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Oscillation Criteria for Forced Second-Order Functional Dynamic Equations with Mixed Nonlinearities on Time Scales

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Abstract

We are concerned with the oscillation of certain forced second-order functional dynamic equation with mixed nonlinearities. Our results in a particular case solve a problem posed by Anderson, and our results in the special cases when the time scale is the set of real numbers and the set of integers involve and improve some oscillation results for second-order differential and difference equations, respectively.

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1 Introduction

We are concerned with the oscillatory behavior of the forced second-order functional dynamic equation with mixed nonlinearities

$$\left(a(t)x^{\Delta}(t)\right)^{\Delta}+\sum_{i=0}^{n}p_i(t)\left|x\left(\tau_i\left(t\right)\right)\right|^{\alpha_i}\ \text{sgn}\,x\left(\tau_i\left(t\right)\right)=e(t) \tag{1.1}$$

on an arbitrary time scale $\mathbb{T}$, where $\alpha_0=1$ and $\alpha_i>0$, $i=1,2,\ldots,n$. We also, assume that $a,\ e,\ p_i\in C_{\text{rd}}\left([0,\infty)_{\mathbb{T}},\mathbb{R}\right)$, $a\left(t\right)>0$, $\tau_i:\mathbb{T}\to\mathbb{T}$ are nondecreasing rd-continuous functions on $\mathbb{R}$, $\tau_i\left(t\right)\leq\sigma\left(t\right)$ and $\lim_{t\to\infty}\tau_i(t)=\infty$, for $i=0,1,\ldots,n$. Our interest is to establish oscillation criteria for equation (1.1) that do not assume that $e\left(t\right)$ and $p_i\left(t\right)$, $i=0,1,2,\ldots,n$ are of definite sign. Since we are interested in the oscillatory and asymptotic behavior of solutions near infinity, we assume that $\sup\mathbb{T}=\infty$, and define the time scale interval $[t_0,\infty)_{\mathbb{T}}$ by $[t_0,\infty)_{\mathbb{T}}:=[t_0,\infty)\cap\mathbb{T}$. By a solution of (1.1) we mean a nontrivial real-valued function $x\in C^1_{\text{rd}}[T_x,\infty)$, $T_x\geq t_0$ which has the property that $ax^{\Delta}\in C^1_{\text{rd}}[T_x,\infty)$ and satisfies equation (1.1) on $[T_x,\infty)$, where C_{rd} is the space of rd-continuous functions. The solutions vanishing in some neighborhood of infinity will be excluded from our consideration. A solution x of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory.

The theory of time scales, which has recently received a lot of recent attention, was introduced by Stefan Hilger in his PhD dissertation written under the direction of Bernd Aulbach (see [26]). Since then a rapidly expanding body of literature has sought to unify, extend, and generalize ideas from discrete calculus, quantum calculus, and continuous calculus to arbitrary time scale calculus. Recall that a time scale $\mathbb{T}$ is a nonempty closed subset of the reals, and the cases when this time scale is the reals or the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist, and they give rise to many applications (see [5]). Not only does the new theory of the so-called “dynamic equations” unify the theories of differential equations and difference equations, but also extends these classical cases to cases “in between”, e.g., to so-called q-difference equations when $\mathbb{T}=q^{\mathbb{N}_0}$, $q>1$ (which has important applications in quantum theory (see [29])) and can be applied on different types of time scales like $\mathbb{T}=h\mathbb{Z}$, $\mathbb{T}=\mathbb{N}_0^2$ and $\mathbb{T}=\mathbb{H}_n$ the space of harmonic numbers. In this work a knowledge and understanding of time scales and time scale notation is assumed; for an excellent introduction to the calculus on time scales, see Bohner and Peterson [5, 6].

