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Discrete Approach on Oscillation of Difference Equations with Continuous Variable

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Abstract

In this study, we introduce a new method for investigation of the delay difference equation

$$\Delta_\alpha x(t) + p(t)x(t-\tau) = 0 \qquad \text{for } t \in [t_0, \infty),$$

where $p \in C\left([t_0, \infty), \mathbb{R}^+\right)$, $\alpha, \tau \in \mathbb{R}^+$ and Δ_α denotes the forward difference operator defined as $\Delta_\alpha x(t) = x(t+\alpha) - x(t)$.

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1 Introduction

In a number of recent papers, oscillatory behavior of the equation

$$\Delta_\alpha x(t) + p(t)x(t-\tau) = 0, \tag{1.1}$$

where $t_0 \le t \in \mathbb{R}$, $p \in C\left([t_0, \infty), \mathbb{R}_0^+\right)$, $\alpha, \tau \in \mathbb{R}^+$ and

$$\Delta_\alpha x(t) = x(t+\alpha) - x(t),$$

has been investigated. To the best of our knowledge, most of these papers depend on integral conditions to test oscillatory behavior of (1.1). We refer readers to [1–11]. Our aim is to make a discrete approach. Namely, we build new tests which do not depend on integral conditions. To do this, we assume $\delta := \dfrac{\tau}{\alpha} \in \mathbb{N}$.

We call a function a solution of (1.1) if this function satisfies (1.1) identically for $t \ge t_0$. We call a solution of (1.1) oscillatory if it has arbitrary large zeros, otherwise we call this solution nonoscillatory. Also, we are not interested in trivial solutions of (1.1).

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2 Main Results

Before stating our results, we introduce the functions

$$\begin{array}{rcccl} c_1: & [t_0, t_0+\alpha) & \to & \mathbb{R}_0^+ & \\ & t & \to & c_1(t) & := \liminf_{n\to\infty} p_n(t) \end{array}$$

and

$$\begin{array}{rcccl} c_2: & [t_0, t_0+\alpha) & \to & \mathbb{R}_0^+ & \\ & t & \to & c_2(t) & := \limsup_{n\to\infty} p_n(t), \end{array}$$

where

$$p_n(t) := p(t + n\alpha).$$

Now, we can give our results.

Theorem 2.1. *Assume that*

$$c_1(t) > 0 \quad \textit{and} \quad c_1(t) + c_2(t) > 1 \qquad \textit{for all } t \in [t_0, t_0+\alpha). \tag{2.1}$$

Then every solution of (1.1) *is oscillatory.*

Proof. For contrary assume that (1.1) has an eventually positive solution x. The case where (1.1) has an eventually negative solution is similar and omitted. So there exists $t_1 \geq t_0$ such that $x(t) > 0$ for all $t \geq t_1$. Then fix $t_2 \geq t_1 + \alpha$ and set

$$N_1 := \left\lfloor \frac{t_2 - t_0}{\alpha} \right\rfloor,$$

where $\lfloor \cdot \rfloor$ denotes the lowest integer function. Clearly there exists $s \in [t_0, t_0+\alpha)$ such that

$$t_2 = s + N_1\alpha$$

holds. Now define the sequence $\{x_n\}$ by

$$x_n := x(s + n\alpha) \qquad \text{for } n \in \mathbb{N},$$

and so $x_n > 0$ for all $n \geq N_1$. In view of (2.1), we have $\varepsilon > 0$ and $N_2 \geq N_1$ such that $c_1(s) > \varepsilon$ and $p_n(s) \geq c_1(s) - \varepsilon > 0$ for all $n \geq N_2$. Then from (1.1), we have

$$\Delta_\alpha x(s + n\alpha) = -p_n(s)x(s + (n-\delta)\alpha) \qquad \text{for all } n \geq N_3 := \max\{N_1 + \delta, N_2\}$$

or

$$\Delta x_n = -p_n(s)x_{n-\delta} < 0 \qquad \text{for all } n \geq N_3, \tag{2.2}$$

where Δ is the usual forward difference operator with

$$\Delta x_n = x_{n+1} - x_n.$$

From (2.2), $\{x_n\}$ is decreasing for all $n \geq N_3$. Then

$$x_n > p_n(s)x_{n-\delta} > [c_1(s) - \varepsilon]\, x_{n-1} \qquad \text{for all } n \geq N_3. \tag{2.3}$$

