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Regularity of the Solutions to a Quasi-linear Problem with Nonlinear Boundary Conditions

Aomar Anane, **Omar Chakrone** and **Najat Moradi**
University Mohammed 1st
Department of Mathematics and Informatics
Oujda, Morocco
ananeomar@yahoo.fr, chakrone@yahoo.fr,
najat_moradi@yahoo.fr

Abstract

In our research we will study the regularity of the solutions to the problem

$$\begin{aligned} \Delta_p u &= |u|^{p-2}\, u \quad \text{in} \quad \Omega \\ |\nabla u|^{p-2}\,\frac{\partial u}{\partial \nu} &= \lambda\, |u|^{p-2}\, u + h \quad \text{on} \quad \partial\Omega \end{aligned}$$

and we show that any solution is indeed in $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$.

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1 Introduction

The problem

$$\begin{aligned} \Delta_p u &= |u|^{p-2}\, u \quad \text{in} \quad \Omega \\ |\nabla u|^{p-2}\,\frac{\partial u}{\partial \nu} &= \lambda\, |u|^{p-2}\, u \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1.1}$$

has been studied by several authors. J. F. Bonder and J. D. Rossi have proved in [3] that (1.1) admits an infinite sequence of eigenvalues (λ_k) such that $\lambda_k \to +\infty$ as $k \to +\infty$. In [5], S. Martinez and J. D. Rossi have proved that the first eigenvalue is isolated, simple and any nontrivial associated eigenfunction does not vanish in $\overline{\Omega}$, but in this result the authors assume that the associated eigenfunction is $C^{1,\alpha}$ for some $\alpha \in (0,1)$, what is not shown yet, and it is the subject of the present paper. In [2], we have treated

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the maximum and the anti-maximum principles for the same problem. We recall that the regularity of solutions to the p-Laplacian with a linear boundary condition has been shown by A. Anane in [1] by using the results of D. DiBenedetto [4]. Our main result is the following.

Theorem 1.1. *Any solution of the problem*

$$\begin{array}{rcll} \Delta_p u & = & |u|^{p-2}\, u & \text{in} \quad \Omega \\ |\nabla u|^{p-2}\, \dfrac{\partial u}{\partial \nu} & = & \lambda\, |u|^{p-2}\, u + h & \text{on} \quad \partial\Omega, \end{array} \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^N$, with a $C^{2,\beta}$ boundary for some $\beta \in (0,1)$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\, \nabla u)$ is the p-Laplacian, $p > 1$, $\dfrac{\partial}{\partial \nu}$ is the outer normal derivative and $h \in L^\infty(\partial\Omega)$, is in $C^{1,\alpha}\left(\overline{\Omega}\right)$, and $\|u\|_{C^{1,\alpha}\left(\overline{\Omega}\right)} \le K$, such that

$$\begin{cases} K \;=\; K\left(N, p, \|u/\partial\Omega\|_{L^{p_0}(\partial\Omega)}, \|h\|_{L^\infty(\partial\Omega)}\right) & \text{for} \quad 1 < p \le N, \\ K \;=\; K\left(N, p, \|u\|_{L^\infty(\Omega)}\right) & \text{for} \quad N < p \end{cases}$$

and

$$p_0 = \begin{cases} \dfrac{p(N-1)}{N-p} & \text{if} \quad 1 < p < N \\ p+1 & \text{if} \quad p = N. \end{cases}$$

2 Proof of the Main Result

Let us consider the problem (1.2) with the hypothesis of Theorem 1.1. We recall that u is a solution if and only if for all $v \in W^{1,p}(\Omega)$ we have

$$\int_\Omega |\nabla u|^{p-2}\, \nabla u \nabla v + \int_\Omega |u|^{p-2}\, uv = \lambda \int_{\partial\Omega} |u|^{p-2}\, uv + \int_{\partial\Omega} hv. \tag{2.1}$$

We first show the following result.

