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Fractional Evolution Integro-Differential Systems with Nonlocal Conditions

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Abstract

In this paper, we use the theory of resolvent operators, the fractional powers of operators, fixed point technique and the Gelfand–Shilov principle to establish the existence and uniqueness of local mild and then local classical solutions of a class of nonlinear fractional evolution integro-differential systems with nonlocal conditions in Banach space. As an application that illustrates the abstract results, a nonlinear nonlocal integro-partial differential system of fractional order is given.

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1 Introduction

The aim of this paper is to study the nonlocal fractional Cauchy problem of the form

$$\frac{d^\alpha u(t)}{dt^\alpha} + A(t)u(t) = f(t, u(t)) + \int_{t_0}^{t} B(t-s)g(s, u(s))ds, \tag{1.1}$$

$$u(t_0) + h(u) = u_0, \tag{1.2}$$

in a Banach space X, where $0 < \alpha \leq 1$, $0 \leq t_0 < t$. Let J denote the closure of the interval $[t_0, T)$, $t_0 < T \leq \infty$. We assume that $-A(t)$ is a closed linear operator defined on a dense domain $D(A)$ in X into X such that $D(A)$ is independent of t. It is assumed also that $-A(t)$ generates an evolution operator in the Banach space X, the function B

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is real valued and locally integrable on $[t_0, \infty)$, the nonlinear maps f and g are defined on $[t_0, \infty) \times X$ into X and $h : C(J, X) \to \overline{D(A)}$ is a given function.

Recently, fractional differential equations have attracted many authors [8, 11, 13–17, 19, 21, 24]. This is mostly because it efficiently describes many phenomena arising in engineering, physics, economy and science. In fact it can find several applications in viscoelasticity, electrochemistry, electromagnetic, and so on. The existence of solutions to evolution equations with nonlocal conditions in Banach space was studied first by Byszewski [9]. Subsequently many authors extended the work to various kind of nonlinear evolution equations [16, 17, 21, 24]. Deng [12] indicated that using the nonlocal condition $u(t_0) + h(u) = u_0$ to describe, for instance, the diffusion phenomenon of a small amount of gas in a transparent tube, can give better results than using the usual local Cauchy problem $u(t_0) = u_0$. Also for several works (first order differential equations with initial conditions) concerned with this kind of research, we refer to [3–7, 10, 20, 28].

The results obtained in this paper are generalizations of the results given by Bahuguna [2], El-Borai [16, 17], Pazy [25] and Yan [27].

Our work is organized as follows. Section 2 is devoted to the review of some essential results in fractional calculus and also to the resolvent operators and the fractional powers of operators which will be used in this work to obtain our main results. In Section 3, we establish the existence of a unique local mild solution of (1.1), (1.2). In Section 4, we study the regularity of the mild solution of the considered problem and show under the additional condition of Hölder continuity on B that this mild solution is in fact the classical solution. In Section 5, as an example, a nonlinear nonlocal evolution integro-partial differential system of fractional order is also provided in order to illustrate the abstract results.

2 Preliminaries

Following Gelfand and Shilov [19], we define the fractional integral of order $\alpha > 0$ as

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds$$

and the fractional derivative of the function f of order $0 < \alpha < 1$ as

$${}_aD_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t f(s)(t-s)^{-\alpha} ds,$$

where f is an abstract continuous function on the interval $[a, b]$ and $\Gamma(\alpha)$ is the Gamma function, see [11].

Definition 2.1. By a classical solution of (1.1), (1.2) on J, we mean a function u with values in X such that:

1) u is continuous function on $[t_0, T)$ and $u(t) \in D(A)$,

2) $\dfrac{d^\alpha u}{dt^\alpha}$ exists and is continuous on (t_0, T), $0 < \alpha < 1$, and u satisfies (1.1) on (t_0, T) and the nonlocal condition (1.2).

By a local classical solution of (1.1), (1.2) on J, we mean that there exist a T_0, $t_0 < T_0 < T$, and a function u defined from $J_0 = [t_0, T_0]$ into X such that u is a classical solution of (1.1), (1.2).

Let E be the Banach space formed from $D(A)$ with the graph norm. Since $-A(t)$ is a closed operator, it follows that $-A(t)$ is in the set of bounded operators from E to X.