On the other hand, considering (2.2)

$$0 > \Delta x_n + p_n(s)x_{n-\delta} > x_{n+1} + [p_n(s) - 1]\, x_n \tag{2.4}$$

for all $n \geq N_3$. Thus from (2.3) and (2.4)

$$[c_1(s) - \varepsilon + p_n(s) - 1]\, x_n < 0 \qquad \text{for all } n \geq N_3,$$

that is,

$$[c_1(s) - \varepsilon + p_n(s) - 1] < 0 \qquad \text{for all } n \geq N_3$$

and taking $\limsup$ on both sides of the above inequality for $n \to \infty$, we see that

$$c_1(s) + c_2(s) \leq 1 + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$c_1(s) + c_2(s) \leq 1,$$

which contradicts with (2.1) and completes the proof. □

Now, we have the following example.

Example 2.2. Assume that $\alpha, \varepsilon \in \mathbb{R}^+$ and $n \in \mathbb{N}$. Then every solution of the difference equation

$$\Delta_\alpha x(t) + (1 + \varepsilon)\, x(t - \alpha n) = 0$$

is oscillatory on $[t_0, \infty)$. Clearly (2.1) holds and so Theorem 2.1 can be applied. Note that all the criteria depending on integral conditions depend on the delay but in this criteria we only need the $\liminf$ and $\limsup$ values of the coefficient.

Theorem 2.3. *Assume that*

$$c_1(t) > \frac{\delta^\delta}{(\delta + 1)^{\delta+1}} \qquad \textit{for all } t \in [t_0, t_0 + \alpha)\,. \tag{2.5}$$

Then every solution of (1.1) *is oscillatory.*

Proof. For contrary assume that (1.1) has an eventually positive solution. Since the equation is linear, the case where (1.1) has an eventually negative solution is omitted. Say this solution is x, so there exists $t_1 \geq t_0$ such that $x(t) > 0$ for all $t \geq t_1$. Fix $t_2 \geq t_1 + \alpha$ and set

$$N_1 := \left\lfloor \frac{t_2 - t_0}{\alpha} \right\rfloor.$$

There exists $s \in [t_0, t_0 + \alpha)$ such that $t_2 = s + N_1\alpha$ holds. Now define

$$r_n := \frac{x(s + n\alpha)}{x(s + (n+1)\alpha)} \qquad \text{for } n \in \mathbb{N}. \tag{2.6}$$

From (1.1) we have

$$x(s + (n+1)\alpha) = x(s + n\alpha) - p_n(s)x(s + (n-\delta)\alpha)$$

or

$$\frac{x(s + (n+1)\alpha)}{x(s + n\alpha)} = 1 - p_n(s)\frac{x(s + (n-\delta)\alpha)}{x(s + n\alpha)},$$

for all $n \in \mathbb{N}$. Considering (2.6), we get

$$\frac{1}{r_n} \leq 1 - p_n(s)\prod_{i=1}^{\delta} r_{n-i} \qquad \text{for all } n \geq N_1 + \delta. \tag{2.7}$$

