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Darboux Problem for Partial Functional Differential Equations with Infinite Delay and Caputo's Fractional Derivative

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Dedicated to Professor S. K. Ntouyas
on the occasion of his 60th birthday.

Abstract

In this paper we provide sufficient conditions for the existence as well as the uniqueness of solutions of two classes of partial hyperbolic differential equations of fractional order with infinite delay. Our results will be obtained using suitable fixed point theorems.

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1 Introduction

This paper deals with the existence of solutions to fractional order initial value problems (IVP for short), for the system

$$({}^cD_0^r u)(x,y) = f(x,y,u_{(x,y)}),\ \text{if } (x,y)\in J, \tag{1.1}$$

$$u(x,y) = \phi(x,y),\ \text{if } (x,y)\in \tilde{J}, \tag{1.2}$$

$$u(x,0) = \varphi(x),\ u(0,y) = \psi(y),\ x\in[0,a],\ y\in[0,b], \tag{1.3}$$

where $J=[0,a]\times[0,b],\ a,b>0,\ \tilde{J}=(-\infty,a]\times(-\infty,b]\backslash(0,a]\times(0,b],\ {}^cD_0^r$ is the standard Caputo's fractional derivative of order $r=(r_1,r_2)\in(0,1]\times(0,1],\ f: J\times B\to\mathbb{R}^n,\ \phi:\tilde{J}\to\mathbb{R}^n$ are given continuous functions, $\varphi:[0,a]\to\mathbb{R}^n$, $\psi:[0,b]\to\mathbb{R}^n$ are given absolutely continuous functions with $\varphi(0)=\psi(0),\ \varphi(x)=\phi(x,0),\ \psi(y)=\phi(0,y)$ for each $(x,y)\in J$ and B is called a phase space that will be specified in Section 3.

We denote by $u_{(x,y)}$ the element of B defined by

$$u_{(x,y)}(s,t) = u(x+s,y+t);\ (s,t)\in(-\infty,0]\times(-\infty,0],$$

here $u_{(x,y)}(\cdot,\cdot)$ represents the history of the state from time $-\infty$ up to the present time x and from time $-\infty$ up to the present time y.

Next we consider the following initial value problem for partial neutral functional differential equations

$${}^cD_0^r\Big(u(x,y)-g(x,y,u_{(x,y)})\Big) = f(x,y,u_{(x,y)}),\ \text{if } (x,y)\in J, \tag{1.4}$$

$$u(x,y) = \phi(x,y),\ \text{if } (x,y)\in \tilde{J}, \tag{1.5}$$

$$u(x,0) = \varphi(x),\ u(0,y) = \psi(y),\ x\in[0,a],\ y\in[0,b], \tag{1.6}$$

where f,ϕ,φ,ψ are as in problem (1.1)–(1.3) and $g: J\times B\to\mathbb{R}^n$ is a given continuous function.

The problem of existence of solutions of Cauchy-type problems for ordinary differential equations of fractional order in spaces of integrable functions without delay was studies in numerous works see [23,36]), a similar problem in spaces of continuous functions was studies in [37]. We can find numerous applications of differential equations of fractional order in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [11, 13, 14, 19, 29, 30, 32]). There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Kilbas [25], Lakshmikantham et al. [26], Miller and Ross [31], Samko [35], the papers of Abbas and Benchohra [1, 2], Agarwal et al. [3], Belarbi et al. [4], Benchohra et al. [5–7], Diethelm [11, 12], Kilbas and Marzan [24], Mainardi [29], Podlubny [34], Vityuk and Golushkov [38], and the references therein.

In this paper, we present existence and uniqueness results for problems (1.1)–(1.3) and (1.4)–(1.6). We give two results for each problem, the first one (Theorem 4.3, Theorem 4.9) is based upon the Banach's contraction principle and the second one (Theorem 4.5, Theorem 4.10) on the nonlinear alternative of Leray–Schauder.

For integer order derivative, various classes of hyperbolic differential equations were considered; see for instance, the book by Kamont [21], the papers by Czlapinski [8, 9], Dawidowski and Kubiaczyk [10], Kamont and Kropielnicka [22], Lakshmikantham and Pandit [27], Pandit [33]. This paper initiates the study of fractional order hyperbolic differential equations with infinite delay involving the Caputo fractional derivative.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. By $L^1(J, \mathbb{R}^n)$ we denote the space of Lebesgue-integrable functions $u : J \to \mathbb{R}^n$ with the norm

$$\|u\|_{L^1} = \int_0^a \int_0^b \|u(x, y)\| dy dx,$$

where $\|.\|$ denotes a suitable complete norm on $\mathbb{R}^n$. $AC(J, \mathbb{R}^n)$ is the space of absolutely continuous valued functions on J.

Definition 2.1 (See [34]). Let $r_1,\ r_2 > 0$ and $r = (r_1, r_2)$. For $u \in L^1(J, \mathbb{R}^n)$, the expression

$$(I_0^r u)(x, y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x - s)^{r_1 - 1}(y - t)^{r_2 - 1} u(s, t) dt ds,$$

where $\Gamma(.)$ is the gamma function, is called the left-sided mixed Riemann–Liouville integral of order r.

Definition 2.2 (See [34]). For $u \in L^1(J, \mathbb{R}^n)$, the Caputo fractional-order derivative of order r is defined by the expression

$$({}^c D_0^r u)(x, y) = \left(I_0^{1-r} \frac{\partial^2}{\partial x \partial y} u \right)(x, y).$$

3 The Phase Space B

The notation of the phase space B plays an important role in the study of both qualitative and quantitative theory for functional differential equations. A usual choice is a seminormed space satisfying suitable axioms, which was introduced by Hale and Kato (see [16]). For further applications see for instance the books [17, 20, 28] and their references.