A great deal of effort has been spent in obtaining criteria for oscillation of dynamic equations on time scales without forcing terms and it is usually assumed that the potential $q(t)$ is a positive function. We refer the reader to the papers [7, 11–20, 24, 35, 36] and the references cited therein. On the other hand very little is known for related equations with forcing terms. Saker [34] considered nonlinear dynamic equations with a forcing term and also with a positive potential function of the form

$$(a(t)x^{\Delta})^{\Delta} + q(t)f(x^{\sigma}(t)) = e(t), \text{ for } t \in [t_0, \infty)_{\mathbb{T}}, \tag{1.2}$$

on a time scale $\mathbb{T}$. The author established some sufficient conditions for oscillation when the forcing term $e(t)$ is small enough for large values of $t \in \mathbb{T}$. Bohner and Tisdell [9] considered (1.2) with a positive potential function and established some sufficient conditions for oscillation when the solutions of the equation without a forcing term are nonoscillatory. Huang and Feng [28] considered the equation

$$x^{\Delta\Delta}(t) + q(t)f(x^{\sigma}(t)) = e(t), \text{ for } t \in [t_0, \infty)_{\mathbb{T}},$$

when $q(t) > 0$, $xf(x) > 0$ for $x \neq 0$, $f'(x) > 0$ and assumed that there exists an oscillatory function $h(t)$ such that $h^{\Delta\Delta}(t) = e(t)$. Anderson [2] studied forced functional dynamic equation with oscillatory potentials of the form

$$(a(t)x^{\Delta})^{\Delta} + p(t)|x(\tau(t))|^{\alpha} \text{ sgn } x(\tau(t)) + q(t)|x(\theta(t))|^{\beta} \text{ sgn } x(\theta(t)) = e(t),$$

for $t \in [t_0, \infty)_{\mathbb{T}}$, where $\alpha, \beta \geq 1$, $\tau, \beta : \mathbb{T} \to \mathbb{T}$ are nondecreasing right-dense continuous functions with $\theta(t) \geq t$ and $\tau(t) \leq t$ such that $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \theta(t) = \infty$ and $a > 0$ is nondecreasing. In that paper, Anderson posed an open question for the case $0 < \alpha, \beta < 1$. In [1], Anderson considered a more general nonlinear dynamic equation of the form

$$(a(t)x^{\Delta})^{\Delta} + q(t)f(t, x^{\sigma}, x^{\Delta}) = 0, \text{ for } t \in [t_0, \infty)_{\mathbb{T}},$$

and established some interval criteria for oscillation based on the information on only a sequence of subintervals of $[t_0, \infty)_{\mathbb{T}}$.

Our results for dynamic equations, in particular, answer the problem that was posed in [2] and improve the results that were established earlier in [1,27,28,34]. As a special case, when $\mathbb{T} = \mathbb{R}$ our results reduce to the results of Hassan, Erbe and Peterson [25] and complement the results that have been established in [10, 22, 23, 31, 37–40], and when $\mathbb{T} = \mathbb{N}$, our results improve the oscillation results that were obtained by Wong and Agarwal [33] and Peng et al. [32] in the sense that our results do not require additional conditions on the unknown solutions and are based on the information on only a sequence of subintervals of $[t_0, \infty)_{\mathbb{T}}$. When $\mathbb{T} = h\mathbb{N}$ or $\mathbb{T} = q^N$, i.e., for generalized difference equations or q-difference equations our results are essentially new and can be applied on different types of time scales. Some examples are considered to illustrate the main results.

2 Main Results

Before stating our main results, we begin with the following lemmas which we will need in the proof of our main results.