From (2.5) there is N_2 with $p_n(s) > 0$ for all $n \geq N_2$. Set $N_3 := \max\{N_1 + \delta, N_2\}$, it follows from (2.7) that $r_n > 1$ for all $n \geq N_3$. Also r_n is bounded from above, otherwise (2.6) and (2.7) implies $r_n < 0$ for all sufficiently large n. Set $\kappa := \liminf_{n\to\infty} r_n$. Then from (2.7) we get

$$\limsup_{n\to\infty} \frac{1}{r_n} = \frac{1}{\kappa} \leq 1 - \liminf_{n\to\infty} p_n(s)\prod_{i=1}^{\delta} r_{n-i}. \tag{2.8}$$

Since

$$\liminf_{n\to\infty} p_n(s)\prod_{i=1}^{\delta} r_{n-i} \geq \left(\liminf_{n\to\infty} p_n(s)\right)\prod_{i=1}^{\delta}\left[\liminf_{n\to\infty} r_{n-i}\right] \geq c_1(s)\kappa^{\delta},$$

we have

$$\frac{1}{\kappa} \leq 1 - c_1(s)\kappa^{\delta}$$

or

$$h(\kappa) := \frac{\kappa - 1}{\kappa^{\delta+1}} \geq c_1(s). \tag{2.9}$$

One can show that

$$\max_{\kappa\geq 1} h(\kappa) = \frac{\delta^{\delta}}{(\delta+1)^{\delta+1}}$$

and hence by (2.9) we obtain

$$\frac{\delta^{\delta}}{(\delta+1)^{\delta+1}} \geq c_1(s),$$

which is a contradiction to (2.5). Therefore the proof is completed. □

We give the following example.

Example 2.4. Consider the difference equation

$$\Delta_\pi x(t) + (|\sin(t)| + \varepsilon)\, x(t - \pi n) = 0,$$

where $\varepsilon > \dfrac{\pi^\pi}{(\pi+1)^{(\pi+1)}}$ and $n \in \mathbb{N}$. Clearly,

$$c_1(t) = \liminf_{n\to\infty} (|\sin(t + n\pi)| + \varepsilon) = |\sin(t)| + \varepsilon > \frac{\pi^\pi}{(\pi+1)^{(\pi+1)}}$$

for all $t \in [t_0, t_0 + \pi)$. Therefore, (2.5) holds and by Theorem 2.3, we see that all solutions are oscillatory on $[t_0, \infty)$.

The following theorem improves the above one.

Theorem 2.5. *Assume that*

$$\liminf_{n\to\infty} \sum_{i=n-\delta}^{n-1} p_i(t) > 0 \qquad \textit{for all } t \in [t_0, t_0 + \alpha)\,. \tag{2.10}$$

Furthermore if

$$c_3(t) := \liminf_{n\to\infty} \inf_{\lambda\in\Gamma(t)} \left\{ \frac{1}{\lambda} \prod_{i=n-\delta}^{n-1} \frac{1}{1 - \lambda p_i(t)} \right\} > 1 \qquad \textit{for all } t \in [t_0, t_0 + \alpha)\,, \tag{2.11}$$

where

$$\Gamma(t) := \{\lambda > 0 : 1 - \lambda p_n(t) > 0 \textit{ for all large } n\}\,,$$

then every solution of (1.1) *is oscillatory on* $[t_0, \infty)$.

Proof. As usual assume for contrary that x is an eventually positive solution of (1.1) and $x(t) > 0$ for all $t \geq t_1$. Set

$$N_1 := \left\lfloor \frac{t_2 - t_0}{\alpha} \right\rfloor,$$

where $t_2 \geq t_1 + \alpha$. Then we have $s \in [t_0, t_0 + \alpha)$ such that $t_2 = s + N_1\alpha$. Now set

$$x_n := x(s + n\alpha) \qquad \text{for } n \in \mathbb{N}.$$

Then $x_n > 0$ for all $n \geq N_1$. From (1.1)

$$\Delta x_n = -p_n(s) x_{n-\delta} < 0 \qquad \text{for all } n \geq N_1 + \delta. \tag{2.12}$$