For any $(x,y)\in J$ denote $E_{(x,y)}:=[0,x]\times\{0\}\cup\{0\}\times[0,y]$, furthermore in case $x=a,\ y=b$ we write simply E. Consider the space $(B,\|(\cdot,\cdot)\|_B)$ is a seminormed linear space of functions mapping $(-\infty,0]\times(-\infty,0]$ into $\mathbb{R}^n$, and satisfying the following fundamental axioms which were adapted from those introduced by Hale and Kato for ordinary differential functional equations:

(A_1) If $z:(-\infty,a]\times(-\infty,b]\to\mathbb{R}^n$ continuous on J and $z_{(x,y)}\in B$, for all $(x,y)\in E$, then there are constants $H,K,M>0$ such that for any $(x,y)\in J$ the following conditions hold:

(i) $z_{(x,y)}$ is in B;

(ii) $\|z(x,y)\|\leq H\|z_{(x,y)}\|_B$,

(iii) $\|z_{(x,y)}\|_B\leq K\sup_{(s,t)\in[0,x]\times[0,y]}\|z(s,t)\|+M\sup_{(s,t)\in E_{(x,y)}}\|z_{(s,t)}\|_B$,

(A_2) For the function $z(\cdot,\cdot)$ in (A_1), $z_{(x,y)}$ is a B-valued continuous function on J.

(A_3) The space B is complete.

Now, we present some examples of phase spaces [8, 9].

Example 3.1. Let B be the set of all functions $\phi:(-\infty,0]\times(-\infty,0]\to\mathbb{R}^n$ which are continuous on $[-\alpha,0]\times[-\beta,0],\ \alpha,\beta\geq 0$, with the seminorm

$$\|\phi\|_B=\sup_{(s,t)\in[-\alpha,0]\times[-\beta,0]}\|\phi(s,t)\|.$$

Then we have $H=K=M=1$. The quotient space $\widehat{B}=B/\|.\|_B$ is isometric to the space $C([-\alpha,0]\times[-\beta,0],\mathbb{R}^n)$ of all continuous functions from $[-\alpha,0]\times[-\beta,0]$ into $\mathbb{R}^n$ with the supremum norm, this means that partial differential functional equations with finite delay are included in our axiomatic model.

Example 3.2. Let C_γ be the set of all continuous functions $\phi:(-\infty,0]\times(-\infty,0]\to\mathbb{R}^n$ for which a limit $\lim_{\|(s,t)\|\to\infty}e^{\gamma(s+t)}\phi(s,t)$ exists, with the norm

$$\|\phi\|_{C_\gamma}=\sup_{(s,t)\in(-\infty,0]\times(-\infty,0]}e^{\gamma(s+t)}\|\phi(s,t)\|.$$

Then we have $H=1$ and $K=M=\max\{e^{-(a+b)},1\}$.

Example 3.3. Let $\alpha,\beta,\gamma\geq 0$ and let

$$\|\phi\|_{CL_\gamma}=\sup_{(s,t)\in[-\alpha,0]\times[-\beta,0]}\|\phi(s,t)\|+\int_{-\infty}^{0}\int_{-\infty}^{0}e^{\gamma(s+t)}\|\phi(s,t)\|dtds$$

be the seminorm for the space CL_γ of all functions $\phi : (-\infty, 0] \times (-\infty, 0] \to \mathbb{R}^n$ which are continuous on $[-\alpha, 0] \times [-\beta, 0]$ measurable on $(-\infty, -\alpha] \times (-\infty, 0] \cup (-\infty, 0] \times (-\infty, -\beta]$, and such that $\|\phi\|_{CL_\gamma} < \infty$. Then

$$H = 1,\ K = \int_{-\alpha}^{0}\int_{-\beta}^{0} e^{\gamma(s+t)}dtds,\ M = 2.$$

4 Main Results

Let us start by defining what we mean by a solution of the problem (1.1)–(1.3). Let the space

$$\Omega := \{u : (-\infty, a] \times (-\infty, b] \to \mathbb{R}^n :\ u_{(x,y)} \in B \text{ for } (x, y) \in E \text{ and } u|_J \in C(J, \mathbb{R}^n)\}.$$

Definition 4.1. A function $u \in \Omega$ is said to be a solution of (1.1)–(1.3) if u satisfies equations (1.1) and (1.3) on J and the condition (1.2) on $\tilde{J}$.

Let $f \in L^1(J, \mathbb{R}^n)$ and consider the problem

$$\begin{cases} ({}^cD_0^r u)(x, y) = f(x, y);\ \text{a.e. } (x, y) \in J, \\ u(x, 0) = \varphi(x),\ u(0, y) = \psi(y),\ \varphi(0) = \psi(0). \end{cases} \tag{4.1}$$

For the existence of solutions for the problem (1.1)–(1.3), we need the following lemma.

Lemma 4.2 (See [1,2]). *A function $u \in AC(J, \mathbb{R}^n)$ is a solution of problem* (4.1) *if and only if $u(x, y)$ satisfies*

$$u(x, y) = \mu(x, y) + (I_0^r f)(x, y);\ \textit{a.e. } (x, y) \in J, \tag{4.2}$$

where

$$\mu(x, y) = \varphi(x) + \psi(y) - \varphi(0).$$

Our first existence result for the IVP (1.1)–(1.3) is based on the Banach contraction principle.