Lemma 2.1. *Any solution of the problem* (1.2) *is in $L^\infty(\Omega)$.*

Proof. If $p > N$, then $W^{1,p}(\Omega) \underset{cpct}{\hookrightarrow} C\left(\overline{\Omega}\right)$. Then any solution of (1.2) is in $L^\infty(\Omega)$. If $1 < p \le N$, we will need the next lemma to complete the proof. The proof of Lemma 2.1 will be completed after proving the following Lemma 2.2. □

Lemma 2.2. *If $u \in W^{1,p}(\Omega)$ is a solution of the problem* (1.2), *then there exists a constant $C > 0$ such that*

$$\left(\|u\|^{p_n}_{L^{p_n}(\Omega)} + \|u/\partial\Omega\|^{p_n}_{L^{p_n}(\partial\Omega)}\right)^{\frac{1}{p_n}} \le C \text{ for all } n \ge 0,$$

where the sequence $\{p_n\}$ *is defined by*

$$p_0 = \begin{cases} \dfrac{p(N-1)}{N-p} & \text{if} \quad 1 < p < N \\ p+1 & \text{if} \quad p = N \end{cases} \quad \text{and } p_{n+1} = \frac{p_0}{p} p_n.$$

In particular, $u \in L^{p_n}(\Omega)$ *and* $u/\partial\Omega \in L^{p_n}(\partial\Omega)$ *for all* $n \geq 0$.

Proof. Let u be a solution of the problem (1.2). So $u \in W^{1,p}(\Omega)$ and, for the choice of p_0, we have

$$W^{1,p}(\Omega) \underset{cont}{\hookrightarrow} L^{p_0}(\Omega) \text{ and } W^{1,p}(\Omega) \underset{cont}{\hookrightarrow} L^{p_0}(\partial\Omega).$$

Then $u \in L^{p_0}(\Omega)$ and $u/\partial\Omega \in L^{p_0}(\partial\Omega)$. We can suppose that $\|u/\partial\Omega\|_{L^{p_0}(\partial\Omega)} \geq 1$; if not, we replace u by $u_0 = \dfrac{u}{\|u/\partial\Omega\|_{L^{p_0}(\partial\Omega)}}$ which is a solution of

$$\begin{aligned} \Delta_p u &= |u|^{p-2} u \quad \text{in} \quad \Omega \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} &= \lambda |u|^{p-2} u + h' \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{2.2}$$

with $h' = \left(\|u/\partial\Omega\|_{L^{p_0}(\partial\Omega)}\right)^{p-1} h \in L^\infty(\partial\Omega)$.

By induction on n, we suppose that $u \in L^{p_n}(\Omega)$, $u/\partial\Omega \in L^{p_n}(\partial\Omega)$ and also $\|u/\partial\Omega\|_{L^{p_n}(\partial\Omega)} \geq 1$, and we show that $u \in L^{p_{n+1}}(\Omega)$, $u/\partial\Omega \in L^{p_{n+1}}(\partial\Omega)$ and $\|u/\partial\Omega\|_{L^{p_{n+1}}(\partial\Omega)} \geq 1$. In what follows, we will indicate the norm $\|u/\partial\Omega\|_{L^p(\partial\Omega)}$ by $\|u\|_{L^p(\partial\Omega)}$. Let us define the sequence $\{v_k\}_{k\geq 0}$ in $W^{1,p}(\Omega)$ by

$$v_k(x) = \begin{cases} k & \text{if} \quad u(x) \geq k \\ u(x) & \text{if} \quad -k \leq u(x) \leq k \\ -k & \text{if} \quad u(x) \leq -k \end{cases} \quad \text{for all } x \in \overline{\Omega},$$

and we put $\delta = p_n - p > 0$. If the test function $|v_k|^\delta v_k$ is taken in the formula (2.1), we find on the one hand

$$\langle \Delta_p u, |v_k|^\delta v_k \rangle = \int_\Omega |u|^{p-2} u |v_k|^\delta v_k dx \geq \int_\Omega |v_k|^{\delta+p} = \int_\Omega |v_k|^{p_n},$$