Definition 2.2 (See [8, 27]). A resolvent operators for problem (1.1), (1.2) is a bounded operator-valued function $R(t, s) \in B(X)$, $0 \leq s \leq t < T$, the space of bounded linear operators on X, having the following properties:

(i) $R(t, s)$ is strongly continuous in s and t, $R(s, s) = I$, $0 \leq s < T$, $\|R(t, s)\| \leq Me^{\beta(t-s)}$ for some constants M and β.

(ii) $R(t, s)E \subset E$, $R(t, s)$ is strongly continuous in s and t on E.

(iii) For $x \in X$, $R(t, s)x$ is continuously differentiable in $s \in [0, T)$ and

$$\frac{\partial R}{\partial s}(t, s)x = R(t, s)A(s)x.$$

(iv) For $x \in X$ and $s \in [0, T)$, $R(t, s)x$ is continuously differentiable in $t \in [s, T)$ and

$$\frac{\partial R}{\partial t}(t, s)x = -A(t)R(t, s)x,$$

with $\dfrac{\partial R}{\partial s}(t, s)x$ and $\dfrac{\partial R}{\partial t}(t, s)x$ are strongly continuous on $0 \leq s \leq t < T$.

Here $R(t, s)$ can be extracted from the evolution operator of the generator $-A(t)$. The resolvent operator is similar to the evolution operator for nonautonomous differential equations in a Banach space.

Definition 2.3 (See [8, 21, 24]). A continuous function $u : J \to X$ is said to be a mild solution of problem (1.1), (1.2) if for all $u_0 \in X$, it satisfies the integral equation

$$\begin{aligned} u(t) &= R(t, t_0)[u_0 - h(u)] \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} R(t, s) \left[f(s, u(s)) + \int_{t_0}^{s} B(s-\tau) g(\tau, u(\tau)) d\tau \right] ds. \end{aligned} \tag{2.1}$$

By a local mild solution of (1.1), (1.2) on J, we mean that there exist a T_0, $t_0 < T_0 < T$, and a function u defined from $J_0 = [t_0, T_0]$ into X such that u is a mild solution of (1.1), (1.2).

We define the fractional power $A^{-q}(t)$ by

$$A^{-q}(t) = \frac{1}{\Gamma(q)} \int_0^\infty x^{q-1} R(x,t)dx, \quad q > 0.$$

For $0 < q \leq 1$, $A^q(t)$ (we denote by A^q for short) is a closed linear operator whose domain $D(A^q) \supset D(A)$ is dense in X. This implies that $D(A^q)$ endowed with the graph norm

$$\|u\|_{D(A)} = \|u\| + \|A^q u\|, u \in D(A^q)$$

is a Banach space as clearly $A^q = (A^{-q})^{-1}$ because A^{-q} is one to one. Since $0 \in \rho(A)$, A^q is invertible, and its graph norm is equivalent to the norm $\|u\|_q = \|A^q u\|$. Thus $D(A^q)$ equipped with the norm $\|\cdot\|_q$ is a Banach space denoted by X_q. For more details we refer to [1,2]. To state and prove the main results of this paper, we shall require the following assumption on the map f and g:

(F) Let U be an open subset of $[0,\infty) \times X_q$, and for every $(t,x) \in U$ there exist a neighborhood $V \subset U$ of (t,x) and constants $L > 0$, $0 < \mu < 1$ such that

$$\|f(s_1,u) - f(s_2,v)\| \leq L[|s_1 - s_2|^\mu + \|u - v\|_q]$$

for all (s_1,u) and (s_2,v) in V.

3 Local Mild Solutions

To establish local existence of the considered problem, we assume that $-A$ is invertible and $t_0 < T < \infty$, see [18,22,23,26]. According to [25, Section 2.6], we can deduce the following.

Lemma 3.1. *Let $A(t)$ be the infinitesimal generator of a resolvent operator $R(t,s)$. We denote by $\rho[A(t)]$ the resolvent set of $A(t)$. If $0 \in \rho[A(t)]$, then*

(a) $R(t,s) : X \longrightarrow D(A^q)$ *for every* $0 \leq s \leq t < T$ *and* $q \geq 0$,

(b) *For every* $u \in D(A^q)$, *we have* $R(t,s)A^q(t)u = A^q(t)R(t,s)u$,

(c) *The operator* $A^q R(t,s)$ *is bounded and* $\|A^q R(t,s)\| \leq M_{q,\beta}(t-s)^{-q}$.