Theorem 4.3. *Assume*

(H_1) *there exists $\ell > 0$ such that*

$$\|f(x, y, u) - f(x, y, v)\| \le \ell\|u - v\|_B,\ \textit{for any } u,\ v \in B \textit{ and } (x, y) \in J.$$

If

$$\frac{\ell K a^{r_1} b^{r_2}}{\Gamma(r_1 + 1)\Gamma(r_2 + 1)} < 1, \tag{4.3}$$

then there exists a unique solution for IVP (1.1)–(1.3) *on $(-\infty, a] \times (-\infty, b]$.*

Proof. Transform the problem (1.1)–(1.3) into a fixed point problem. Consider the operator $N : \Omega \to \Omega$ defined by,

$$N(u)(x,y) = \begin{cases} \phi(x,y), & (x,y) \in \tilde{J}, \\ \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} & \\ \qquad \times f(s,t,u_{(s,t)})dtds, & (x,y) \in J. \end{cases}$$

Let $v(\cdot,\cdot) : (-\infty, a] \times (-\infty, b] \to \mathbb{R}^n$ be a function defined by

$$v(x,y) = \begin{cases} \phi(x,y), & (x,y) \in \tilde{J}, \\ \mu(x,y), & (x,y) \in J. \end{cases}$$

Then $v_{(x,y)} = \phi$ for all $(x,y) \in E$. For each $w \in C(J, \mathbb{R}^n)$ with $w(x,y) = 0$ for each $(x,y) \in E$ we denote by $\overline{w}$ the function defined by

$$\overline{w}(x,y) = \begin{cases} 0, & (x,y) \in \tilde{J}, \\ w(x,y) & (x,y) \in J. \end{cases}$$

If $u(\cdot,\cdot)$ satisfies the integral equation,

$$u(x,y) = \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,u_{(s,t)})dtds,$$

we can decompose $u(\cdot,\cdot)$ as $u(x,y) = \overline{w}(x,y) + v(x,y);\ (x,y) \in J$, which implies $u_{(x,y)} = \overline{w}_{(x,y)} + v_{(x,y)}$, for every $(x,y) \in J$, and the function $w(\cdot,\cdot)$ satisfies

$$w(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,\overline{w}_{(s,t)} + v_{(s,t)})dtds.$$

Set

$$C_0 = \{w \in C(J, \mathbb{R}^n) :\ w(x,y) = 0 \ \text{ for } (x,y) \in E\},$$

and let $\|\cdot\|_{(a,b)}$ be the seminorm in C_0 defined by

$$\|w\|_{(a,b)} = \sup_{(x,y)\in E} \|w_{(x,y)}\|_B + \sup_{(x,y)\in J} \|w(x,y)\| = \sup_{(x,y)\in J} \|w(x,y)\|,\ w \in C_0.$$

C_0 is a Banach space with norm $\|\cdot\|_{(a,b)}$. Let the operator $P :\ C_0 \to C_0$ be defined by

$$(Pw)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,\overline{w}_{(s,t)} + v_{(s,t)})dtds, \quad (4.4)$$

for each $(x,y) \in J$. Then the operator N has a fixed point is equivalent to P has a fixed point, and so we turn to proving that P has a fixed point. We shall show that

$P: C_0 \to C_0$ is a contraction map. Indeed, consider $w, w^* \in C_0$. Then we have for each $(x,y) \in J$

$$\begin{aligned}
\|P(w)(x,y) \quad - \quad & P(w^*)(x,y)\| \leq \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \\
& \times\|f(s,t,\overline{w}_{(s,t)}+v_{(s,t)}) - f(s,t,\overline{w^*}_{(s,t)}+v_{(s,t)})\|dtds \\
\leq \quad & \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\ell\|\overline{w}_{(s,t)} - \overline{w^*}_{(s,t)}\| \\
\leq \quad & \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\ell K \\
& \times \sup_{(s,t)\in[0,x]\times[0,y]}\|\overline{w}(s,t) - \overline{w^*}(s,t)\|dtds \\
\leq \quad & \frac{\ell K}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}dtds\|\overline{w} - \overline{w^*}\|_{(a,b)} \\
\leq \quad & \frac{\ell K x^{r_1}y^{r_2}}{\Gamma(r_1+1)\Gamma(r_2)+1}\|\overline{w} - \overline{w^*}\|_{(a,b)}.
\end{aligned}$$

Therefore

$$\|P(w) - P(w^*)\|_{(a,b)} \leq \frac{\ell K a^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2)+1}\|\overline{w} - \overline{w^*}\|_{(a,b)},$$

and hence P is a contraction. Therefore, P has a unique fixed point by Banach's contraction principle. □

In the sequel we will make use of the following generalization of Gronwall's lemma for two independent variables and singular kernel.

Lemma 4.4 (See [18]). *Let $\upsilon : J \to [0,\infty)$ be a real function and $\omega(\cdot,\cdot)$ be a nonnegative, locally integrable function on J. If there are constants $c > 0$ and $0 < r_1, r_2 < 1$ such that*

$$\upsilon(x,y) \leq \omega(x,y) + c\int_0^x\int_0^y \frac{\upsilon(s,t)}{(x-s)^{r_1}(y-t)^{r_2}}dtds,$$

then there exists a constant $\delta = \delta(r_1, r_2)$ such that

$$\upsilon(x,y) \leq \omega(x,y) + \delta c\int_0^x\int_0^y \frac{\omega(s,t)}{(x-s)^{r_1}(y-t)^{r_2}}dtds,$$

for every $(x,y) \in J$.