and on the other hand we have

$$\begin{aligned} \langle \Delta_p u, |v_k|^\delta v_k \rangle &= -\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \left(|v_k|^\delta v_k\right) + \int_{\partial\Omega} |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} |v_k|^\delta v_k \partial\sigma \\ &= -S_n \int_\Omega \left|\nabla \left(|v_k|^{\frac{\delta}{p}} v_k\right)\right|^p + \lambda \int_{\partial\Omega} |u|^{p-2} u |v_k|^\delta v_k + \int_{\partial\Omega} h |v_k|^\delta v_k \\ &\leq \lambda \int_{\partial\Omega} |u|^{p_n} + \|h\|_{L^\infty(\partial\Omega)} \|v_k\|^{\delta+1}_{L^{\delta+1}(\partial\Omega)} - S_n \left\|\nabla \left(|v_k|^{\frac{\delta}{p}} v_k\right)\right\|^p_{L^p(\Omega)}, \end{aligned}$$

where $S_n = (\delta+1)\left(\dfrac{p}{p_n}\right)^p$. Then

$$\int_{\Omega} |v_k|^{p_n} \leq \lambda \int_{\partial\Omega} |u|^{p_n} + \|h\|_{L^{\infty}(\partial\Omega)} \|v_k\|^{\delta+1}_{L^{\delta+1}(\partial\Omega)} - S_n \left\| \nabla \left(|v_k|^{\frac{\delta}{p}} v_k \right) \right\|^p_{L^p(\Omega)}. \tag{2.3}$$

With the imbedding $W^{1,p}(\Omega) \underset{cont}{\hookrightarrow} L^{p_0}(\Omega)$ we obtain

$$\begin{aligned} \left\| \nabla \left(|v_k|^{\frac{\delta}{p}} v_k \right) \right\|^p_{L^p(\Omega)} &\geq C_1 \left\| |v_k|^{\frac{\delta+p}{p}} \right\|^p_{L^{p_0}(\Omega)} - \left\| |v_k|^{\frac{\delta+p}{p}} \right\|^p_{L^p(\Omega)} \\ &\geq C_1 \|v_k\|^{p_n}_{L^{p_{n+1}}(\Omega)} - \|v_k\|^{\delta+p}_{L^{\delta+p}(\Omega)}. \end{aligned}$$

It follows that

$$\|v_k\|^{p_n}_{L^{p_{n+1}}(\Omega)} \leq \frac{1}{C_1 S_n} \left(\lambda \|u\|^{p_n}_{L^{p_n}(\partial\Omega)} + \|h\|_{\infty} \|v_k\|^{\delta+1}_{L^{\delta+1}(\partial\Omega)} + (S_n - 1) \|v_k\|^{p_n}_{L^{p_n}(\Omega)} \right).$$

With the results $\delta + 1 < p_n$, $\|u\|_{L^{p_n}(\partial\Omega)} \geq 1$ and supposing that $\mathrm{mes}_{\sigma}(\partial\Omega) \leq 1$, we obtain

$$\begin{aligned} \|v_k\|^{\delta+1}_{L^{\delta+1}(\partial\Omega)} &\leq \|u\|^{\delta+1}_{L^{\delta+1}(\partial\Omega)} \\ &\leq \|u\|^{\delta+1}_{L^{p_n}(\partial\Omega)} \left(\mathrm{mes}_{\sigma}(\partial\Omega)^{1-\frac{\delta+1}{p_n}} \right) \\ &\leq \|u\|^{p_n}_{L^{p_n}(\partial\Omega)}. \end{aligned}$$