Lemma 2.1 (See [25]). *Let* $(\alpha_1, \alpha_2, \ldots \alpha_n)$ *be an* n*-tuple satisfying* $\alpha_1 > \alpha_2 > \ldots > \alpha_m > 1 > \alpha_{m+1} > \ldots > \alpha_n > 0$. *Then there exists an* n*-tuple* $(\eta_1, \eta_2, \ldots, \eta_n)$ *with* $0 < \eta_i < 1$ *satisfying*

(a) $\sum_{i=1}^{n} \alpha_i \eta_i = 1,$

and which also satisfies either

(b) $\sum_{i=1}^{n} \eta_i < 1,$

or

(c) $\sum_{i=1}^{n} \eta_i = 1.$

Lemma 2.2. *Let* A *be an arbitrary real number,* B *a positive real number and let* γ *be a quotient of odd positive integers. Then*

$$Au + Bu^{\frac{\gamma+1}{\gamma}} \geq -\left(\frac{A}{\gamma+1}\right)^{\gamma+1}\left(\frac{\gamma}{B}\right)^{\gamma}.$$

Proof. We let $f(u) = Au + Bu^{\frac{\gamma+1}{\gamma}}$. It is easy to see that $f(u)$ obtains its minimum at $u = \left(\frac{-A\gamma}{B(\gamma+1)}\right)^{\gamma}$ and $f_{\min} = -\left(\frac{A}{\gamma+1}\right)^{\gamma+1}\left(\frac{\gamma}{B}\right)^{\gamma}$. □

Lemma 2.3. *Assume* $T \in [t_0, \infty)_{\mathbb{T}}$ *and there exist constants* $a_k,\ b_k \in [T, \infty)_{\mathbb{T}}$ *such that* $a_k < b_k,\ k = 1, 2$, *with*

$$p_i(t) \geq 0, \quad \text{for } t \in [\tau(a_k), b_k)_{\mathbb{T}}, \text{ for } i = 0, 1, 2, \ldots, n \text{ and } k = 1, 2,$$

and

$$e(t)\begin{cases} \leq 0, & t \in [\tau(a_1), b_1)_{\mathbb{T}}, \\ \geq 0, & t \in [\tau(a_2), b_2)_{\mathbb{T}}, \end{cases}$$

where $\tau(t) := \min\{t, \tau_0(t), \ldots, \tau_n(t)\}$. *Assume equation* (1.1) *has a solution* x *such that* $x(t)$ *and* $x(\tau_i(t)),\ i = 0, 1, 2, \ldots, n$ *are of one sign on* $[T, \infty)_{\mathbb{T}}$. *Then, for* $t \in [a_k, b_k)_{\mathbb{T}},\ k = 1, 2$, *we have*

$$\left(a(t)x^{\Delta}(t)\right)^{\Delta} + \sum_{i=0}^{n} p_i(t)\delta_{i,k}^{\alpha_i}(t)\ \left|x^{\sigma}(t)\right|^{\alpha_i}\ \text{sgn}\, x^{\sigma}(t) \leq e(t),$$

where, for $i = 0, 1, 2, \ldots, n$ *and* $k = 1, 2$, *we have*

$$\delta_{i,k}(t) := \int_{\tau_i(a_k)}^{\tau_i(t)} \frac{\Delta s}{a(s)}\left(\int_{\tau_i(a_k)}^{\sigma(t)} \frac{\Delta s}{a(s)}\right)^{-1},$$

and when $\tau_i(t) = \sigma(t)$, *we set* $\delta_{i,k}(t) = 1$.

Proof. We consider the case where $x(t)$ and $x(\tau_i(t)),\ i = 1, 2, \ldots, n$ are positive on $[T, \infty)_{\mathbb{T}}$ (when $x(t)$ and $x(\tau_i(t))$ are negative on $[T, \infty)_{\mathbb{T}}$, the proof follows the same argument using the interval $[\tau(a_2), b_2)_{\mathbb{T}}$ instead of $[\tau(a_1), b_1)_{\mathbb{T}}$). From (1.1), we find that $a(t)x^{\Delta}(t)$ is nonincreasing on $[\tau(a_1), b_1)_{\mathbb{T}}$. Then, for $t \in [a_1, b_1)_{\mathbb{T}}$

$$\begin{aligned} x(\sigma(t)) - x(\tau_i(t)) &= \int_{\tau_i(t)}^{\sigma(t)} \frac{a(s)x^{\Delta}(s)}{a(s)} \Delta s \\ &\le \left(ax^{\Delta}\right)(\tau_i(t)) \int_{\tau_i(t)}^{\sigma(t)} \frac{\Delta s}{a(s)}, \end{aligned}$$

and so

$$\frac{x(\sigma(t))}{x(\tau_i(t))} \le 1 + \frac{\left(ax^{\Delta}\right)(\tau_i(t))}{x(\tau_i(t))} \int_{\tau_i(t)}^{\sigma(t)} \frac{\Delta s}{a(s)}. \tag{2.1}$$