Thus x_n is decreasing. Define the set Λ (see [7]) by

$$\Lambda := \{\lambda > 0 : \Delta x_n + \lambda p_n(s) x_n \leq 0 \text{ for all large } n\}. \tag{2.13}$$

Since $1 \in \Lambda$, $\Lambda \neq \emptyset$. And one can easily show that $\Lambda \subset \Gamma(s)$. Considering (2.10), set

$$\kappa := \frac{1}{6} \liminf_{n\to\infty} \sum_{i=n-\delta}^{n-1} p_i(s) > 0.$$

Then there exists $N_2 \geq N_1$ such that

$$\sum_{i=n-\delta}^{n-1} p_i(s) > 3\kappa \qquad \text{for all } n \geq N_2.$$

Thus there exists an increasing divergent sequence $\{s_n\}$ on $[N_2, \infty)$ such that

$$\sum_{i=s_n-\delta}^{n-1} p_i(s) > \kappa \quad \text{and} \quad \sum_{i=n}^{s_n-1} p_i(s) > \kappa$$

with $s_n \geq n$ for all $n \geq N_2$. Therefore by (2.12) we get

$$\begin{aligned} x_n &> x_n - x_{s_n} = -\sum_{i=n}^{s_n-1} \Delta x_i = \sum_{i=n}^{s_n-1} p_i(s) x_{i-\delta} \geq x_{s_n-\delta} \sum_{i=n}^{s_n-1} p_i(s) > \kappa x_{s_n-\delta} \\ &> \kappa \left[x_{s_n-\delta} - x_{n+1}\right] = -\kappa \sum_{i=s_n-\delta}^{n} \Delta x_i = \kappa \sum_{i=s_n-\delta}^{n} p_i(s) x_{i-\delta} > \kappa^2 x_{n-\delta}, \end{aligned}$$

which implies

$$\frac{x_{n-\delta}}{x_n} < \frac{1}{\kappa^2} \qquad \text{for all } n \geq N_2.$$

Hence

$$x_n - x_{n-\delta} = \sum_{i=n-\delta}^{n-1} \Delta x_i = -\sum_{i=n-\delta}^{n-1} p_i(s) x_{i-\delta} \leq -x_{n-\delta} \sum_{i=n-\delta}^{n-1} p_i(s) \leq -2\kappa x_{n-\delta}$$

for all $n \geq N_2$. Thus

$$2\kappa x_{n-\delta} \geq x_{n-\delta}$$

and

$$\Delta x_n = -p_n(s) x_{n-\delta} > -\frac{1}{2\kappa} x_{n-\delta} \geq -\frac{1}{2\kappa^3} x_{n-\delta},$$

which implies $\frac{1}{2\kappa^3} \notin \Lambda$. Therefore $\Lambda \subset \mathbb{R}$ is a bounded interval. From (2.11), there exist a constant $c > 1$ and $N_3 \geq N_2$ such that

$$\inf_{\lambda \in \Gamma(s)} \left\{ \frac{1}{\lambda} \prod_{i=n-\delta}^{n-1} \frac{1}{1 - \lambda p_i(s)} \right\} \geq c \tag{2.14}$$

for all $n \geq N_3$. Let $\sigma := \frac{c+1}{2} \sup \Lambda$. Since $\sigma \in \Lambda \subset \Gamma(s)$, we have

$$\Delta x_n + \sigma p_n(s) x_n \leq 0 \qquad \text{for all } n \geq N_3. \tag{2.15}$$