Let $Y = C([t_0,t_1]; X_q)$ be endowed with the supremum norm

$$\|y\|_\infty = \sup_{t_0 \leq t \leq t_1} \|y(t)\|_q, \quad y \in Y.$$

Then Y is a Banach space. The function $h : Y \to X_q$ is continuous and there exists a number b such that $\|R(t,t_0)\| < 1/2b$ and

$$\|h(x) - h(y)\|_q \leq b\|x - y\|_\infty$$

for all $x, y \in Y$. Note that, if $z \in Y$, then $A^{-q}z \in Y$.

Theorem 3.2. *Suppose that the operator $-A(t)$ generates the resolvent operator $R(t,s)$ with $\|R(t,s)\| \leq Me^{\beta(t-s)}$ and that $0 \in \rho[-A(t)]$. If the maps f and g satisfy* (F) *and the real-valued map B is integrable on J, then* (1.1), (1.2) *has a unique local mild solution for every $u_0 \in X_q$.*

Proof. We fix a point (t_0, u_0) in the open subset U of $[0,\infty) \times X_q$ and choose $t_1' > t_0$ and $\epsilon > 0$ such that (F) holds for the functions f and g on the set

$$V = \{(t,x) \in U : t_0 \leq t \leq t_1', \|x - u_0\|_q \leq \epsilon\}. \tag{3.1}$$

Let

$$N_1 = \sup_{t_0 \leq t \leq t_1'} \|f(t,u_0)\|, \quad N_2 = \sup_{t_0 \leq t \leq t_1'} \|g(t,u_0)\|. \tag{3.2}$$

Set $\lambda = \sup\limits_{x \in Y} \|h(x)\|_q$ and choose $t_1 > t_0$ such that

$$\|R(t,t_0) - I\|[\|u_0\|_q + \lambda] \leq \frac{\epsilon}{2},\ t_0 \leq t \leq t_1 \tag{3.3}$$

and

$$t_1 - t_0 < \min\left\{t_1' - t_0, \left[\frac{\epsilon}{2} M_{q,\beta}^{-1} \Gamma(\alpha)(\alpha - q)\{(L\epsilon + N_1) + a_T(L\epsilon + N_2)\}^{-1}\right]^{\frac{1}{\alpha - q}}\right\}, \tag{3.4}$$

where

$$a_T = \int_0^T |B(s)| ds. \tag{3.5}$$

We define a map on Y by $\Phi y = \tilde{y}$, where $\tilde{y}$ is given by

$$\begin{aligned}
&\tilde{y}(t) = R(t,t_0)A^q[u_0 - h(A^{-q}y)] \\
&+\frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} A^q R(t,s) \left[f(s, A^{-q}y(s)) + \int_{t_0}^s B(s-\tau) g(\tau, A^{-q}y(\tau)) d\tau\right] ds.
\end{aligned}$$

For every $y \in Y,\ \Phi y(t_0) = A^q[u_0 - h(A^{-q}y)]$, and for $t_0 \leq s \leq t \leq t_1$ we have

$$\begin{aligned}
\Phi y(t) - \Phi y(s) &= [R(t,t_0) - R(s,t_0)]A^q[u_0 - h(A^{-q}y)] \\
&+ \frac{1}{\Gamma(\alpha)} \int_s^t (t-\tau)^{\alpha-1} A^q R(t,\tau) \\
&\quad \times \left[f(\tau, A^{-q}y(\tau)) + \int_{t_0}^{\tau} B(\tau - \eta) g(\eta, A^{-q}y(\eta)) d\eta\right] d\tau \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^s (t-\tau)^{\alpha-1} A^q [R(t,\tau) - R(s,\tau)] \\
&\quad \times \left[f(\tau, A^{-q}y(\tau)) + \int_{t_0}^{\tau} B(\tau - \eta) g(\eta, A^{-q}y(\eta)) d\eta\right] d\tau.
\end{aligned}$$

It follows from (F) on the functions f and g, Lemma 3.1.c and (3.5) that $\Phi : Y \to Y$. Let S be the nonempty closed and bounded set given by

$$S = \{y \in Y : y(t_0) = A^q[u_0 - h(A^{-q}y)], \|y(t) - A^q[u_0 - h(A^{-q}y)]\| \leq \epsilon\}. \quad (3.6)$$