So

$$\begin{aligned} \|v_k\|^{p_n}_{L^{p_{n+1}}(\Omega)} &\leq \frac{1}{C_1 S_n} \left((\lambda + \|h\|_{\infty}) \|u\|^{p_n}_{L^{p_n}(\partial\Omega)} + |S_n - 1| \|u\|^{p_n}_{L^{p_n}(\Omega)} \right) \\ &\leq \frac{R_n}{C_1 S_n} \left(\|u\|^{p_n}_{L^{p_n}(\partial\Omega)} + \|u\|^{p_n}_{L^{p_n}(\Omega)} \right), \end{aligned}$$

where $R_n = \max\left(\lambda + \|h\|_{L^{\infty}(\partial\Omega)}, |S_n - 1| \right)$. This gives

$$\begin{aligned} \|u\|^{p_n}_{L^{p_{n+1}}(\Omega)} &\leq \liminf_{|k| \to +\infty} \left(\|v_k\|^{p_n}_{L^{p_{n+1}}(\Omega)} \right) \\ &\leq \frac{R_n}{C_1 S_n} \left(\|u\|^{p_n}_{L^{p_n}(\partial\Omega)} + \|u\|^{p_n}_{L^{p_n}(\Omega)} \right), \end{aligned} \tag{2.4}$$

and therefore $u \in L^{p_{n+1}}(\Omega)$.

We must now prove that $u/\partial\Omega \in L^{p_{n+1}}(\partial\Omega)$ and $\|u\|_{L^{p_{n+1}}(\partial\Omega)} \geq 1$. Using (2.3), we find

$$\int_{\Omega} |v_k|^{p_n} + S_n \left\| \nabla (|v_k|^{\frac{\delta}{p}} v_k) \right\|^p_{L^p(\Omega)} \leq \lambda \int_{\partial\Omega} |u|^{p_n} + \|h\|_{\infty} \|v_k\|^{\delta+1}_{L^{\delta+1}(\partial\Omega)},$$

and with the result $W^{1,p}(\Omega) \underset{cont}{\hookrightarrow} L^{p_0}(\partial\Omega)$, we have

$$\begin{aligned} \left\| \nabla(|v_k|^{\frac{\delta}{p}} v_k) \right\|_{L^p(\Omega)}^p &\geq C_2 \left\| |v_k|^{\frac{\delta+p}{p}} \right\|_{L^{p_0}(\partial\Omega)}^p - \left\| |v_k|^{\frac{\delta+p}{p}} \right\|_{L^p(\Omega)}^p \\ &\geq C_2 \|v_k\|_{L^{p_{n+1}}(\partial\Omega)}^{p_n} - \|v_k\|_{L^{\delta+p}(\Omega)}^{\delta+p}. \end{aligned}$$

So

$$S_n \left(C_2 \|v_k\|_{L^{p_{n+1}}(\partial\Omega)}^{p_n} - \|v_k\|_{L^{\delta+p}(\Omega)}^{\delta+p} \right) \leq \lambda \int_{\partial\Omega} |u|^{p_n} + \|h\|_\infty \|v_k\|_{L^{\delta+1}(\partial\Omega)}^{\delta+1} - \int_\Omega |v_k|^{p_n}.$$

We conclude that

$$\|v_k\|_{L^{p_{n+1}}(\partial\Omega)}^{p_n} \leq \frac{1}{C_2 S_n} \left(\lambda \|u\|_{L^{p_n}(\partial\Omega)}^{p_n} + \|h\|_\infty \|v_k\|_{L^{\delta+1}(\partial\Omega)}^{\delta+1} + |S_n - 1| \|v_k\|_{L^{p_n}(\Omega)}^{p_n} \right).$$

Then

$$\|v_k\|_{L^{p_{n+1}}(\partial\Omega)}^{p_n} \leq \frac{R_n}{C_2 S_n} \left(\|u\|_{L^{p_n}(\partial\Omega)}^{p_n} + \|u\|_{L^{p_n}(\Omega)}^{p_n} \right)$$

and

$$\begin{aligned} \|u\|_{L^{p_{n+1}}(\partial\Omega)}^{p_n} &\leq \liminf_{|k|\to+\infty} \left(\|v_k\|_{L^{p_{n+1}}(\partial\Omega)}^{p_n} \right) \\ &\leq \frac{R_n}{C_2 S_n} \left(\|u\|_{L^{p_n}(\partial\Omega)}^{p_n} + \|u\|_{L^{p_n}(\Omega)}^{p_n} \right). \end{aligned} \tag{2.5}$$

Consequently $u/\partial\Omega \in L^{p_{n+1}}(\partial\Omega)$ and $\|u\|_{L^{p_{n+1}}(\partial\Omega)} \geq \|u\|_{L^{p_n}(\partial\Omega)} \geq 1$. Hence we have $u \in L^{p_n}(\Omega)$, $u/\partial\Omega \in L^{p_n}(\partial\Omega)$ and $\|u\|_{L^{p_n}(\partial\Omega)} \geq 1$ for all $n \geq 0$.