Now we give an existence result based upon the nonlinear alternative of Leray–Schauder type [15].

Theorem 4.5. *Assume*

(H_2) *There exist* $p, q \in C(J, \mathbb{R}_+)$ *such that*

$$\|f(x, y, u)\| \le p(x, y) + q(x, y)\|u\|_B, \text{ for } (x, y) \in J \text{ and each } u \in B.$$

Then the IVP (1.1)–(1.3) *has at least one solution on* $(-\infty, a] \times (-\infty, b]$.

Proof. Let $P : C_0 \to C_0$ defined as in (4.4). We shall show that the operator P is continuous and completely continuous.

Step 1: P is continuous. Let $\{w_n\}$ be a sequence such that $w_n \to w$ in C_0. Then

$$\|P(w_n)(x, y) - P(w)(x, y)\| \le \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x - s)^{r_1 - 1}(y - t)^{r_2 - 1} \times \|f(s, t, \overline{w_{n(s,t)}} + v_{n(s,t)}) - f(s, t, \overline{w}_{(s,t)} + v_{(s,t)})\| dtds.$$

Since f is a continuous function, we have

$$\begin{aligned}
\|P(w_n) - P(w)\|_\infty &\le \frac{x^{r_1} y^{r_2} \|f(\cdot, \cdot, \overline{w_n}_{(\cdot,\cdot)} + v_{n(\cdot,\cdot)}) - f(\cdot, \cdot, \overline{w}_{(\cdot,\cdot)} + v_{(\cdot,\cdot)})\|_\infty}{\Gamma(r_1 + 1)\Gamma(r_2 + 1)} \\
&\le \frac{a^{r_1} b^{r_2} \|f(\cdot, \cdot, \overline{w_n}_{(\cdot,\cdot)} + v_{n(\cdot,\cdot)}) - f(\cdot, \cdot, \overline{w}_{(\cdot,\cdot)} + v_{(\cdot,\cdot)})\|_\infty}{\Gamma(r_1 + 1)\Gamma(r_2 + 1)} \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}$$

Step 2: P maps bounded sets into bounded sets in C_0. Indeed, it is enough show that, for any $\eta > 0$, there exists a positive constant $\widetilde{\ell}$ such that, for each $w \in B_\eta = \{w \in C_0 : \|w\|_{(a,b)} \le \eta\}$, we have $\|P(w)\|_\infty \le \widetilde{\ell}$. Let $w \in B_\eta$. By (H_2) we have for each $(x, y) \in J$,

$$\begin{aligned}
&\|P(w)(x, y)\| \\
&\le \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x - s)^{r_1 - 1}(y - t)^{r_2 - 1} \|f(s, t, \overline{w}_{(s,t)} + v_{(s,t)})\| dtds \\
&\le \frac{1}{\Gamma(r_1)\Gamma(r_2)} \| \int_0^x \int_0^y (x - s)^{r_1 - 1}(y - t)^{r_2 - 1} p(s, t) dtds \| \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x - s)^{r_1 - 1}(y - t)^{r_2 - 1} q(s, t) \|\overline{w}_{(s,t)} + v_{(s,t)}\|_B dtds \\
&\le \frac{\|p\|_\infty}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^x (x - s)^{r_1 - 1}(y - t)^{r_2 - 1} dtds \\
&\quad + \frac{\|q\|_\infty \eta^*}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x - s)^{r_1 - 1}(y - t)^{r_2 - 1} dtds \\
&\le \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1 + 1)\Gamma(r_2 + 1)} x^{r_1} y^{r_2} \\
&\le \frac{\|p\|_\infty + \|q\|_\infty \eta^*}{\Gamma(r_1 + 1)\Gamma(r_2 + 1)} a^{r_1} b^{r_2} := \ell^*,
\end{aligned}$$

where

$$\begin{aligned} \|\overline{w}_{(s,t)} + v_{(s,t)}\|_B &\leq \|\overline{w}_{(s,t)}\|_B + \|v_{(s,t)}\|_B \\ &\leq K\eta + K\|\phi(0,0)\| + M\|\phi\|_B := \eta^*. \end{aligned}$$

Hence $\|P(w)\|_\infty \leq \ell^*$.

Step 3: P maps bounded sets into equicontinuous sets in C_0. Let $(x_1, y_1), (x_2, y_2) \in (0, a] \times (0, b]$, $x_1 < x_2$, $y_1 < y_2$, B_η be a bounded set as in Step 2, and let $w \in B_\eta$. Then