Then for $y \in S$, we have

$$\begin{aligned}
&\|\Phi y(t) - A^q[u_0 - h(A^{-q}y)]\| \leq \|R(t,t_0) - I\| \, \|A^q[u_0 - h(A^{-q}y)]\| \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \|A^q R(t,s)\| \, \|f(s, A^{-q}y(s)) - f(s,u_0)\| ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \|A^q R(t,s)\| \int_{t_0}^{t} |B(s-t)| \, \|g(\tau, A^{-q}y(\tau)) - g(\tau,u_0)\| d\tau ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \|A^q R(t,s)\| \, \|f(s,u_0)\| ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \|A^q R(t,s)\| \int_{t_0}^{t} |B(s-t)| \, \|g(\tau,u_0)\| d\tau ds.
\end{aligned}$$

Using Lemma 3.1.c, (3.3) and (3.4) we get

$$\begin{aligned}
&\|\Phi y(t) - A^q[u_0 - h(A^{-q}y)]\| \\
&\quad \leq \frac{\epsilon}{2} + \frac{M_{q,\beta}}{\Gamma(\alpha)} (\alpha - q)^{-1} \{(L\epsilon + N_1) + a_T(L\epsilon + N_2)\}(t_1 - t_0)^{\alpha - q} \leq \epsilon. \quad (3.7)
\end{aligned}$$

Thus $\Phi : S \to S$. Now we shall show that Φ is a strict contraction on S which will ensure the existence of a unique continuous function satisfying (1.1) and (1.2). If y and z two elements in S, then

$$\begin{aligned}
&\|\Phi y(t) - \Phi z(t)\| = \|\tilde{y}(t) - \tilde{z}(t)\| \\
&\leq \|R(t,t_0)\| \, \|h(A^{-q}z) - h(A^{-q}y)\|_q \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \|A^q R(t,s)\| \, \|f(s, A^{-q}y(s)) - f(s, A^{-q}z(s))\| ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \|A^q R(t,s)\| \\
&\times \left[\int_{t_0}^{s} |B(s-\tau)| \, \|g(\tau, A^{-q}y(\tau)) - g(\tau, A^{-q}z(\tau))\| d\tau \right] ds.
\end{aligned}$$

Using assumption (F) on f and g, (3.5), Lemma 3.1.c and (3.4) respectively, we get

$$\begin{aligned}
&\|\Phi y(t) - \Phi z(t)\| \\
&\leq b\|R(t,t_0)\| \, \|y - z\|_\infty + \frac{L}{\Gamma(\alpha)} [(1 + a_T) \int_{t_0}^{t} (t-s)^{\alpha-1} \|A^q R(t,s)\| ds] \|y - z\|_Y
\end{aligned}$$

$$
\begin{aligned}
&\le \; b\|R(t,t_0)\| \, \|y-z\|_Y + \frac{L}{\Gamma(\alpha)}(1+a_T)M_{q,\beta}\,(\alpha-q)^{-1}(t_1-t_0)^{\alpha-q}\|y-z\|_Y \\
&< \; \frac{1}{2}\|y-z\|_Y \\
&\quad + \frac{1}{\epsilon\Gamma(\alpha)}[(L\epsilon+N_1)+a_T(L\epsilon+N_2)]M_{q,\beta}(\alpha-q)^{-1}(t_1-t_0)^{\alpha-q}\|y-z\|_Y \\
&< \; \frac{1}{2}\|y-z\|_Y + \frac{1}{2}\|y-z\|_Y.
\end{aligned}
$$

Thus Φ is a strict contraction map from S into S and therefore by the Banach contraction principle there exists a unique fixed point y in S such that

$$\Phi y = y = \tilde{y}. \tag{3.8}$$

Let $u = A^{-q}y$. Using Lemma 3.1.b, we have

$$
\begin{aligned}
u(t) &= R(t,t_0)[u_0 - h(u)] \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}R(t,s)[f(s,u(s)) + \int_{t_0}^{s} B(s-\tau)g(\tau,u(\tau))d\tau]ds
\end{aligned} \tag{3.9}
$$

for every $t \in [t_0, t_1]$. Hence u is a unique local mild solution of (1.1), (1.2). □

4 Local Classical Solutions

In this section we establish the regularity of the mild solutions of (1.1), (1.2). Let J denote the closure of the interval $[t_0, T)$, $t_0 < T \le \infty$. In addition to the hypotheses mentioned in the earlier sections, we assume on the kernel B, that

(H) There exist constants $L_0 \ge 0$ and $0 < p \le 1$ such that

$$|B(t_1) - B(t_2)| \le L_0|t_1 - t_2|^p, \text{ for all } t_1, t_2 \in J.$$

Theorem 4.1. *Suppose that $-A(t)$ generates the resolvent operator $R(t,s)$ such that $\|R(t,s)\| \le Me^{\beta(t-s)}$ and $0 \in \rho[-A(t)]$. Further, suppose that the maps f and g satisfy* (*F*) *and the kernel B satisfies* (*H*). *Then* (1.1), (1.2) *has a unique local classical solution for each $u_0 \in X_q$.*