It remains to show that there exists $C > 0$ such that

$$\left(\|u\|_{L^{p_n}(\Omega)}^{p_n} + \|u\|_{L^{p_n}(\partial\Omega)}^{p_n} \right)^{\frac{1}{p_n}} \leq C \text{ for all } n \geq 0.$$

From the two preceding inequalities (2.4) and (2.5), we conclude that

$$\|u\|_{L^{p_{n+1}}(\Omega)}^{p_{n+1}} + \|u\|_{L^{p_{n+1}}(\partial\Omega)}^{p_{n+1}} \leq D_n \left(\|u\|_{L^{p_n}(\partial\Omega)}^{p_n} + \|u\|_{L^{p_n}(\Omega)}^{p_n} \right)^{\frac{p_0}{p}},$$

where $D_n = \left(\frac{1}{C_1} + \frac{1}{C_2} \right)^{\frac{p_0}{p}} \frac{1}{(\delta+1)} \left(\frac{p_n}{p} \right)^{p_0} (R_n)^{\frac{p_0}{p}}$. Since $S_n \underset{n\to+\infty}{\longrightarrow} 0$ one has $|S_n - 1| \underset{n\to+\infty}{\longrightarrow} 1$. Thus for rather large values of n we have $|S_n - 1| \leq 2$. Consequently

$$\|u\|_{L^{p_{n+1}}(\Omega)}^{p_{n+1}} + \|u\|_{L^{p_{n+1}}(\partial\Omega)}^{p_{n+1}} \leq C (p_n)^{p_0} \left(\|u\|_{L^{p_n}(\partial\Omega)}^{p_n} + \|u\|_{L^{p_n}(\Omega)}^{p_n} \right)^{\frac{p_0}{p}},$$

where

$$C = \frac{1}{p^{p_0}} \left(\left(\frac{1}{C_1} + \frac{1}{C_2} \right) \max\left(\lambda + \|h\|_{L^\infty(\partial\Omega)}, 2 \right) \right)^{\frac{p_0}{p}}.$$

Taking $v_n = \left(\|u\|^{p_n}_{L^{p_n}(\partial\Omega)} + \|u\|^{p_n}_{L^{p_n}(\Omega)}\right)^{\frac{1}{p_n}}$, we have for large values of n

$$v_{n+1}^{p_{n+1}} \leq C(p_n)^{p_0}(v_n^{p_n})^{\frac{p_0}{p}} \text{ and } \ln(v_{n+1}) \leq \frac{A}{p_{n+1}} + p\frac{\ln(p_n)}{p_n} + \ln(v_n),$$

where $A = \ln(C)$. Thus

$$\ln(v_{n+1}) \leq A \sum_{n_0+1\leq k\leq n+1} \left(\frac{1}{p_k}\right) + p \sum_{n_0\leq k\leq n} \left(\frac{\ln(p_k)}{p_k}\right) + \ln(v_{n_0})$$

for all $n \geq n_0$. The sequence $\left\{\frac{1}{p_k}\right\}_{k\geq 0}$ is geometric and $0 < \frac{p}{p_0} < 1$, so

$$\sum_{n_0+1\leq k\leq n+1} \left(\frac{1}{p_k}\right) \leq \frac{p_0}{p_0 - p}.$$