Also, we see (using $\tau_i(t)$ is nondecreasing) that, for $t \in [a_1, b_1)_{\mathbb{T}}$

$$\begin{aligned} x(\tau_i(t)) &> x(\tau_i(t)) - x(\tau_i(a_1)) = \int_{\tau_i(a_1)}^{\tau_i(t)} \frac{\left(a(s)x^{\Delta}(s)\right)}{a(s)} \Delta s \\ &\ge \left(ax^{\Delta}\right)(\tau_i(t)) \int_{\tau_i(a_1)}^{\tau_i(t)} \frac{\Delta s}{a(s)}, \end{aligned}$$

which implies for $t \in (a_1, b_1)_{\mathbb{T}}$, that

$$\frac{\left(ax^{\Delta}\right)(\tau_i(t))}{x(\tau_i(t))} < \left(\int_{\tau_i(a_1)}^{\tau_i(t)} \frac{\Delta s}{a(s)}\right)^{-1}. \tag{2.2}$$

Therefore, (2.1) and (2.2) imply, for $t \in (a_1, b_1)_{\mathbb{T}}$

$$\frac{x(\sigma(t))}{x(\tau_i(t))} < \int_{\tau_i(a_1)}^{\sigma(t)} \frac{\Delta s}{a(s)} \left(\int_{\tau_i(a_1)}^{\tau_i(t)} \frac{\Delta s}{a(s)}\right)^{-1} = \frac{1}{\delta_{i,1}(t)}.$$

Hence

$$x(\tau_i(t)) > \delta_{i,1}(t)\, x(\sigma(t)), \quad \text{for } i \in I_1 \text{ and } t \in [a_1, b_1)_{\mathbb{T}}. \tag{2.3}$$

Then, by (1.1) and (2.3), we get, for $t \in [a_1, b_1)_{\mathbb{T}}$

$$\left(a(t)x^{\Delta}(t)\right)^{\Delta} + \sum_{i=0}^{n} p_i(t)\delta_{i,1}^{\alpha_i}(t)\, x^{\alpha_i}(\sigma(t)) \le e(t).$$

This completes the proof. □

First, we state and prove an oscillation criterion for equation (1.1).

Theorem 2.4. *Suppose that for any* $T \in [t_0, \infty)_{\mathbb{T}}$ *there exist constants* $a_k,\ b_k \in [T, \infty)_{\mathbb{T}}$ *such that* $a_k < b_k$*,* $k = 1, 2$*, with*

$$p_i(t) \geq 0, \quad \text{for } t \in [\tau(a_k), b_k)_{\mathbb{T}}, \text{ for } i = 0, 1, 2, \ldots, n, \text{ and } k = 1, 2.$$

Assume further

$$e(t) \begin{cases} \leq 0, & t \in [\tau(a_1), b_1)_{\mathbb{T}}, \\ \geq 0, & t \in [\tau(a_2), b_2)_{\mathbb{T}}, \end{cases}$$

where $\tau(t)$ *is defined as in Lemma 2.3. Furthermore, suppose there exists a* C^1_{rd} *function* u *such that for* $k = 1, 2,$ $u(t) \not\equiv 0$ *on* (a_k, b_k) *and* $u(a_k) = u(b_k) = 0$ *such that, for* $k = 1, 2,$

$$\int_{a_k}^{b_k} [P_{1,k}(t)(u^\sigma(t))^2 - a(t)(u^\Delta(t))^2]\Delta t \geq 0, \tag{2.4}$$