$$\begin{aligned} &\|P(w)(x_2, y_2) - P(w)(x_1, y_1)\| \\ &\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \left\| \int_0^{x_1}\int_0^{y_1} [(x_2-s)^{r_1-1}(y_2-t)^{r_2-1} - (x_1-s)^{r_1-1}(y_1-t)^{r_2-1}] \right. \\ &\quad \times f(s,t,u_{(s,t)})dtds + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2}\int_{y_1}^{y_2} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1} \\ &\quad \left. \times\, f(s,t,\overline{w}_{(s,t)} + v_{(s,t)})dtds \right\| \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \left\| \int_0^{x_1}\int_{y_1}^{y_2} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1} f(s,t,\overline{w}_{(s,t)} + v_{(s,t)})dtds \right\| \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \left\| \int_{x_1}^{x_2}\int_0^{y_1} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1} f(s,t,\overline{w}_{(s,t)} + v_{(s,t)})dtds \right\| \\ &\leq \frac{\|p\|_\infty + \|q\|_\infty\eta}{\Gamma(r_1)\Gamma(r_2)} \int_0^{x_1}\int_0^{y_1} [(x_1-s)^{r_1-1}(y_1-t)^{r_2-1} \\ &\quad -(x_2-s)^{r_1-1}(y_2-t)^{r_2-1}]dtds \\ &\quad + \frac{\|p\|_\infty + \|q\|_\infty\eta}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2}\int_{y_1}^{y_2} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1}dtds \\ &\quad + \frac{\|p\|_\infty + \|q\|_\infty\eta}{\Gamma(r_1)\Gamma(r_2)} \int_0^{x_1}\int_{y_1}^{y_2} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1}dtds \\ &\quad + \frac{\|p\|_\infty + \|q\|_\infty\eta}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2}\int_0^{y_1} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1}dtds \\ &\leq \frac{\|p\|_\infty + \|q\|_\infty\eta}{\Gamma(r_1+1)\Gamma(r_2+1)} [y_2^{r_2}(x_2-x_1)^{r_1} + x_2^{r_1}(y_2-y_1)^{r_2} \\ &\quad -(x_2-x_1)^{r_1}(y_2-y_1)^{r_2} + x_1^{r_1}y_1^{r_2} - x_2^{r_1}y_2^{r_2}] \\ &\quad + \frac{\|p\|_\infty + \|q\|_\infty\eta}{\Gamma(r_1+1)\Gamma(r_2+1)} (x_2-x_1)^{r_1}(y_2-y_1)^{r_2} \\ &\quad + \frac{\|p\|_\infty + \|q\|_\infty\eta}{\Gamma(r_1+1)\Gamma(r_2+1)} [x_2^{r_1} - (x_2-x_1)^{r_1}](y_2-y_1)^{r_2} \\ &\quad + \frac{\|p\|_\infty + \|q\|_\infty\eta}{\Gamma(r_1+1)\Gamma(r_2+1)} (x_2-x_1)^{r_1}[y_2^{r_2} - (y_2-y_1)^{r_2}] \\ &\leq \frac{\|p\|_\infty + \|q\|_\infty\eta}{\Gamma(r_1+1)\Gamma(r_2+1)} [2y_2^{r_2}(x_2-x_1)^{r_1} + 2x_2^{r_1}(y_2-y_1)^{r_2} \end{aligned}$$

$$+x_1^{r_1}y_1^{r_2} - x_2^{r_1}y_2^{r_2} - 2(x_2 - x_1)^{r_1}(y_2 - y_1)^{r_2}].$$

As $x_1 \to x_2$, $y_1 \to y_2$ the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3, together with the Arzela–Ascoli theorem, we can conclude that $P : C_0 \to C_0$ is continuous and completely continuous.

Step 4 (A priori bounds): We now show there exists an open set $U \subseteq C_0$ with $w \neq \lambda P(w)$, for $\lambda \in (0,1)$ and $w \in \partial U$. Let $w \in C_0$ and $w = \lambda P(w)$ for some $0 < \lambda < 1$. Thus for each $(x,y) \in J$,

$$w(x,y) = \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,u_{(s,t)})dtds.$$

This implies by (H_2) that, for each $(x,y) \in J$, we have

$$\begin{aligned}\|w(x,y)\| &\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}[p(s,t) \\ &\quad + q(s,t)\|\overline{w}_{(s,t)} + v_{(s,t)}\|_B]dtds \\ &\leq \frac{\|p\|_\infty a^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} q(s,t)\|\overline{w}_{(s,t)} + v_{(s,t)}\|_B dtds.\end{aligned}$$

But

$$\begin{aligned}\|\overline{w}_{(s,t)} + v_{(s,t)}\|_B &\leq \|\overline{w}_{(s,t)}\|_B + \|v_{(s,t)}\|_B \\ &\leq \sup\{w(\tilde{s},\tilde{t}) : (\tilde{s},\tilde{t}) \in [0,s] \times [0,t]\} + M\|\phi\|_B + K\|\phi(0,0)\|. \quad (4.5)\end{aligned}$$

If we name $z(s,t)$ the right-hand side of (4.5), then we have

$$\|\overline{w}_{(s,t)} + v_{(s,t)}\|_B \leq z(x,y),$$

and therefore, for each $(x,y) \in J$ we obtain

$$\begin{aligned}\|w(x,y)\| &\leq \frac{\|p\|_\infty a^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} q(s,t)z(s,t)dtds. \quad (4.6)\end{aligned}$$

Using the above inequality and the definition of z for each $(x,y) \in J$ we have

$$\begin{aligned}z(x,y) &\leq M\|\phi\|_B + K\|\phi(0,0)\| + \frac{K\|p\|_\infty a^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} \\ &\quad + \frac{K\|p\|_\infty}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} z(s,t)dtds.\end{aligned}$$

Then by Lemma 4.4, there exists $\delta = \delta(r_1, r_2)$ such that we have

$$\|z(x,y)\| \leq R + \delta \frac{K\|q\|_\infty}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} R dt ds,$$

where

$$R = M\|\phi\|_B + K\|\phi(0,0)\| + \frac{K\|p\|_\infty a^{r_1} b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}.$$

Hence

$$\|z\|_\infty \leq R + \frac{R\delta K\|q\|_\infty a^{r_1} b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} := \widetilde{M}.$$

Then, (4.6) implies that

$$\|w\|_\infty \leq \frac{a^{r_1} b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} (\|p\|_\infty + \widetilde{M}\|q\|_\infty) := M^*.$$

Set

$$U = \{w \in C_0 : \|w\|_{(a,b)} < M^* + 1\}.$$

$P : \overline{U} \to C_0$ is continuous and completely continuous. By our choice of U, there is no $w \in \partial U$ such that $w = \lambda P(w)$, for $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray–Schauder type [15], we deduce that N has a fixed point which is a solution to problem (1.1)–(1.3). □

Now we present two similar existence results for the problem (1.4)–(1.6).