It is checked easily that

$$\frac{\ln(p_k)}{p_k} = (\mu + \gamma k)\left(\frac{p}{p_0}\right)^k$$

and

$$\sum_{n_0\leq k\leq n} \left(\frac{\ln(p_k)}{p_k}\right) \leq \sum_{0\leq k}(\mu + \gamma k)\left(\frac{p}{p_0}\right)^k = \frac{\mu p_0}{p_0 - p} + \gamma\frac{pp_0}{(p_0 - p)^2},$$

where $\mu = \frac{\ln(p_0)}{p_0}$ and $\gamma = \frac{\ln(p_0) - \ln(p)}{p_0}$. Then

$$\ln(v_n) \leq \frac{p_0}{p_0 - p}(A + p\mu) + \gamma\frac{p^2 p_0}{(p_0 - p)^2} + \ln(v_{n_0}) = D \text{ for all } n \geq n_0.$$

Consequently $v_n = \left(\|u\|^{p_n}_{L^{p_n}(\partial\Omega)} + \|u\|^{p_n}_{L^{p_n}(\Omega)}\right)^{\frac{1}{p_n}} \leq \exp(D)$. □

Completion of Proof of Lemma 2.1. Let us now return to the proof of Lemma 2.1. By Lemma 2.2, we conclude that

$$\|u\|_{L^{p_n}(\Omega)} \leq \exp(D) \text{ and } \|u/\partial\Omega\|_{L^{p_n}(\partial\Omega)} \leq \exp(D) \text{ for all } n \geq n_0.$$

Then

$$\|u\|_{L^\infty(\Omega)} \leq \limsup_{n\to+\infty} \|u\|_{L^{p_n}(\Omega)} \leq \exp(D)$$

and

$$\|u/\partial\Omega\|_{L^\infty(\partial\Omega)} \leq \limsup_{n\to+\infty} \|u/\partial\Omega\|_{L^{p_n}(\partial\Omega)} \leq \exp(D).$$

Thus we have proved that $u \in L^\infty(\Omega)$ and $u/\partial\Omega \in L^\infty(\partial\Omega)$. The proof is complete. □

The continuation of the proof of Theorem 1.1 is based on the following result of Tolksdorf and DiBenedetto (see [4, 6]).

Lemma 2.3. *Let u be in $W^{1,p}(\Omega) \cap L^{\infty}(\Omega)$ such that $\Delta_p u \in L^r(\Omega)$, where r is in* $[1, +\infty]$.

(i) *If $r = \infty$ and $\partial\Omega$ is $C^{2,\beta}$ with $\beta \in (0,1)$, then $u \in C^{1,\alpha}\left(\overline{\Omega}\right)$ for some $\alpha \in (0,1)$ and $\|u\|_{C^{1,\alpha}\left(\overline{\Omega}\right)} \leq k_1\left(N, p, \|u\|_{L^{\infty}(\Omega)}, \|\Delta_p u\|_{L^{\infty}(\Omega)}\right)$.*

(ii) *If $r > Np'$, then, for all open subsets w of Ω such that $\overline{w} \subset \Omega$, u is in $C^{1,\alpha}(\overline{w})$ for some $\alpha \in (0,1)$ and $\|u\|_{C^{1,\alpha}(\overline{w})} \leq k_2\left(N, p, \|u\|_{L^{\infty}(\Omega)}, \|\Delta_p u\|_r, w\right)$.*

Proof of Theorem 1.1. Using Lemma 2.3, u is in $C^{1,\alpha}\left(\overline{\Omega}\right)$ and $\|u\|_{C^{1,\alpha}\left(\overline{\Omega}\right)} \leq K$, where $\alpha \in (0,1)$ and K depends on N, p, and $\|u\|_{L^{\infty}(\Omega)}$ because $\|\triangle_p u\|_{L^{\infty}(\Omega)} = \|u\|_{L^{\infty}(\Omega)}^{p-1}$. Moreover for $1 < p \leq N$, we have $\|u\|_{L^{\infty}(\Omega)} \leq C$, where C depends only on $\|u/\partial\Omega\|_{L^{p_0}(\partial\Omega)}$ and $\|h\|_{\infty}$, so $K = K\left(N, p, \|u/\partial\Omega\|_{L^{p_0}(\partial\Omega)}, \|h\|_{\infty}\right)$. □

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