where

$$\eta_0 := 1 - \sum_{i=1}^n \eta_i,\ P_{1,k}(t) := p_0(t)\,\delta_{0,k}(t) + \left(\eta_0^{-1}|e(t)|\right)^{\eta_0} \prod_{i=1}^n \left(\eta_i^{-1} p_i(t)\delta_{i,k}^{\alpha_i}(t)\right)^{\eta_i},$$

and where $\alpha_1 > \alpha_2 > \ldots > \alpha_m > 1 > \alpha_{m+1} > \ldots > \alpha_n > 0$ *and* $\eta_i > 0,\ i = 1, 2, \ldots, n$ *satisfy* (a) *and* (b) *of Lemma 2.1 and* $\delta_{i,k}$*,* $i = 0, 1, 2, \ldots, n$ *and* $k = 1, 2$ *are defined as in Lemma 2.3. Then every solution of equation* (1.1) *is oscillatory.*

Proof. Assume (1.1) has a nonoscillatory solution x on $[t_0, \infty)_{\mathbb{T}}$. Then, without loss of generality, there is a solution x of (1.1) and a $T \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t),\ x(\tau_i(t)) > 0,\ i = 0, 1, 2, \ldots, n$ on $[T, \infty)_{\mathbb{T}}$ (when $x(t), x(\tau_i(t)) < 0$ on $[T, \infty)_{\mathbb{T}}$, the proof follows the same argument using the interval $[\tau(a_2), b_2)_{\mathbb{T}}$ instead of $[\tau(a_1), b_1)_{\mathbb{T}}$). From Lemma 2.3, we get, for $t \in [a_1, b_1)_{\mathbb{T}}$

$$\left(a(t)x^\Delta(t)\right)^\Delta + \sum_{i=0}^n p_i(t)\delta_{i,1}^{\alpha_i}(t)\, x^{\alpha_i \sigma}(t) \leq e(t). \tag{2.5}$$

Define

$$z(t) := -\frac{a(t)x^\Delta(t)}{x(t)}.$$

Then, from (2.5), we have, for $t \in [a_1, b_1)_{\mathbb{T}}$

$$z^\Delta(t) \geq \sum_{i=0}^n p_i(t)\delta_{i,1}^{\alpha_i}(t)\,(x^\sigma(t))^{\alpha_i - 1} + \frac{|e(t)|}{x^\sigma(t)} + \frac{x(t)z^2(t)}{x^\sigma(t)a(t)}.$$

But, since $x(t) > 0$, we have

$$0 < \frac{x(t)}{a(t)x^\sigma(t)} = \frac{1}{a(t) + \mu(t)\frac{a(t)x^\Delta(t)}{x(t)}} = \frac{1}{a(t) - \mu(t)z(t)},$$

which implies, for $t \in [a_1, b_1)_{\mathbb{T}}$

$$z^{\Delta}(t) \geq \sum_{i=0}^{n} p_i(t)\delta_{i,1}^{\alpha_i}(t)\left(x^{\sigma}(t)\right)^{\alpha_i-1} + \frac{|e(t)|}{x^{\sigma}(t)} + \frac{z^2(t)}{a(t)-\mu(t)z(t)}. \tag{2.6}$$

Corresponding to the exponents $\alpha_i,\ 1 \leq i \leq n$, in equation (1.1), let $\eta_i,\ 1 \leq i \leq n$, be chosen to satisfy (a) and (b) in Lemma 2.1 and $\eta_0 := 1 - \sum_{i=1}^{n} \eta_i$. Using the arithmetic-geometric mean inequality, see [4, Page 17],

$$\sum_{i=0}^{n} \eta_i u_i \geq \prod_{i=0}^{n} u_i^{\eta_i}, \quad \text{where } u_i \geq 0,$$