Definition 4.6. A function $u \in \Omega$ is said to be a solution of (1.4)–(1.6) if u satisfies equations (1.4) and (1.6) on J and the condition (1.5) on $\tilde{J}$.

Let $f \in L^1(J, \mathbb{R}^n)$ and $g \in AC(J, \mathbb{R}^n)$ and consider the following linear problem

$$^cD_0^r\Big(u(x,y) - g(x,y)\Big) = f(x,y); \text{ a.e. } (x,y) \in J, \tag{4.7}$$

$$u(x,0) = \varphi(x),\ u(0,y) = \psi(y);\ (x,y) \in J, \tag{4.8}$$

with $\varphi(0) = \psi(0)$. For the existence of solutions for the problem (1.4)–(1.6), we need the following lemma.

Lemma 4.7. *A function* $u \in AC(J, \mathbb{R}^n)$ *is a solution of problem* (4.7)–(4.8) *if and only if* $u(x,y)$ *satisfies*

$$u(x,y) = \mu(x,y) + g(x,y) - g(x,0) - g(0,y) + g(0,0) + I_0^r(f)(x,y), \tag{4.9}$$

for a.e. $(x,y) \in J$.

Proof. Let $u(x,y)$ be a solution of problem (4.7)–(4.8). Then, taking into account the definition of the fractional Caputo derivative, we have

$$I_0^{1-r}\Big(D_{xy}^2(u(x,y)-g(x,y)\Big)=f(x,y).$$

Hence, we obtain

$$I_0^r\Big(I_0^{1-r}D_{xy}^2u)(u(x,y)-g(x,y)\Big)=(I_0^rf)(x,y),$$

then

$$I_0^1D_{xy}^2\Big(u(x,y)-g(x,y)\Big)=(I_0^rf)(x,y).$$

Since

$$\begin{aligned}I_0^1(D_{xy}^2)\Big(u(x,y)-g(x,y)\Big) &= u(x,y)-u(x,0)-u(0,y)+u(0,0)\\ &\quad -[g(x,y)-g(x,0)-g(0,y)+g(0,0)],\end{aligned}$$

we have

$$u(x,y)=\mu(x,y)+g(x,y)-g(x,0)-g(0,y)+g(0,0)+I_0^r(f)(x,y).$$

Now let $u(x,y)$ satisfy (4.9). It is clear that $u(x,y)$ satisfies (4.7)–(4.8). □

As a consequence of Lemma 4.7 we have the following auxiliary result

Corollary 4.8. *The function* $u\in\Omega$ *is a solution of problem* (1.4)–(1.6) *if and only if* u *satisfies the equation*

$$\begin{aligned}u(x,y) &= \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}f(s,t,u_{(s,t)})dsdt\\ &\quad +\mu(x,y)+g(x,y,u_{(x,y)})-g(x,0,u_{(x,0)})\\ &\quad -g(0,y,u_{(0,y)})+g(0,0,u_{(0,0)}),\end{aligned}$$

for all $(x,y)\in J$ *and the condition* (1.5) *on* $\tilde{J}$.

Theorem 4.9. *Assume that* (H_1) *holds and moreover*

(H_1') *there exists a nonnegative constant* ℓ' *such that*

$$\|g(x,y,u)-g(x,y,v)\|\le\ell'\|u-v\|_B,\ \textit{for each}\ (x,y)\in J,\ \textit{and}\ u,\ v\in B.$$

If

$$K\left[4\ell'+\frac{\ell a^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}\right]<1, \tag{4.10}$$

then there exists a unique solution for IVP (1.4)–(1.6) *on* $(-\infty,a]\times(-\infty,b]$.

Proof. Consider the operator $N_1 : \Omega \to \Omega$ defined by,

$$N_1(u)(x,y) = \begin{cases} \phi(x,y), & (x,y) \in \tilde{J}, \\ \mu(x,y) + g(x,y,u_{(x,y)}) - g(x,0,u_{(x,0)}) & \\ -g(0,y,u_{(0,y)}) + g(0,0,u_{(0,0)}) & \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} & \\ \times f(s,t,u_{(s,t)})dtds, & (x,y) \in J. \end{cases}$$

In analogy to Theorem 4.3, we consider the operator $P_1 : C_0 \to C_0$ defined by

$$\begin{aligned} P_1(x,y) &= g(x,y,\overline{w}_{(x,y)} + v_{(x,y)}) - g(x,0,\overline{w}_{(s,0)} + v_{(s,0)}) \\ &- \quad g(0,y,\overline{w}_{(0,y)} + v_{(0,y)}) + g(0,0,\overline{w}_{(0,0)} + v_{(0,0)}) \\ &+ \quad \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,\overline{w}_{(s,t)} + v_{(s,t)})dtds, (x,y) \in J. \end{aligned}$$