Proof. From Theorem 3.2, it follows that there exist T_0, $t_0 < T_0 < T$ and a function u such that u is a unique mild solution of (1.1), (1.2) on $J_0 = [t_0, T_0)$ given by (3.9). Let $v(t) = A^q u(t)$. Then

$$
\begin{aligned}
v(t) &= R(t,t_0)A^q[u_0 - h(u)] \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}A^qR(t,s)[\tilde{f}(s) + \int_{t_0}^{s} B(s-\tau)\tilde{g}(\tau)d\tau]ds,
\end{aligned} \tag{4.1}
$$

where $\tilde{f}(t) = f(t, A^{-q}v(t))$, $\tilde{g}(t) = g(t, A^{-q}v(t))$. Since $u(t)$ is continuous on J_0 and the maps f and g satisfy (F), it follows that $\tilde{f}$ and $\tilde{g}$ are continuous, and therefore bounded on J_0. Let

$$M_1 = \sup_{t\in J_0} \|\tilde{f}(t)\| \text{ and } M_2 = \sup_{t\in J_0} \|\tilde{g}(t)\|. \tag{4.2}$$

By using the same method as in [13, Theorem 3.2], we can prove that $v(t)$ is locally Hölder continuous on J_0. Then there exist a constant C such that for every $t_0' > t_0$, we have

$$\|v(t_1) - v(t_2)\| \leq C|t_1 - t_2|^p, \tag{4.3}$$

for all $t_0 < t_0' < t_1, t_2 < T_0$. Now, assumption (F) with (4.3) implies that there exist constants $k_1, k_2 \geq 0$ and $0 < \gamma, \eta < 1$ such that for all $t_0 < t_0' < t_1, t_2 < T_0$, we have

$$\|\tilde{f}(t_1) - \tilde{f}(t_2)\| \leq k_1|t_1 - t_2|^\gamma,$$

$$\|\tilde{g}(t_1) - \tilde{g}(t_2)\| \leq k_2|t_1 - t_2|^\eta,$$

which shows that $\tilde{f}$ and $\tilde{g}$ are locally Hölder continuous on J_0. Let

$$\omega(t) = \tilde{f}(t) + \int_{t_0}^{t} B(t-\tau)\tilde{g}(\tau)d\tau.$$

It is clear that $\omega(t)$ is locally Hölder continuous on J_0. For $t_2 \leq t_1$, we have

$$\|\omega(t_1) - \omega(t_2)\| \leq C^*|t_1 - t_2|^\beta,$$

for some constants $C^* \geq 0$ and $0 < \beta < 1$. Consider the Cauchy problem

$$\frac{d^\alpha v(t)}{dt^\alpha} + A(t)v(t) = \omega(t), t > t_0, \tag{4.4}$$

$$v(t_0) = u_0 - h(u). \tag{4.5}$$

From Pazy [25], the problem (4.4), (4.5) has a unique solution v on J_0 into$X$ given by

$$v(t) = R(t, t_0)[u_0 - h(u)] + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} R(t,s)\omega(s)ds, \tag{4.6}$$

for $t > t_0$. Each term on the right-hand side belongs to $D(A)$, hence belongs to $D(A^q)$. Applying A^q on both sides of (4.6) and using the uniqueness of $v(t)$, we have that $A^q v(t) = u(t)$. It follows that u is the classical solution of (1.1), (1.2) on J_0. Thus u is the unique local classical solution of (1.1), (1.2) on J. $\square$

5 Application

Consider the nonlinear integro-partial differential equation of fractional order

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \sum_{|q|=2m} a_q(x,t) D_x^q u(x,t) = F(x,t,u) + \int_{t_0}^{t} B(t-s) G(x,s,u) ds, \quad (5.1)$$

with the nonlocal condition

$$u(x,t_0) + \sum_{i=1}^{N} c_i u(x,t_i) = g(x), \quad (5.2)$$

where $t_0 \le t_1 < t_2 < \cdots < t_N < T$, $t \in [t_0, T) \subset \mathbb{R}^+$, $x \in \mathbb{R}^n$, where $\mathbb{R}^n$ is the n-dimensional Euclidean space, $D_x^q = D_{x_1}^{q_1} \cdots D_{x_n}^{q_n}$, $D_{x_i} = \dfrac{\partial}{\partial x_i}$, $q = (q_1, \cdots, q_n)$ is an n-dimensional multi-index, $|q| = q_1 + \cdots + q_n$. The operators F, G are defined as