we get

$$\begin{aligned}
&|e(t)|\left(x^{\sigma}(t)\right)^{-1} + \sum_{i=1}^{n} p_i(t)\delta_{i,1}^{\alpha_i}(t)\left(x^{\sigma}(t)\right)^{\alpha_i-1} \\
&= \eta_0\left(\eta_0^{-1}|e(t)|\left(x^{\sigma}(t)\right)^{-1}\right) + \sum_{i=1}^{n} \eta_i\left(\eta_i^{-1} p_i(t)\delta_{i,1}^{\alpha_i}(t)\left(x^{\sigma}(t)\right)^{\alpha_i-1}\right) \\
&\geq \left(\eta_0^{-1}|e(t)|\right)^{\eta_0}\left(x^{\sigma}(t)\right)^{-\eta_0} \prod_{i=1}^{n} \eta_i^{-\eta_i}\left(p_i(t)\delta_{i,1}^{\alpha_i}(t)\right)^{\eta_i} x^{\eta_i(\alpha_i-1)}(t) \\
&= \left(\eta_0^{-1}|e(t)|\right)^{\eta_0} \prod_{i=1}^{n} \eta_i^{-\eta_i}\left(p_i(t)\delta_{i,1}^{\alpha_i}(t)\right)^{\eta_i},
\end{aligned}$$

which implies

$$z^{\Delta}(t) \geq P_{1,1}(t) + \frac{z^2(t)}{a(t)-\mu(t)z(t)}. \tag{2.7}$$

Multiplying (2.7) by $(u^{\sigma}(t))^2$ and integrating from a_1 to b_1, we have

$$\int_{a_1}^{b_1} (u^{\sigma}(t))^2 z^{\Delta}(t)\Delta t \geq \int_{a_1}^{b_1} P_{1,1}(t)(u^{\sigma}(t))^2\Delta t + \int_{a_1}^{b_1} \frac{z^2(t)(u^{\sigma}(t))^2}{a(t)-\mu(t)z(t)}\Delta t.$$

Using integration by parts on the first integral, we get

$$\begin{aligned}
&u^2(t)z(t)]_{a_1}^{b_1} - \int_{a_1}^{b_1} [u(t)+u^{\sigma}(t)]u^{\Delta}(t)z(t)\Delta t \\
&\qquad \geq \int_{a_1}^{b_1} P_{1,1}(t)(u^{\sigma}(t))^2\Delta t + \int_{a_1}^{b_1} \frac{z^2(t)(u^{\sigma}(t))^2}{a(t)-\mu(t)z(t)}\Delta t.
\end{aligned}$$

Rearranging and using $u(a_1) = 0 = u(b_1)$, we get

$$\begin{aligned} 0 &\geq \int_{a_1}^{b_1} \frac{z^2(t)(u^\sigma(t))^2}{a(t)-\mu(t)z(t)}\Delta t + \int_{a_1}^{b_1} [u(t)+u^\sigma(t)]u^\Delta(t)z(t)\Delta t \\ &\quad + \int_{a_1}^{b_1} P_{1,1}(t)(u^\sigma(t))^2\Delta t \\ &= \int_{a_1}^{b_1} \frac{z^2(t)(u^\sigma(t))^2}{a(t)-\mu(t)z(t)}\Delta t + \int_{a_1}^{b_1} [2u^\sigma(t)-\mu(t)u^\Delta(t)]u^\Delta(t)z(t)\Delta t \\ &\quad + \int_{a_1}^{b_1} P_{1,1}(t)(u^\sigma(t))^2\Delta t. \end{aligned}$$

Adding and subtracting the term $\int_{a_1}^{b_1} a(t)(u^\Delta(t))^2\Delta t$ and then using (2.4), we have

$$\begin{aligned} 0 &\geq \int_{a_1}^{b_1} \left(\frac{z^2(t)(u^\sigma(t))^2}{a(t)-\mu(t)z(t)} + 2u^\sigma(t)u^\Delta(t)z(t) + [a(t)-\mu(t)z(t)](u^\Delta(t))^2\right)\Delta t \\ &\quad + \int_{a_1}^{b_1} [P_{1,1}(t)(u^\sigma(t))^2 - a(t)(u^\Delta(t))^2]\Delta t \\ &\geq \int_{a_1}^{b_1} \left(\frac{u^\sigma(t)z(t)}{\sqrt{a(t)-\mu(t)z(t)}} + \sqrt{a(t)-\mu(t)z(t)}\,u^\Delta(t)\right)^2 \Delta t. \end{aligned}$$