Set

$$r_n := \frac{x_n}{x_{n+1}} \qquad \text{for all } n \geq N_3.$$

Then from (2.15), we see that

$$r_n \geq \frac{1}{1 - \sigma p_n(s)} \qquad \text{for all } n \geq N_3,$$

which yields

$$\frac{x_{n-\delta}}{x_n} = \prod_{i=n-\delta}^{n-1} r_i \geq \prod_{i=n-\delta}^{n-1} \frac{1}{1 - \sigma p_i(s)} = \left(\frac{1}{\sigma} \prod_{i=n-\delta}^{n-1} \frac{1}{1 - \sigma p_i(s)} \right) \sigma \geq c\sigma \tag{2.16}$$

for all $n \geq N_3 + \delta$. Then from (2.12) and (2.16), we get

$$\Delta x_n + c\sigma p_n(s) x_n \leq 0 \qquad \text{for all } n \geq N_3 + \delta,$$

which implies $c\sigma \in \Lambda$. Since $c\sigma = \frac{c(c+1)}{2} \sup \Lambda > \sup \Lambda$, this leads to a contradiction. Therefore the proof is completed. □

Corollary 2.6. *If*

$$c_4(t) := \liminf_{n \to \infty} \sum_{i=n-\delta}^{n-1} p_i(t) > \left(\frac{\delta}{\delta + 1} \right)^{\delta+1} \qquad \textit{for all } t \in [t_0, t_0 + \alpha)$$

holds, then every solution of (1.1) *is oscillatory on* $[t_0, \infty)$.

Proof. By the arithmetic and geometric mean inequalities, we have

$$c_3(t) \geq \left(\frac{\delta + 1}{\delta} \right)^{\delta+1} c_4(t) > 1 \qquad \text{for all } t \in [t_0, t_0 + \alpha),$$

which implies (2.10) and (2.11) holds. Therefore the claim follows by Theorem 2.5. □

Remark 2.7. Since $c_4(t) \geq \delta c_1(t)$ for all $t \in [t_0, t_0 + \alpha)$, Corollary 2.6 improves Theorem 2.3.

Remark 2.8. Our assumptions also guarantee that there are no positive solutions of inequalities of the form

$$\Delta_\alpha x(t) + p(t) x(t - \tau) \leq 0.$$

References

[1] Ravi P. Agarwal. *Difference equations and inequalities*, volume 228 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker Inc., New York, second edition, 2000. Theory, methods, and applications.

[2] Ming Po Ch'ên and B. G. Zhang. Oscillation and comparison theorems of difference equations with positive and negative coefficients. *Bull. Inst. Math. Acad. Sinica*, 22(4):295–306, 1994.

[3] L. H. Erbe and B. G. Zhang. Oscillation of discrete analogues of delay equations. *Differential Integral Equations*, 2(3):300–309, 1989.

[4] J. Jaroš and I. P. Stavroulakis. Necessary and sufficient conditions for oscillations of difference equations with several delays. *Utilitas Math.*, 45:187–195, 1994.

[5] G. Ladas, Ch. G. Philos, and Y. G. Sficas. Sharp conditions for the oscillation of delay difference equations. *J. Appl. Math. Simulation*, 2(2):101–111, 1989.

[6] G. Ladas and C. Qian. Oscillatory behavior of difference equations with positive and negative coefficients. *Matematiche (Catania)*, 44(2):293–309 (1990), 1989.

[7] G. Ladas, Y. G. Sficas, and I. P. Stavroulakis. Necessary and sufficient conditions for oscillations. *Amer. Math. Monthly*, 90(9):637–640, 1983.

[8] Jian Hua Shen. Comparison theorems for oscillations of difference equations with continuous variables. *Kexue Tongbao (Chinese)*, 41(16):1441–1444, 1996.

[9] X. H. Tang, J. S. Yu, and D. H. Peng. Oscillation and nonoscillation of neutral difference equations with positive and negative coefficients. *Comput. Math. Appl.*, 39(7-8):169–181, 2000.

[10] J. S. Yu, B. G. Zhang, and X. Z. Qian. Oscillations of delay difference equations with oscillating coefficients. *J. Math. Anal. Appl.*, 177(2):432–444, 1993.

[11] B. G. Zhang. Oscillations of a class of difference equations with continuous arguments. *Appl. Math. Lett.*, 14(5):557–561, 2001.