We shall show that the operator P_1 is a contraction. Let $w,\ w_* \in C_0$, then following the steps of Theorem 4.3, we have

$$\begin{aligned} \|P_1(w)(x,y) - P_1(w_*)(x,y)\| &\leq \|g(x,y,\overline{w}_{(x,y)} + v_{(x,y)}) - g(x,y,\overline{w_*}_{(x,y)} + v_{(x,y)})\| \\ &+ \|g(x,0,\overline{w}_{(x,0)} + v_{(x,0)}) - g(x,0,\overline{w_*}_{(x,0)} + v_{(x,0)})\| \\ &+ \|g(0,y,\overline{w}_{(0,y)} + v_{(0,y)}) - g(0,y,\overline{w_*}_{(0,y)} + v_{(0,y)})\| \\ &+ \|g(0,0,\overline{w} + v) - g(0,0,\overline{w_*}_{(0,0)} + v_{(0,0)})\| \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \\ &\times \|f(s,t,\overline{w}_{(s,t)} + v_{(s,t)}) - f(s,t,\overline{w_*}_{(s,t)} + v_{(s,t)})\|dtds \\ \leq \quad & 4\ell' K\|\overline{w} - \overline{w_*}\|_{(a,b)} + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \\ &\times \ell K\|\overline{w} - \overline{w_*}\|dtds. \end{aligned}$$

Therefore

$$\|P_1(w) - P_1(w_*)\|_{(a,b)} \leq K\left[4\ell' + \frac{\ell a^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2)+1}\right]\|\overline{w} - \overline{w_*}\|_{(a,b)},$$

which implies by (4.10) that P_1 is a contraction. Hence P_1 has a unique fixed point by Banach's contraction principle. □

Our last existence result for the IVP (1.4)–(1.6) is based on the nonlinear alternative of Leray–Schauder type.

Theorem 4.10. *Assume* (H_2) *and the following conditions:*

(H_3) *the function* g *is continuous and completely continuous, and for any bounded set* D *in* Ω, *the set* $\{(x,y) \to g(x,y,u_{(x,y)}) :\ u \in D\}$, *is equicontinuous in* $C(J,\mathbb{R}^n)$.

(H_4) *There exist constants* $0 \leq d_1 K < \frac{1}{4},\ d_2 \geq 0$ *such that*

$$\|g(x,y,u)\| \leq d_1\|u\|_B + d_2,\ (x,y) \in J,\ u \in B.$$

Then the IVP (1.4)–(1.6) *has at least one solution on* $(-\infty, a] \times (-\infty, b]$.

Proof. Let $P_1 : C_0 \to C_0$ defined as in Theorem 4.9. We shall show that the operator P_1 is continuous and completely continuous. Using (H_3) it suffices to show that the operator $P_2 : C_0 \to C_0$ defined by

$$\begin{aligned} P_2(w)(x,y) &= g(x,y,\overline{w}_{(x,y)} + v_{(x,y)}) - g(x,0,\overline{w}_{(x,0)} + v_{(x,0)}) \\ &\quad -g(0,y,\overline{w}_{(0,y)} + v_{(0,y)}) + g(0,0,\overline{w}_{(0,0)} + v_{(0,0)}) \\ &\quad +\frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,\overline{w}_{(s,t)} + v_{(s,t)}) dtds, \end{aligned}$$

is continuous and completely continuous. This was proved in Theorem 4.5. We now show there exists an open set $U \subseteq C_0$ with $w \neq \lambda P_2(w)$, for $\lambda \in (0,1)$ and $w \in \partial U$. Let $w \in C_0$ and $w = \lambda p_2(w)$ for some $0 < \lambda < 1$. Thus for each $(x,y) \in J$,

$$\begin{aligned} w(x,y) &= \lambda[g(x,y,\overline{w}_{(x,y)} + v_{(x,y)}) - g(x,0,\overline{w}_{(x,0)} + v_{(x,0)}) \\ &\quad -g(0,y,\overline{w}_{(0,y)} + v_{(0,y)}) + g(0,0,\overline{w}_{(0,0)} + v_{(0,0)})] \\ &\quad +\frac{\lambda}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,\overline{w}_{(s,t)} + v_{(s,t)}) dtds, \end{aligned}$$

and

$$\begin{aligned} \|w(x,y)\| &= 4d_1\|\overline{w}_{(x,y)} + v_{(x,y)}\|_B + \frac{\|p\|_\infty a^{r_1} b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} q(s,t) \|\overline{w}_{(s,t)} + v_{(s,t)}\|_B dtds. \end{aligned}$$

Using the above inequality and the definition of z we have that

$$\|z\|_\infty \leq R_1 + \frac{R_1 \delta K \|q^*\|_\infty a^{r_1} b^{r_2}}{(1-4d_1K)\Gamma(r_1+1)\Gamma(r_2+1)} := L,$$

where

$$R_1 = \frac{1}{1-4d_1K}\left[8d_2K + \frac{K\|p\|_\infty a^{r_1} b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}\right]$$

and

$$\|q^*\|_\infty = \frac{\|q\|_\infty}{1-4d_1K}.$$

Then

$$\begin{aligned}\|w\|_\infty &\le 4d_1\|\phi\|_B + 8d_2 + 4Ld_1 \\ &\quad + \frac{a^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}(\|p\|_\infty + L\|q\|_\infty) \\ &:= L^*.\end{aligned}$$