It follows that

$$\int_{a_1}^{b_1} \left(\frac{u^\sigma(t)z(t)}{\sqrt{a(t)-\mu(t)z(t)}} + \sqrt{a(t)-\mu(t)z(t)}\,u^\Delta(t)\right)^2 \Delta t = 0.$$

This implies that

$$\frac{u^\sigma(t)z(t)}{\sqrt{a(t)-\mu(t)z(t)}} + \sqrt{a(t)-\mu(t)z(t)}\,u^\Delta(t) = 0, \qquad \text{for } t \in [a_1, b_1)_{\mathbb{T}}.$$

Solving for u^Δ, we get that u solves the IVP

$$u^\Delta = \frac{-z}{a-\mu z}u^\sigma, \quad u(a_1) = 0, \qquad \text{for } t \in [a_1, b_1)_{\mathbb{T}}.$$

Since $\frac{z}{a-\mu z} \in \mathcal{R}$, we get from [5, Theorem 2.71] that $u(t) \equiv 0$ on $[a_1, b_1)_{\mathbb{T}}$. This completes the proof. □

Theorem 2.5. *Suppose that for any $T \in [t_0, \infty)_{\mathbb{T}}$ there exist constants $a_k,\ b_k \in [T, \infty)_{\mathbb{T}}$ such that $a_k < b_k,\ k = 1, 2$, with*

$$p_i(t) \geq 0, \quad \textit{for } t \in [\tau(a_k), b_k)_{\mathbb{T}}, \textit{ for } i = 0, 1, 2, \ldots, n \textit{ and } k = 1, 2,$$

and

$$e(t)\begin{cases} \le 0, & t\in[\tau(a_1),b_1)_{\mathbb{T}},\\ \ge 0, & t\in[\tau(a_2),b_2)_{\mathbb{T}},\end{cases}$$

where $\tau(t)$ is defined as in Lemma 2.3. Furthermore, suppose there exists a C^1_{rd} function u such that for $k=1,2,$ $u(t)\not\equiv 0$ on (a_k,b_k) and $u(a_k)=u(b_k)=0$ such that, for $k=1,2,$

$$\int_{a_k}^{b_k}[P_{2,k}(t)(u^{\sigma}(t))^2-a(t)(u^{\Delta}(t))^2]\Delta t\ge 0,$$

where

$$P_{2,k}(t):=p_0(t)\,\delta_{0,k}(t)+\prod_{i=1}^{n}\left(\eta_i^{-1}p_i(t)\delta_{i,k}^{\alpha_i}(t)\right)^{\eta_i},$$

and where $\alpha_1>\alpha_2>\ldots>\alpha_m>1>\alpha_{m+1}>\ldots>\alpha_n>0$ and $\eta_i>0,\ i=1,2,\ldots,n$ satisfy (a) *and* (c) *of Lemma 2.1 and $\delta_{i,k}$, $i=0,1,2,\ldots,n$ and $k=1,2$ are defined as in Lemma 2.3. Then every solution of equation* (1.1) *is oscillatory.*

Proof. The proof is the same as the proof of Theorem 2.4 (just put $\eta_0=0$ and then apply conditions (a) and (c) of Lemma 2.1). □

Example 2.6. Assume there is a strictly increasing sequence $\{s_m\}\subset\mathbb{T}$ with $s_m<\sigma(s_m)<\sigma^2(s_m)<\sigma^3(s_m)\le s_{m+1}$, $m\in\mathbb{N}$ and $\lim_{m\to\infty}s_m=\infty$. Assume the forcing function e in (1.1) is rd-continuous on $\mathbb{T}$ and satisfies

$$e(t)=\begin{cases}0, & t=s_m,\ \ \sigma^2(s_m),\\ -1, & t=\sigma(s_{2m})\\ 1 & t=\sigma(s_{2m+1}).\end{cases}$$