$$F(x,t,u) = f(x,t,u(x,t), Du(x,t), \cdots, D^{2m-1}u(x,t)), \quad (5.3)$$

$$G(x,t,u) = g(x,t,u(x,t), Du(x,t), \cdots, D^{2m-1}u(x,t)). \quad (5.4)$$

Let $L_2(\mathbb{R}^n)$ be the set of all square integrable functions on $\mathbb{R}^n$. We denote by $C^m(\mathbb{R}^n)$ the set of all continuous real-valued functions defined on $\mathbb{R}^n$ which have continuous partial derivatives of order less than or equal to m. By $C_0^m(\mathbb{R}^n)$ we denote the set of all functions $f \in C^m(\mathbb{R}^n)$ with compact supports. Let $H_0^m(\mathbb{R}^n)$ be the completion of $C_0^m(\mathbb{R}^n)$ with respect to the norm

$$\|f\|_m^2 = \sum_{|q| \le m} \int_{R^n} |D_x^q f(x)|^2 dx.$$

It is supposed that

(i) The operator $A = -\sum_{|q|=2m} a_q(x,t) D_x^q$ is uniformly elliptic on $\mathbb{R}^n$. In other words, all the coefficients a_q, $|q| = 2m$, are continuous and bounded on $\mathbb{R}^n$ and there is a positive number c such that

$$(-1)^{m+1} \sum_{|q|=2m} a_q(x,t) \xi^q \ge c|\xi|^{2m},$$

for all $x \in \mathbb{R}^n$ and all $\xi \ne 0$, $\xi \in \mathbb{R}^n$, where $\xi^q = \xi_1^{q_1} \cdots \xi_n^{q_n}$ and $|\xi|^2 = \xi_1^2 + \cdots + \xi_n^2$.

(ii) All the coefficients a_q, $|q| = 2m$, satisfy a uniform Hölder condition on $\mathbb{R}^n$.

Under these conditions, the operator A defined by (i) with domain of definition $D(A) = H^{2m}(\mathbb{R}^n)$ generates a resolvent operator $R(t,s)$ defined on $L_2(\mathbb{R}^n)$, and it is well known that $H^{2m}(\mathbb{R}^n)$ is dense in $X = L_2(\mathbb{R}^n)$. If the nonlocal function $g(x)$ is an element in Hilbert space $H^{2m}(\mathbb{R}^n)$, then we can write

$$R(t,s)g(x) = \int_{\mathbb{R}^n} \Upsilon(x,y,t-s)g(y)dy,$$

where Υ is the fundamental solution of the nonlocal Cauchy problem

$$\frac{\partial u(x,t)}{\partial t} + \sum_{|q|=2m} a_q(x,t)D_x^q u(x,t) = 0,$$

$$u(x,t_0) + \sum_{i=1}^{N} c_i u(x,t_i) = g(x).$$

It can be proved that

$$\|D_x^q R(t,s)g\| \leq \frac{M}{t^\beta}\|g\|,$$

where $0 < \beta < 1$, M is a positive constant, $|q| \leq 2m-1$, $t > 0$ and $\|g\|^2 = \int_{\mathbb{R}^n} g^2(x)dx$. This achieves the proof of the existence of mild solutions of the problem (5.1), (5.2). The operators F and G defined in (5.3), (5.4) satisfy

(iii) There are numbers $L \geq 0$ and $0 \leq \lambda \leq 1$ such that

$$\sum_{|q|\leq 2m-1} \int_{\mathbb{R}^n} |f(x,t,D_x^q u) - f(x,s,D_x^q v)|^2 dx \leq L(|t-s|^\lambda + \sum_{|q|\leq 2m-1} \int_{\mathbb{R}^n} |A^\lambda D_x^q (u-v)|^2 dx).$$

for all $(t,u),(s,v) \in \mathbb{R}^+ \times X_q$ and all $x \in \mathbb{R}^n$.

If the kernel B is integrable on $t_0 < t < T < \infty$, applying Theorem 3.2 stated above, we deduce that (5.1), (5.2) has a unique local mild solution. In addition, if the real-valued map B satisfies the assumption (H), again by applying Theorem 4.1, we conclude that the considered problem has a unique local classical solution.

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