\
= \;& \deg\left\{-\frac{1}{2kT}\int_{-kT}^{kT}[-a_1(t)x'(t) + a_2(t)x(t) + f_k(t)\right. \\
& +g(t, x(t-\tau_1), x(t-\tau_2), \cdots, x(t-\tau_n))]dt, \Omega_k \cap \operatorname{Ker}(L_k), 0\} \\
= \;& \deg\{-x, \Omega_k \cap \operatorname{Ker}(L_k), 0\} \\
\neq \;& 0.
\end{aligned}$$

From Lemma 2.5, for any $x \in \partial\Omega_k \cap \operatorname{Dom}(L_k)$ and $\lambda \in (0,1)$ we have $L_kx \neq \lambda N_kx$. By Theorem 1.1, the equation $L_kx = N_kx$ has at least one solution in $\operatorname{Dom}(L) \cap \overline{\Omega_k}$. So there exists a $2kT$-periodic solution x_k of the system (2.1). The proof is complete. □

Lemma 2.7. *Let $\{x_k\}_{k\in\mathbb{N}}$ be the sequence given by Lemma 2.6. Then there exists x_0 and a subsequence of $\{x_n\}_{n\in\mathbb{N}}$ (again we call it $\{x_n\}_{n\in\mathbb{N}}$) such that $x_k \to x_0$ in $C^1_{\rm loc}(\mathbb{R}, \mathbb{R})$ as $k \to +\infty$.*

Proof. By (2.18), (2.21) and the Arzelà–Ascoli theorem, we obtain that a subsequence of $\{x_k\}_{k\in\mathbb{N}}$ converges in $C^1_{\rm loc}(\mathbb{R}, \mathbb{R})$ to a solution x_0 of (1.1) satisfying

$$\int_{-\infty}^{\infty}(|x_0(t)|^2 + |x_0'(t)|^2)dt < \infty. \tag{2.25}$$

To see this note from (2.1) that

$$\begin{aligned}
&\lim_{k\to\infty}[x_k''(t) + a_1(t)x_k'(t) - a_2(t)x_k(t) \\
&\quad -g(t, x_k(t-\tau_1), x_k(t-\tau_2), \cdots, x_k(t-\tau_n))] \\
= \;& x_0''(t) + a_1(t)x_0'(t) - a_2(t)x_0(t) - g(t, x_0(t-\tau_1), x_0(t-\tau_2), \cdots, x_0(t-\tau_n)) \\
= \;& \lim_{k\to\infty} f_k(t) = f(t),
\end{aligned}$$

so x_0 is a solution of (1.1). Also, we have

$$\int_{-\infty}^{\infty}[|x_0'(t)|^2 + |x_0(t)|^2]dt = \lim_{k\to\infty}\int_{-kT}^{kT}[|x_k'(t)|^2 + |x_k(t)|^2]dt < \infty.$$

This shows that (2.25) holds. □

Lemma 2.8. *The function x_0 determined by Lemma 2.7 is the desired homoclinic solution of* (1.1).

Proof. The proof will be divided into two steps.

Step 1: We prove that $x_0(t) \to 0$, as $t \to \pm\infty$. By (2.25), we have

$$\lim_{j\to\infty}\int_{|t|\geq j}[|x_0(t)|^2 + |x_0'(t)|^2]dt = 0. \tag{2.26}$$

Hence (2.18) and (2.26) shows that our claim holds.

Step 2: We now show that $x_0'(t) \to 0$ as $t \to \pm\infty$. By (2.16), (2.18) and (2.26), it suffices to prove that

$$\int_j^{j+1} |x_0''(t)|^2 dt \to 0, \quad \text{as} \quad j \to +\infty. \tag{2.27}$$

On the other hand, we obtain from (1.1) that

$$\int_j^{j+1} |x_0''(t)|^2 dt = \int_j^{j+1} | - a_1(t)x_0'(t) + a_2(t)x_0(t) + f(t) + g(t, x_0(t-\tau_1), x_0(t-\tau_2), \cdots, x_0(t-\tau_n))|^2 dt.$$

Since $g(t, 0, 0, \cdots, 0) = 0$ for all $t \in \mathbb{R}$, $x_0(t) \to 0$ as $t \to \pm\infty$, $\int_j^{j+1} |x_0'(t)|^2 dt \to 0$ and $\int_j^{j+1} |f(t)|^2 dt \to 0$ as $j \to \pm\infty$, so (2.27) follows. □

Proof of Theorem 2.1. The result follows now from Lemma 2.8. □

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