Assume the function $\tau_i:\mathbb{T}\to\mathbb{T}$ satisfies $\tau_0(s_m)=\sigma(s_m)$ and $s_m\le\tau_i(s_m)\le\sigma(s_m)$, $i=1,2,\ldots,n$, $m\in\mathbb{N}$. Note that $e(t)\le 0$ on $\left[\tau(s_{2m}),\sigma^2(s_{2m})\right)_{\mathbb{T}}$ and $e(t)\ge 0$ on $\left[\tau(s_{2m+1}),\sigma^2(s_{2m+1})\right)_{\mathbb{T}}$. Further assume $p_i(t)\ge 0$, $i=0,1,2,\ldots,n$ on $\left[\tau(s_m),\sigma^2(s_m)\right)_{\mathbb{T}}$, $m\in\mathbb{N}$. Let u be a delta differentiable function on $\mathbb{T}$ such that $u(t)=e(t)$ on $\cup_{m=1}^{\infty}[s_m,\sigma^2(s_m)]$. Then

$$\begin{aligned}&\int_{s_{2m}}^{\sigma^2(s_{2m})}\left\{P_{1,1}(t)(u^{\sigma}(t))^2-a(t)(u^{\Delta}(t))^2\right\}\Delta t\\ &= \left\{P_{1,1}(s_{2m})(u^{\sigma}(s_{2m}))^2-a(s_{2m})(u^{\Delta}(s_{2m}))^2\right\}\mu(s_{2m})\\ &\quad+\left\{P_{1,1}(\sigma(s_{2m}))(u(\sigma^2(s_{2m})))^2-a(\sigma(s_{2m}))(u^{\Delta}(\sigma(s_{2m})))^2\right\}\mu(\sigma(s_{2m}))\\ &= p_0(s_{2m})\mu(s_{2m})-\frac{a(s_{2m})}{\mu(s_{2m})}-\frac{a(\sigma(s_{2m}))}{\mu(\sigma(s_{2m}))}\ge 0,\end{aligned}$$

provided

$$p_0(s_{2m})\mu(s_{2m})\ge\frac{a(s_{2m})}{\mu(s_{2m})}+\frac{a(\sigma(s_{2m}))}{\mu(\sigma(s_{2m}))}.$$

Similarly,

$$\int_{s_{2m+1}}^{\sigma^2(s_{2m+1})} \left\{P_{1,2}(t)(u^{\sigma}(t))^2 - a(t)(u^{\Delta}(t))^2\right\} \Delta t \geq 0,$$

provided

$$p_0(s_{2m+1})\mu(s_{2m+1}) \geq \frac{a(s_{2m+1})}{\mu(s_{2m+1})} + \frac{a(\sigma(s_{2m+1}))}{\mu(\sigma(s_{2m+1}))}.$$

In conclusion we have by Theorem 2.4 that (1.1) is oscillatory if

$$p_0(s_i)\mu(s_i) \geq \frac{a(s_i)}{\mu(s_i)} + \frac{a(\sigma(s_i))}{\mu(\sigma(s_i))},$$

for $i = m, m+1, m \in \mathbb{N}$.

Remark 2.7. Note that the results of this paper can be extended to the more general dynamic equation

$$\left(a(t)x^{\Delta}(t)\right)^{\Delta} + \sum_{i=0}^{n} p_i(t) \left|x(g_i(t))\right|^{\alpha_i} \text{ sgn } x(g_i(t)) = e(t),$$

where the functions $g_i : \mathbb{T} \to \mathbb{T}$ are nondecreasing rd-continuous functions on $\mathbb{R}$ and $\lim_{t\to\infty} g_i(t) = \infty$, for $i = 0, 1, \ldots, n$ (see [25]). We leave this to interested reader.

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