Set

$$U_1 = \{w \in C_0 : \|w\|_{(a,b)} < L^* + 1\}.$$

By our choice of U_1, there is no $w \in \partial U$ such that $w = \lambda P_2(w)$, for $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray–Schauder type [15], we deduce that N_1 has a fixed point which is a solution to problem (1.4)–(1.6). □

5 An Example

As an application of our results we consider the following partial hyperbolic functional differential equations of the form

$$({}^cD_0^r u)(x,y) = \frac{ce^{x+y-\gamma(x+y)}\|u_{(x,y)}\|}{(e^{x+y}+e^{-x-y})(1+\|u_{(x,y)}\|)}, \text{ if } (x,y) \in J := [0,1]\times[0,1], \quad (5.1)$$

$$u(x,0) = x,\ u(0,y) = y^2,\ x \in [0,1],\ y \in [0,1], \quad (5.2)$$

$$u(x,y) = x + y^2,\ (x,y) \in \tilde{J}, \quad (5.3)$$

where $\tilde{J} := (-\infty,1]\times(-\infty,1]\backslash(0,1]\times(0,1],\ c = \dfrac{2}{\Gamma(r_1+1)\Gamma(r_2+1)}$ and γ a positive real constant. Let

$$B_\gamma = \left\{u \in C((-\infty,0]\times(-\infty,0],\mathbb{R}) : \lim_{\|(\theta,\eta)\|\to\infty} e^{\gamma(\theta+\eta)}u(\theta,\eta) \text{ exists in } \mathbb{R}\right\}.$$

The norm of B_γ is given by

$$\|u\|_\gamma = \sup_{(\theta,\eta)\in(-\infty,0]\times(-\infty,0]} e^{\gamma(\theta+\eta)}|u(\theta,\eta)|.$$

Let

$$E := [0,1]\times\{0\} \cup \{0\}\times[0,1],$$

and $u : (-\infty,1]\times(-\infty,1] \to \mathbb{R}$ such that $u_{(x,y)} \in B_\gamma$ for $(x,y) \in E$. Then

$$\begin{aligned}\lim_{\|(\theta,\eta)\|\to\infty} e^{\gamma(\theta+\eta)}u_{(x,y)}(\theta,\eta) &= \lim_{\|(\theta,\eta)\|\to\infty} e^{\gamma(\theta-x+\eta-y)}u(\theta,\eta) \\ &= e^{\gamma(x+y)} \lim_{\|(\theta,\eta)\|\to\infty} u(\theta,\eta) \\ &< \infty.\end{aligned}$$

Hence $u_{(x,y)} \in B_\gamma$. Finally we prove that

$$\begin{aligned}\|u_{(x,y)}\|_\gamma &= K\sup\{|u(s,t)| : (s,t) \in [0,x]\times[0,y]\}\\ &\quad +M\sup\{\|u_{(s,t)}\|_\gamma : (s,t)\in E_{(x,y)}\},\end{aligned}$$

where $K = M = 1$ and $H = 1$. If $x+\theta \le 0,\ y+\eta \le 0$, then we get

$$\|u_{(x,y)}\|_\gamma = \sup\{|u(s,t)| : (s,t) \in (-\infty,0]\times(-\infty,0]\},$$

and if $x+\theta \ge 0,\ y+\eta \ge 0$, then we have

$$\|u_{(x,y)}\|_\gamma = \sup\{|u(s,t)| : (s,t) \in [0,x]\times[0,y]\}.$$

Thus for all $(x+\theta, y+\eta) \in [0,1]\times[0,1]$, we get

$$\begin{aligned}\|u_{(x,y)}\|_\gamma &= \sup\{|u(s,t)| : (s,t) \in (-\infty,0]\times(-\infty,0]\}\\ &\quad +\sup\{|u(s,t)| : (s,t) \in [0,x]\times[0,y]\}.\end{aligned}$$

Then

$$\begin{aligned}\|u_{(x,y)}\|_\gamma &= \sup\{\|u_{(s,t)}\|_\gamma : (s,t) \in E\}\\ &\quad +\sup\{|u(s,t)| : (s,t) \in [0,x]\times[0,y]|\}.\end{aligned}$$

$(B_\gamma, \|.\|_\gamma)$ is a Banach space. We conclude that B_γ is a phase space. Set

$$f(x,y,u_{(x,y)}) = \frac{ce^{x+y-\gamma(x+y)}\|u_{(x,y)}\|}{(e^{x+y}+e^{-x-y})(1+\|u_{(x,y)})\|},\ (x,y)\in[0,1]\times[0,1].$$

For each $u,\ \overline{u} \in B_\gamma$ and $(x,y) \in [0,1]\times[0,1]$ we have

$$\begin{aligned}|f(x,y,u_{(x,y)}) - f(x,y,\overline{u}_{(x,y)})| &\le \frac{e^{x+y}\|u-\overline{u}\|_B}{c(e^{x+y}+e^{-x-y})}\\ &\le \frac{1}{c}\|u-\overline{u}\|_B.\end{aligned}$$

Hence condition (H$_1$) is satisfied with $\ell = \dfrac{1}{c}$. Since $a = b = K = 1$ we get

$$\frac{\ell a^{r_1}b^{r_2}K}{\Gamma(r_1+1)\Gamma(r_2+1)} = \frac{1}{c\Gamma(r_1+1)\Gamma(r_2+1)} = \frac{1}{2} < 1,$$

for each $(r_1, r_2) \in (0,1]\times(0,1]$. Consequently Theorem 4.3 implies that problem (5.1)–(5.3) has a unique solution defined on $(-\infty,1]\times(-\infty,1]$.

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